Integrable models: Intrinsic properties and Classification.

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Intrinsic properties.

Main concepts:
Symmetries,
Conservation laws,
Backlund transformations,
Rucursion operators,
Hamiltonian operators,

Main problem: Nonlocalities.

Technique: Jet calculus.

Our main task is to formulate constructive necessary integrability conditions for evolution equations with two independent variables.

Example. Consider equations of the form

$$u_t = u_3 + f(u_1, u), \qquad u_1 = u_x,\ u_2 = u_{xx}, \ldots$$

Then if the equation is integrable then

$$\rho = \frac{\partial f}{\partial u_1}$$

is a local conserved density, i.e. $\rho_t = \sigma_x$.

For example, for the mKdV-equation $u_t = u_3 + 3u^2 u_1$ we expect that $\rho = 3u^2$ is a conserved density. Indeed,

$$(3u^2)_t = \left(6uu_2 - 3u_1^2 + \frac{3}{2}u^4\right)_x.$$

ODE case.

Suppose we have a dynamical system

$$\frac{d\,u_i}{dt} = F_i(u_1, \ldots, u_n), \qquad i = 1, \ldots, n\,. \tag{1}$$

Then any function $G(u_1, \ldots, u_n)$ can be differentiated in time in virtue of the system (1) as

$$\frac{d\,G}{dt} = \sum_{k=1}^{n} F_k(u_1, \ldots, u_n)\frac{\partial G}{\partial u_k}. \tag{2}$$

Now we can forget that $u_1, \ldots u_n$ are functions of time t and regard them as the set of *independent* variables. Denote by $\mathfrak{F}$ the ring of "all" functions of these variables.

We can rewrite (2) as $\frac{d\,G}{dt} = X_F(G)$, where

$$X_F = \sum_{k=1}^{n} F_k \frac{\partial}{\partial u_k}. \tag{3}$$

Definition. Linear homogeneous differential operator of the form

$$X = \sum_{k=1}^{n} X_k(u_1, \ldots, u_n) \frac{\partial}{\partial u_k}, \tag{4}$$

is called a **vector field**.

Remark 1. We have $X(fg) = fX(g) + gX(f)$ i.e. any vector field defines a derivation of $\mathfrak{F}$.

Remark 2. All vector fields form a Lie algebra w.r.t. the Lie bracket

$$[X, Y] = X \circ Y - Y \circ X.$$

First integrals

First integrals of a dynamical system can be defined as elements of the kernal space for the corresponding vector field.

Definition. A function $I = I(u_1, \ldots, u_n)$ is a *first integral* of the dynamical system (1) if $\frac{dI}{dt} = X_F(I) = 0$.

Any function of first integrals is a first integral. Only functionally independent first integrals are to be counted.

Symmetries.

The next fundamental concept of the local theory of nonlinear ODEs is the infinitesimal symmetry.

Definition. A vector field

$$X_G = \sum_{k=1}^{n} G_k(u_1, u_2, \ldots, u_n)\frac{\partial}{\partial u_k}, \qquad (5)$$

is called (infinitesimal) symmetry of dynamical system (1) iff

$$[X_F,\, X_G] = 0. \qquad (6)$$

Condition (6) is equivalent to the fact that the dynamical systems (1) and

$$\frac{d\,u_i}{d\tau} = G_i(u_1, \ldots, u_n), \qquad i = 1, \ldots, n\,. \qquad (7)$$

are compatible. It means that for any initial data $\mathbf{u}_0$ there exists a common solution $\mathbf{u}(t, \tau)$ of systems (1) and (7) such that $\mathbf{u}(0, 0) = \mathbf{u}_0$.

The symmetry condition (6) can also be written in the following two equivalent forms:

$$\frac{d\mathbf{G}}{dt} = \mathbf{F}_*(\mathbf{G}). \tag{8}$$

or

$$\mathbf{F}_*(\mathbf{G}) - \mathbf{G}_*(F) = 0. \tag{9}$$

Here $\mathbf{F} = (F_1, \ldots, F_n)$ and $\mathbf{F}_*$ is a matrix with entries

$$F_{*i,j} = \frac{\partial F_i}{\partial u_j}.$$

The matrix $\mathbf{F}_*$ is called the Fréchet derivative of the vector–function $\mathbf{F} = (F_1, \ldots, F_n)$. Relation (8) means that $\mathbf{G}$ satisfies the linearization of dynamical system (1).

Hamiltonian structures.

Any Poisson bracket between functions $f(u_1, \ldots, u_m)$ and $g(u_1, \ldots, u_m)$ is given by

$$\{f,\, g\} = \sum_{i,j} P_{i,j}(u_1, \ldots, u_m) \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial u_j},$$

where $P_{i,j} = \{u_i, u_j\}$. The equivalent form is

$$\{f,\, g\} = < \mathrm{grad}\, f,\ P(\mathrm{grad}\, g) > .$$

The entries P_{ij} of the Hamiltonian operator P are not arbitrary since by definition

$$\{f, g\} = -\{g, f\},$$

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0.$$

The Hamiltonian dynamics is defined by

$$\frac{du_i}{dt} = \{H, u_i\},$$

or

$$\frac{d\vec{u}}{dt} = P(\mathrm{grad}\, H),$$

where H is a Hamiltonian function.

If $\{K, H\} = 0$, then K is an integral of motion for the dynamical system. Moreover, the vector fields corresponding to Hamiltonians H and K commute each other.

If $\{J, f\} = 0$ for any f, then J is called the *Casimir function* of the Poisson bracket. The Casimir functions exist if the bracket is degenerate (i.e. $Det\, P = 0$).

For the symplectic manifold the coordinates are q_i and p_i, $i = 1, \ldots N$. The standard Poisson bracket is given by

$$\{p_i, p_j\} = \{q_i, q_j\} = 0, \quad \{p_i, q_j\} = \delta_{i,j}.$$

The corresponding dynamical system has the usual Hamiltonian form

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}.$$

For the spinning tops the Hamiltonian structure is defined by a linear Poisson bracket. In this case

$$P_{ij} = C_{ij}^k u_k.$$

It is well-known that this formula defines a Poisson bracket iff C_{ij}^k are structure constants of a Lie algebra.

The class of quadratic Poisson brackets

$$P_{ij} = C_{ij}^{kl} \, u_k u_l$$

is of a great importance for the modern mathematical physics.

Two Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are said to be compatible if

$$\{\cdot,\cdot\}_\lambda = \{\cdot,\cdot\}_1 + \lambda\{\cdot,\cdot\}_2$$

is a Poisson bracket for any λ.

Theorem. Let

$$H(\lambda) = H_0 + \lambda H_1 + \lambda^2 H_2 + \cdots$$

be a Casimir function for the bracket $\{\cdot,\cdot\}_\lambda$. Then the coefficients H_i commute each other with respect to both brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$.

The dynamical equation for any Hamiltonian H_i can be written in two Hamiltonian forms:

$$\frac{du_i}{dt} = \{H_i, u_i\}_1 = \{H_{i-1}, u_i\}_2.$$

PDE case. Independent jet variables.

Let $x_1, ..., x_n$ be independent variables and u is the dependent variable.

Suppose we have no differential equation at all. All symbols

$$u, \quad \text{and} \quad u_\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_n} u}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n} \tag{10}$$

where $\alpha = (\alpha_1, ..., \alpha_n)$, are regarded as *independent* variables.

In this case, $\mathfrak{F}$ is the ring of "all" functions depending on a finite number of variables (10).

We have the total derivative operators

$$D_i = \sum_\alpha u_{(\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_n)} \frac{\partial}{\partial\, u_{(\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n)}}$$

They are derivations of $\mathfrak{F}$ such that $[D_i, D_j] = 0$.

If we consider a differential equation, there are relations between variables (10) and we must choose a complete set of independent jet variables. This set plays role of coordinates for the equation.

The procedure looks very simple for the evolutionary equations

$$u_t = F(u, u_x, u_{xx}, \dots, \frac{\partial^n u}{\partial x^n}) \qquad (11)$$

with one dependent and two independent variables. All partial derivatives of u, containing differentiations w.r.t. t, can be eliminated in virtue of the equation and it's differential consequences. For example,

$$u_{xt} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial u} u_x + \dots + \frac{\partial F}{\partial u_n} u_{n+1}.$$

So, one can represent any mixed derivative as a function depending on a finite number of the following variables

$$u_0 = u,\ u_1 = u_x,\ u_2 = u_{xx},\ \dots,\ u_i = \frac{\partial^i u}{\partial x^i},\ \dots\ .$$

We know how to differentiate all these variables w.r.t. x:

$$(u_0)_x = u_1,\ \dots,\ (u_i)_x = u_{i+1},\ \dots .$$

This dynamical system coincides with the total x-derivative for the jet variables with one dependent and one independent variables. The corresponding vector field is given by:

$$D_x = \frac{\partial}{\partial x} + \sum_0^{\infty} u_{i+1} \frac{\partial}{\partial u_i}. \tag{12}$$

Total t-derivative depends on r.h.s. F of evolution equation:

$$(u_0)_t = F(u, u_1, \dots, u_n),\ \dots,\ (u_i)_t = D_x^i(F),\ \dots.$$

The corresponding vector field is as follows:

$$D_t = \sum_0^{\infty} D_x^i(F) \frac{\partial}{\partial u_i}. \tag{13}$$

However, for some problems a different choice of independent jet variables turns out to be more suitable.

Example. Consider the KdV equation

$$u_t = u_{xxx} + uu_x$$

and take

$$u, \quad u_1 = u_t, \quad u_2 = u_{tt} \cdots$$

$$v = u_x, \quad v_1 = v_t, \quad v_2 = v_{tt} \cdots$$

$$w = u_{xx}, \quad w_1 = w_t, \quad w_2 = w_{tt} \cdots$$

for independent jet variables. Then

$$D_t = \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i} + \sum_{i=0}^{\infty} v_{i+1} \frac{\partial}{\partial v_i} + \sum_{i=0}^{\infty} w_{i+1} \frac{\partial}{\partial w_i},$$

$$D_x = v \frac{\partial}{\partial u} + w \frac{\partial}{\partial v} + (u_1 - uv) \frac{\partial}{\partial w} + \cdots$$

Let us consider now the hyperbolical equations of the form

$$u_{xy} = G(u, u_x, u_y).$$

The most natural choice of independent variables is

$$\begin{aligned} u_0 = \bar{u}_0 = u,\ \ u_1 = u_x,\ \ u_2 = u_{xx}, \ldots, \\ \bar{u}_1 = u_y,\ \ \bar{u}_2 = u_{yy}, \ldots, \end{aligned}$$

It is not difficult to prove by induction that all mixed derivatives of u can be expressed through them.

The corresponding dynamical systems have the form

$$(u_i)_x = u_{i+1},\ i \in \mathbb{Z}_+,$$

$$(\bar{u}_i)_x = a_i(x, y, u, u_1, \bar{u}_1, \dots, \bar{u}_i), \qquad i \in \mathbb{N},$$

and

$$(\bar{u}_i)_y = \bar{u}_{i+1},\ i \in \mathbb{Z}_+,$$

$$(u_i)_x = \bar{a}_i(x, y, u, \bar{u}_1, u_1, \dots, u_i), \qquad i \in \mathbb{N},$$

where the functions a_i and $\bar{a}_i$ are defined recursively in the following way:

$$a_1 = \bar{a}_1 = G(u, u_1, \bar{u}_1),$$

$$a_2 = (a_1)_y = \frac{\partial G}{\partial u}\bar{u}_1 + \frac{\partial G}{\partial \bar{u}_1}G + \frac{\partial G}{\partial \bar{u}_1}\bar{u}_2,$$

$$\bar{a}_2 = (\bar{a}_1)_x = \frac{\partial G}{\partial u}u_1 + \frac{\partial G}{\partial u_1}u_2 + \frac{\partial G}{\partial \bar{u}_1}G,$$

$$a_3 = (a_2)_y, \quad \bar{a}_3 = (\bar{a}_2)_x, \dots .$$

The corresponding total derivatives are given by the formulas

$$D_x = \sum_0^\infty u_{i+1}\frac{\partial}{\partial u_i} + \sum_1^\infty D_y^{i-1}(G)\frac{\partial}{\partial \bar{u}_i}$$

and

$$D_y = \sum_0^\infty \bar{u}_{i+1}\frac{\partial}{\partial \bar{u}_i} + \sum_1^\infty D_x^{i-1}(G)\frac{\partial}{\partial u_i}.$$

It seems that the definition of D_x is based on the definition of D_y and vice verse. However, these vector fields are well-defined.

Example. Consider the Liouville equation $u_{xy} = \exp(u)$. Then

$$D_x = \sum_0^\infty u_{i+1}\frac{\partial}{\partial u_i}+$$

$$\exp(u)\left(\frac{\partial}{\partial \bar{u}_1} + \bar{u}_1\frac{\partial}{\partial \bar{u}_2} + (\bar{u}_2 + \bar{u}_1^2)\frac{\partial}{\partial \bar{u}_3} + \cdots\right).$$

It is easy to verify that $D_x(\bar{u}_2 - \frac{1}{2}\bar{u}_1^2) = 0$.

Scalar evolution equations.

Main notions: Denote by $\mathfrak{F}$ the ring of "all" functions depending on a finite number of *independent* jet variables

$$u,\, u_1 = u_x,\, u_2 = u_{xx}, \ldots, \tag{14}$$

In these variables the vector field

$$D_x = u_1 \frac{\partial}{\partial u_0} + u_2 \frac{\partial}{\partial u_1} + u_3 \frac{\partial}{\partial u_2} + \cdots\,, \tag{15}$$

represents the total derivative operator with respect to x.

Remark. Not any function $f(u, u_1, ..., u_k)$ belongs to $\operatorname{Im} D_x$. If $f \in \operatorname{Im} D_x$, then $\frac{\delta f}{\delta u} = 0$, where

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x \circ \frac{\partial}{\partial u_1} + D_x^2 \circ \frac{\partial}{\partial u_2} - \cdots$$

Generalized symmetries.

Consider evolution equation

$$u_t = F(u, u_x, u_{xx}, \ldots, u_n), \quad u_i = \frac{\partial^i u}{\partial x^i}. \quad (16)$$

The corresponding total t-derivative is given by:

$$D_t = \sum_0^\infty D_x^i(F)\frac{\partial}{\partial u_i}. \quad (17)$$

The generalized (higher) symmetry is an evolution equation

$$u_\tau = G(u, u_x, u_{xx}, \ldots, u_m), \quad (18)$$

that is compatible with (16).

More rigorously, the total τ-derivative is given by:

$$D_\tau = \sum_0^\infty D_x^i(G)\frac{\partial}{\partial u_i}. \quad (19)$$

Definition. Equation (18) is called *infinitesimal local symmetry* for (16) if

$$[D_t, D_\tau] = 0.$$

Calculating the coefficients of $\frac{\partial}{\partial u}$, we find that

$$D_t(G) = F_*(G), \qquad (20)$$

or

$$F_*(G) - G_*(F) = 0, \qquad (21)$$

where

$$a_* = \sum_k \frac{\partial a}{\partial u_k} D_x^k$$

denotes the Fréchet derivative of element $a \in \mathcal{F}$.

The left hand side of (21) must be identically zero w.r.t. jet variables (14).

Cosymmetries.

The dual objects for symmetries are cosymmetries which satisfy the equation

$$D_t(g) + F_*^t(g) = 0,$$

where

$$F_*^t = \sum_k (-1)^k D_x^k \circ \frac{\partial F}{\partial u_k}$$

is the differential operator adjoint to F_*. The product $g\,G$ of any cosymmetry g and symmetry G is a total x-derivative.

Example 1. For any m equation $u_\tau = u_m$ is a symmetry for $u_t = u_n$.

Example 2. The Burgers equation

$$u_t = u_{xx} + 2uu_x$$

has the following third order symmetry

$$u_\tau = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x.$$

Example 3. The simplest higher symmetry for the Korteweg-de Vries equation

$$u_t = u_{xxx} + 6uu_x$$

has the following form

$$u_\tau = u_5 + 10uu_3 + 20u_1u_2 + 30u^2u_1.$$

Recursion operators.

The simplest symmetry for any equation (16) is u_x. Indeed, the total derivative related to the equation $u_\tau = u_x$ coincides with D_x.

The usual way to get other symmetries is to act to u_x by a recursion operator $\mathcal{R}$. By definition, the recursion operator is a ratio of two differential operators that satisfies the identity

$$[D_t - F_*, \mathcal{R}] = \mathcal{R}_t - [F_*, \mathcal{R}] = 0. \qquad (22)$$

It follows from (20) and (22) that for any symmetry G the expression $\mathcal{R}(G)$ is a symmetry as well.

For example, for the Korteweg-de Vries equation $u_t = u_{xxx} + 6\,u\,u_x$ the simplest recursion operator

$$\mathcal{R} = D_x^2 + 4u + 2u_x D_x^{-1} \tag{23}$$

is the ratio of two differential operators

$$\mathcal{H}_1 = D_x, \qquad \mathcal{H}_2 = D_x^3 + 4uD_x + 2u_x.$$

Most of known recursion operators have the following special form

$$\mathcal{R} = R + \sum_{i=1}^{k} G_i D_x^{-1} g_i, \qquad (24)$$

where R is a differential operator, G_i and g_i are some fixed symmetries and cosymmetries common for all members of the hierarchy. We call recursion operators (24) *quasilocal*.

Applying such operator to any symmetry, we get a local expression, (i.e. a function of finite number of variables $u, u_x, \ldots u_i, \ldots$) since the product of any symmetry and cosymmetry belongs to $Im\, D_x$.

Moreover, a different choice of integration constants gives rise to an additional linear combination of the symmetries $G_1, \ldots, G_k$.

It is possible to prove that for the Korteweg-de Vries equation the associative algebra $\mathbf{A}$ of all quasilocal recursion operators is generated by one operator (23). In other words, this algebra is isomorphic to the algebra of all polynomials in one variable.

However, it is not true for the Krichever-Novikov equation

$$u_{t_1} = u_{xxx} - \frac{3}{2}\frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x}, \qquad P^{(V)} = 0.$$

It turns out that there exist two quasilocal recursion operators $\mathcal{R}_1$ and $\mathcal{R}_2$ of orders 4 and 6 related by the elliptic curve equation

$$\mathcal{R}_2^2 = \mathcal{R}_1^3 - \phi\mathcal{R}_1 - \theta.$$

Conservation laws.

The notion of first integrals, in contrast to symmetries, cannot be generalized to the case of PDEs. It is replaced by the concept of local conservation laws, which are also related to constants of motion.

Definition. A function $\rho \in \mathfrak{F}$ is called a *density* of a local conservation law for equation (16) if there exist a function $\sigma \in \mathfrak{F}$ such that

$$D_t(\rho) = D_x(\sigma). \tag{25}$$

The function σ is called a *flux* of the conservation law.

We can eliminate function σ applying to (25) the Euler operator to get

$$\frac{\delta D_t(\rho)}{\delta u} = 0.$$

Example. Functions

$$\rho_1 = u, \quad \rho_2 = u^2, \quad \rho_3 = -u_1^2 + 2u^3$$

are conserved densities of the Korteweg - de Vries equation $u_t = u_3 + 6uu_1$. Indeed,

$$D_t(u) = D_x(u_2 + 3u^2),$$

$$D_t(u^2) = D_x(2uu_2 - u_1^2 + 4u^3),$$

$$D_t(\rho_3) = D_x(9u^4 + 6u^2u_2 + u_2^2 - 12uu_1^2 - 2u_1u_3)\,.$$

Function u^3 is not a density of a conservation law for the Korteweg de Vries equation. Indeed, $D_t(u^3) = 3u^2u_3 + 18u^3u_1$. In order to check that the right-hand side is not a total derivative we apply the Euler operator

$$\frac{\delta}{\delta u}(3u^2u_3 + 18u^3u_1) = -18u_1u_2 \neq 0\,.$$

If u is a function periodic in space with period L, then $I_k = \int_0^L \rho_k\, dx$ do not depend on time and are constants of motion.

Relation (25) is evidently satisfied if $\rho = D_x(h)$ for any $h \in \mathfrak{F}$. In this case $\sigma = D_t(h)$. Such "conservation laws" we call trivial.

Definition. Two conserved densities ρ_1, ρ_2 are called equivalent $\rho_1 \sim \rho_2$ if the difference $\rho_1 - \rho_2$ is a trivial density (i.e. $\rho_1 - \rho_2 \in \mathrm{Im} D_x$).

Lemma. For any conserved density ρ, function $g = \dfrac{\delta \rho}{\delta u}$ is a cosymmetry.

Hamiltonian operators.

Most of known integrable equations can be written in a Hamiltonian form

$$u_t = \mathcal{H}\left(\frac{\delta\rho}{\delta u}\right),$$

where ρ is a conserved density and $\mathcal{H}$ is a Hamiltonian operator. It is known that this operator satisfies the equation

$$(D_t - F_*)\,\mathcal{H} = \mathcal{H}(D_t + F_*^t), \qquad (26)$$

which means that $\mathcal{H}$ takes cosymmetries to symmetries. Besides (26), the Hamiltonian operator should satisfy relations equivalent to the skew-symmetricity and the Jacobi identity for the corresponding Poisson bracket

$$\{f,\, g\} = \frac{\delta f}{\delta u}\ \mathcal{H}\left(\frac{\delta g}{\delta u}\right).$$

It is easy to see that the ratio $\mathcal{H}_2\mathcal{H}_1^{-1}$ of any two Hamiltonian operators is a recursion operator.

As the rule, the Hamiltonian operators are local (i.e. differential) or quasilocal operators. The latter means that

$$\mathcal{H} = H + \sum_{i=1}^{m} G_i D_x^{-1} \bar{G}_i, \tag{27}$$

where H is a differential operator and $G_i, \bar{G}_i$ are fixed symmetries. It is clear that acting by the operator (27) on any cosymmetry, we get a local symmetry.

For example, the Korteweg-de Vries equation can be represent in the Hamiltonian form in two different ways:

$$u_t = \mathcal{H}_1 \left(\frac{\delta \rho_3}{\delta u}\right) = \mathcal{H}_2 \left(\frac{\delta \rho_2}{\delta u}\right),$$

where $\mathcal{H}_1 = D_x, \quad \mathcal{H}_2 = D_x^3 + 4uD_x + 2u_x.$

For the Krichever-Novikov equation the simplest quasilocal Hamiltonian operator is given by

$$\mathcal{H}_1 = u_x D_x^{-1} u_x.$$

Symmetry approach to classification of integrable equations.

1979-2006

Was developed by: A.Shabat, A.Zhiber, N.Ibragimov, A.Fokas, V.Sokolov, S.Svinolupov, A.Mikhailov, R.Yamilov, V.Adler, P.Olver, J.Sanders, J.P.Wang, V.Novikov, A.Meshkov, D.Demskoy, H.Chen, Y.Lee, C.Liu, I.Khabibullin, B.Magadeev, R.Heredero, V.Marikhin ...

Definition. PDE is integrable if it possesses infinitely many generalized symmetries.

Why integrable equations possess higher symmetries?

"Explanation". A linear equation has infinitely many higher symmetries. Integrable nonlinear equation is related to a linear one by some transformation. The same transformation produces higher symmetries for nonlinear equation starting from symmetries of the linear one.

For instance, the Burgers equation is integrable because of the Cole-Hopf substitution

$$u = \frac{v_x}{v},$$

which relates the equation to $v_t = v_{xx}$. Moreover, the same substitution maps the third order symmetry of the Burgers equation to

$$v_\tau = v_{xxx},$$

etc.

The first classification result in frames of the symmetry approach was:

Theorem. (Shabat-Zhiber 1979)
Nonlinear hyperbolic equation of the form

$$u_{xy} = F(u)$$

possesses higher symmetries iff (up to scalings and shifts)

$F(u) = e^u$, $F(u) = e^u + e^{-u}$, or $F(u) = e^u + e^{-2u}$.

The complete classification of integrable hyperbolic equations of the form

$$u_{xy} = F(u, u_x, u_y)$$

is an open problem till now.

Example:

$$u_{xy} = S(u)\sqrt{1 - u_x^2}\sqrt{1 - u_y^2},$$

$$S'' - 2S^3 + c\,S = 0;$$

Integrability conditions for evolution equations

For further consideration we will need formal pseudo-differential series, which for simplicity we shall call formal series

$$A = a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_0 + a_{-1} D_x^{-1} +$$

$$a_{-2} D^{-2} + \cdots \qquad\qquad a_k \in \mathfrak{F}, \quad m \in \mathbb{Z}.$$

The product of two formal series is defined by

$$D_x^k \circ b D_x^m = b D_x^{m+k} + C_k^1 D_x(b) D_x^{k+m-1} +$$

$$C_k^2 D_x^2(b) D_x^{k+m-2} + \cdots ,$$

where $k, m \in \mathbb{Z}$ and C_n^j is the binomial coefficient

$$C_k^j = \frac{k(k-1)(k-2)\cdots(k-j+1)}{j!}.$$

This product is associative.

For any series

$$A = a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_0 + a_{-1} D_x^{-1} +$$

we can find uniquely the inverse element

$$B = b_{-m} D_x^{-m} + b_{-m-1} D_x^{-m-1} + \cdots , \quad b_k \in \mathfrak{F}$$

such that $A \circ B = B \circ A = 1$.

Moreover we can find a series

$$C = c_1 D_x + c_0 + c_{-1} D_x^{-1} + c_{-2} D_x^{-2} + \cdots$$

such that $C^m = A$. If we know first k coefficients of the series A we can find the first k coefficients of the series C.

Example. Let $A = D_x^2 + u$. Then

$$C = A^{1/2} = D_x + \frac{u}{2} D_x^{-1} - \frac{u_1}{4} D_x^{-2} + \cdots .$$

We can easily find as many coefficients of C as required.

Definition. The *residue* of a formal series $A = \sum_{k \le n} a_k D_x^k,\ \ a_k \in \mathfrak{F}$ is by definition the coefficient at D_x^{-1}:

$$\mathrm{res}\,(A) = a_{-1}\,.$$

The *logarithmic residue* of A is defined as

$$\mathrm{res}\log A = \frac{a_{n-1}}{a_n}\,.$$

We will use the following important Adler's **Theorem.** For any two formal series A, B the residue of the commutator belongs to $\mathrm{Im}\, D_x$:

$$\mathrm{res}[A, B] = D_x(\sigma(A, B)),$$

where

$$\sigma(A, B) = \sum_{p \le \mathrm{ord}(B),\ q \le \mathrm{ord}(A)}^{p+q+1>0} C_q^{p+q+1} \times$$

$$\sum_{s=0}^{p+q} (-1)^s D_x^s(a_q) D_x^{p+q-s}(b_q)\,.$$

Definition. A pseudo-differential symbol

$$L = l_1 D_x + l_0 + l_{-1} D_x^{-1} + \cdots ,$$

where $l_k = l_k(u_{s_k}, \ldots, u)$, is called a formal recursion operator (or formal symmetry) for the equation

$$u_t = F(u_n, u_{n-1}, \ldots, u)$$

if

$$L_t = [F_*,\, L], \quad \text{where} \quad F_* = \sum_{k=0}^{n} \frac{\partial F}{\partial u_k} D_x^k$$

Theorem 1 (Ibragimov-Shabat 1980). If equation $u_t = F$ possesses an infinite hierarchy of higher symmetries

$$u_{\tau_i} = G_i(u_{m_i}, \ldots, u), \qquad m_i \to \infty$$

then the equation has a formal recursion operator.

Theorem 2 (Svinolupov-VS 1982). If equation $u_t = F$ possesses an infinite hierarchy of higher conserved densities

$$\rho_i(u_{m_i}, \ldots, u)_t \in Im\, D_x, \qquad \frac{\partial^2 \rho_i}{\partial u_{m_i}^2} \neq 0, \qquad m_i \to \infty$$

then the equation has a formal recursion operator.

Theorem 3 (Svinolupov-VS 1982). If equation $u_t = F$ is related to the linear equation $v_t = v_n$ by a differential substitution

$$v = \varphi(u_k, \cdots, u)$$

then the equation has a formal recursion operator.

The formal recursion operator allows us to construct local conservation laws for the equation $u_t = F$:

Proposition. The functions

$$\rho_i = res(L^i), \quad i = -1, 1, 2, \ldots, \text{ and } \quad \rho_0 = \frac{l_0}{l_1}$$

are conserved densities.

Example. For the Korteweg-de Vries equation $u_t = u_3 + 6uu_1$ we can take

$$L = \left(D_x^2 + 4u + 2u_1 D_x^{-1}\right)^{1/2}$$

and

$$\rho_1 = 2u, \quad \rho_2 = 2u_1, \quad \rho_2 = 2u_2 + u^2, \ldots$$

Theorem 4 (Svinolupov-VS 1982).
i). Under assumptions of Theorem 2 all even canonical densities ρ_{2j} are trivial.
ii). Under assumptions of Theorem 3 all canonical densities are trivial.

We call ρ_i **canonical densities**.

Classification of KdV-type equations (Ibragimov-Shabat, Fokas, 1980)

Consider equations of the form

$$u_t = u_3 + f(u_1, u). \qquad \text{(kdvt)}$$

Let us find the simplest canonical density ρ_1.
Equating the coefficients of $D_x^3, D_x^2, \ldots$ in

$$L_t - [F_*,\, L] = 0,$$

where

$$L = l_1 D_x + l_0 + l_{-1} D_x^{-1} + \cdots\,,$$
$$F_* = D_x^3 + \tfrac{\partial f}{\partial u_1} D_x + \tfrac{\partial f}{\partial u},$$

we get:

$$D_x^3:\ 3D_x(l_1) = 0; \qquad D_x^2:\ 3D_x^2(l_1) + 3D_x(l_0) = 0;$$

$$D_x:\ D_x^3(l_1) + 3D_x^2(l_0) + 3D_x(l_{-1}) + \tfrac{\partial f}{\partial u_1}\, D_x(l_1) = (l_1)_t + l_1\, D_x\left(\tfrac{\partial f}{\partial u_1}\right).$$

If we put $l_1 = 1$ then

$$\rho_1 = l_{-1} = \frac{1}{3}\,\frac{\partial f}{\partial u_1}$$

Thus we discovered a very remarkable fact:

$$\left(\frac{\partial f}{\partial u_1}\right)_t = D_x(\sigma_1)$$

for any integrable equation !

Example. For the mKdV-equation $u_t = u_3 + 3u^2u_1$ we expect that $\rho_1 = u^2$ is a conserved density. Indeed,

$$(u^2)_t = D_x(2uu_2 - u_1^2 + \frac{1}{2}u^4).$$

We can eliminate unknown σ_1 applying the Euler operator

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_x \circ \frac{\partial}{\partial u_1} + D_x^2 \circ \frac{\partial}{\partial u_2} - \cdots$$

As the result we get the first integrability condition

$$0 = \frac{\delta}{\delta u}\left(\frac{\partial f}{\partial u_1}\right)_t = 3u_4\left(u_2 \frac{\partial^4 f}{\partial u_1^4} + u_1 \frac{\partial^4 f}{\partial u_1^3 \partial u}\right) + \cdots$$

It implies

$$f(u_1, u) = \mu u_1^3 + A(u) u_1^2 + B(u) u_1 + C(u).$$

For such f the first condition is equivalent to

$$\mu A' = 0, \qquad B''' + 8\mu B' = 0,$$
$$(B'C)' = 0, \qquad AB' + 6\mu C' = 0.$$

It is almost enough to complete the classification.

The second integrability condition has the form

$$\left(\frac{\partial f}{\partial u}\right)_t = D_x(\sigma_2)$$

Using this fact we derive several more differential relations between $A(u)$, $B(u)$, $C(u)$. Solving them alltogether we obtain the following list of equations

$$\begin{aligned}
u_t &= u_{xxx} + (c_1 u^2 + c_2 u + c_3) u_x \\
u_t &= u_{xxx} - \frac{1}{2} u_x^3 + \left(c_1 e^{2u} + c_2 e^{-2u} + c_3\right) u_x \\
u_t &= u_{xxx} + c_1 u_x^3 + c_2 u_x^2 + c_3 u_x + c_4
\end{aligned}$$

For more general class of equations

$$u_t = u_3 + f(u_2, u_1, u) \qquad (28)$$

several simplest canonical densities have the form

$$\begin{aligned}
\rho_0 &= \frac{\partial f}{\partial u_2}, \\
\rho_1 &= 3\frac{\partial f}{\partial u_1} + \left(\frac{\partial f}{\partial u_2}\right)^2, \\
\rho_2 &= 9\sigma_0 + 27\frac{\partial f}{\partial u} - 9\frac{\partial f}{\partial u_2}\frac{\partial f}{\partial u_1} + 2\left(\frac{\partial f}{\partial u_2}\right)^3, \\
&\cdot \quad \cdot \quad \cdot
\end{aligned}$$

The point transformations

$$u = \psi(\hat{u})$$

preserve this class of equations.

Description of integrable equations (28). (Svinolupov-VS 1982)

1. Equation of the form

$$u_t = u_{xxx} + f(u_{xx}, u_x, u)$$

is integrable iff it satisfies integrability conditions $(\rho_i)_t = D(\sigma_i)$, $i = 0, 1, 2, 3$.

2. A complete list (up to "almost invertible" transformations) of equations with infinite hierarchy of conservation laws can be written as:

$$\begin{array}{rcll}
u_t & = & u_{xxx} + u\,u_x, & \text{KdV} \\
u_t & = & u_{xxx} + u^2\,u_x, & \text{mKdV} \\
u_t & = & u_{xxx} - \frac{1}{2}u_x^3 + (\alpha e^{2u} + \beta e^{-2u})u_x, & \text{CD1} \\
u_t & = & u_{xxx} - \frac{1}{2}Q''\,u_x + \frac{3}{8}\frac{((Q - u_x^2)_x)^2}{u_x\,(Q - u_x^2)}, & \text{CD2} \\
u_t & = & u_{xxx} - \frac{3}{2}\frac{u_{xx}^2 + Q(u)}{u_x} & \text{KN}\ ,
\end{array}$$

where $Q'''''(u) = 0$.

3. Equations KdV and KN form a complete list up to differential substitutions.

All integrable equations of the form

$$u_t = F(u_2, u_1, u, x, t)$$

were listed by **Svinolupov 1985** (see also **VS-Svinolupov 1991.**)

The answer is:

$$u_t = u_2 + 2uu_x + h(x),$$
$$u_t = u^2u_2 - \lambda x u_1 + \lambda u$$
$$u_t = u^2u_2 + \lambda u^2$$
$$u_t = u^2u_2 - \lambda x^2 u_1 + 3\lambda x u$$

This is a complete list up to the contact transformations

$$\hat{x} = \varphi(x, u, u_1), \qquad \hat{u} = \psi(x, u, u_1),$$

$$\hat{u}_i = \left(\frac{1}{D_x(\varphi)} D_x\right)^i (\psi),$$

where

$$D_x(\varphi)\frac{\partial \psi}{\partial u_1} = D_x(\psi)\frac{\partial \varphi}{\partial u_1}.$$

All equations of the form

$$u_t = u_5 + F(u_4, u_3, u_2, u_1, u),$$

possessing higher conservation laws were found by **Drinfeld-VS-Svinolupov 1985**.

Examples: Well-known equations

$$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1,$$
$$u_t = u_5 + 5uu_3 + \tfrac{25}{2}u_1u_2 + 5u^2u_1$$
$$u_t = u_5 + 5(u_1 - u^2)u_3 + 5u_2^2 - 20uu_1u_2$$
$$-5u_1^3 + 5u^4u_1$$

A new equation

$$\begin{aligned} u_t \;=\; & u_5 + 5(u_2 - u_1^2 + \lambda_1 e^{2u} - \lambda_2^2 e^{-4u})\, u_3 \\ & -5u_1u_2^2 + 15(\lambda_1 e^{2u} + 4\lambda_2^2 e^{-4u})\, u_1u_2 + u_1^5 \\ & -90\lambda_2^2 e^{-4u}\, u_1^3 + 5(\lambda_1 e^{2u} - \lambda_2^2 e^{-4u})^2\, u_1 \end{aligned}$$

Classification of systems.

The most significant work has been done by **Mikhailov-Shabat-Yamilov 1987**. All systems of the form

$$u_t = u_2 + F(u, v, u_1, v_1),$$

$$v_t = -v_2 + G(u, v, u_1, v_1)$$

possessing higher conservation laws, were listed.

Example 1: Well-known NLS-equation

$$u_t = u_2 + u^2 v,$$

$$v_t = -v_2 - v^2 u,$$

Example 2. The Landau-Lifshitz equation (after stereographic projection)

$$u_t = u_2 - \frac{2u_1^2}{u+v} - \frac{4(\,p(u,v)u_1 + r(u)v_1\,)}{(u+v)^2}$$

$$v_t = -v_2 + \frac{2v_1^2}{u+v} - \frac{4(\,p(u,v)v_1 + r(-v)u_1\,)}{(u+v)^2},$$

where

$$r(y) = c_4y^4 + c_3y^3 + c_2y^2 + c_1y + c_0$$

and

$$p(u,v) = 2c_4u^2v^2 + c_3(uv^2 - vu^2) -$$
$$2c_2uv + c_1(u-v) + 2c_0.$$

Multi-component systems.

In several papers by **Svinolupov 1991-1994** remarkable relations between special types of polynomial N-component systems and non-associative algebras were established.

Theorem 1. If C^i_{jk} are structural constants of any left-symmetric algebra then the system

$$u^i_t = u^i_{xx} + 2C^i_{jk}u^k u^j_x + A^i_{jkm}u^k u^j u^m,$$

where $\quad i, j, k = 1, \ldots, N$ and

$$A^i_{jkm} = \frac{1}{3}(C^i_{jr}C^r_{km} + C^i_{kr}C^r_{mj} + C^i_{mr}C^r_{jk} \\ -C^i_{rj}C^r_{km} - C^i_{rk}C^r_{mj} - C^i_{rm}C^r_{jk}),$$

possesses higher symmetries.

Theorem 2. If C^i_{jk} are structural constants of any Jordan algebra then the KdV-type system

$$u^i_t = u^i_{xxx} + C^i_{jk}u^k u^j_x, \qquad i, j, k = 1, \ldots, N$$

possesses higher symmetries.

Theorem 3. If C^i_{jkm} are structural constants of any Jordan triple system then the mKdV-type system

$$u^i_t = u^i_{xxx} + C^i_{jkm} u^k u^j u^m_x, \quad i, j, k = 1, \ldots, N$$

possesses higher symmetries.

Theorem 4. If C^i_{jkm} are structural constants of any Jordan triple system then the nonlinear Schroedinger-type system

$$\begin{aligned} u^i_t &= u^i_{xx} + C^i_{jkm} u^j v^k u^m, \qquad i, j, k = 1, \ldots, N \\ v^i_t &= -v^i_{xx} - C^i_{jkm} v^j u^k v^m \end{aligned}$$

possesses higher symmetries.

Theorem 5. If C^i_{jkm} are structural constants of any Jordan triple system then the nonlinear derivative Schroedinger-type system

$$\begin{aligned} u^i_t &= u^i_{xx} + C^i_{jkm} (u^j v^k u^m)_x, \qquad i, j, k = 1, \ldots, N \\ v^i_t &= -v^i_{xx} - C^i_{jkm} (v^j u^k v^m)_x \end{aligned}$$

possesses higher symmetries.

Definition of left-symmetric algebra:

$$As(X, Y, Z) = As(Y, X, Z),$$

where

$$As(X, Y, Z) = (X \circ Y) \circ Z - X \circ (Y \circ Z).$$

Definition of Jordan algebra:

$$X \circ Y = Y \circ X, \qquad X^2 \circ (Y \circ X) = (X^2 \circ Y) \circ X.$$

If $*$ is a multiplication in an associative algebra then $X \circ Y = X * Y + Y * X$ is a Jordan operation.

Definition of Jordan triple system:

$$\{X, Y, Z\} = \{Z, Y, X\},$$

$$\{X, Y, \{V, W, Z\}\} - \{V, W, \{X, Y, Z\}\} = \\ \{\{X, Y, V\}, W, Z\} - \{V, \{Y, X, W\}, Z\}.$$

Example of left-symmetric algebra.
The set of all N-dimensional vectors w.r.t.

$$X \circ Y =< X, C > Y+ < X, Y > C,$$

where C is a fixed (constant) vector.

Examples of simple Jordan algebras.
a) The set of all $N \times N$ matrices w.r.t.

$$X \circ Y = XY + YX$$

b) The set of all N-dimensional vectors w.r.t.

$$X \circ Y =< X, C > Y+ < Y, C > X- < X, Y > C.$$

Examples of simple triple Jordan systems.
a) The set of all $N \times N$ matrices w.r.t.

$$\{X, Y, Z\} = XYZ + ZYX$$

b) The set of all N-dimensional vectors w.r.t.

$$\{X, Y, Z\} =< X, Y > Z+ < Y, Z > X- < X, Z > Y.$$

c) The set of all N-dimensional vectors w.r.t.

$$\{X, Y, Z\} =< X, Y > Z+ < Y, Z > X.$$

Examples of corresponding integrable systems: Svinolupov-VS 1994.

The matrix Burgers equation

$$u_t = u_2 + uu_1;$$

the matrix KdV-equation

$$u_t = u_1 + uu_1 + u_1 u;$$

the matrix mKdV equation

$$u_t = u_3 + u^2 u_1 + u_1 u^2;$$

the vector Burgers equation (new)

$$\begin{aligned} u_t = u_2 + 2 < u, u_x > C + 2 < C, u > u_x + \\ < u, u >< C, u > C - < u, u >< C, C > u; \end{aligned}$$

the vector KdV equation (new)

$$u_t = u_3 + < C, u > u_1 + < C, u_1 > u - < u, u_1 > C;$$

the matrix NLS equation

$$\begin{aligned} u_t &= u_2 + 2\,uvu, \\ v_t &= -v_2 - 2\,vuv; \end{aligned}$$

the vector NLS equation 1 (Manakov)

$$\begin{aligned} u_t &= u_2 + <u,v> u, \\ v_t &= -v_2 - <u,v> v; \end{aligned}$$

the vector NLS equation 2 (Kulish-Sklyanin)

$$\begin{aligned} u_t &= u_2 + 2 <u,v> u - <u,u> v, \\ v_t &= -v_2 - 2 <u,v> v + <v,v> u; \end{aligned}$$

Classification of integrable matrix evolution equations.

Olver and Sokolov 1998 listed integrable non-abelian polynomial evolution equations having higher symmetries. One of the lists:

$$
\begin{aligned}
u_t &= u_3 + 3u^2u_1 + 3u_1u^2, \\
u_t &= u_3 + 3uu_2 - 3u_2u - 6uu_1u, \\
u_t &= u_3 + 3u_1^2.
\end{aligned}
$$

Second order non-abelian systems of NLS- and DNLS-types also were listed and several new integrable models were found.

Examples.

$$u_t = u_2 + 2(u+v)u_1, \qquad v_t = -v_2 + 2v_1(u+v);$$

$$u_t = u_2 + 2u_1vu, \qquad v_t = -v_2 + 2vuv_1.$$

Non-abelian Painleve equations:

$$u_2 + 3u^2 = xE + C,$$

$$u_2 + 2u^3 + xu = \lambda E,$$

$$u_2 + \frac{1}{x}u_1 = u_1 u^{-1} u_1.$$

Classification of integrable matrix ODEs.

Polynomial non-abelian ODEs have been considered by **Mikhailov-VS, 2000** and some partial classification results have been obtained.

For example the following system

$$u_t = v^2, \qquad v_t = u^2$$

possesses infinitely many symmetries

$$u_{\tau_i} = P_i(u, v), \qquad v_{\tau_i} = Q_i(u, v)$$

and first integrals

$$\rho_i = \mathrm{trace}\, R_i(u, v).$$

There exists two interesting integrable non-abelian equations containing arbitrary constant element C:

$$u_t = Cu^2 - u^2 C$$

and

$$u_t = uCu^2 - u^2 Cu.$$

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