ON A GENERALIZED FROBENIUS–STICKELBERGER ADDITION FORMULA

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Abstract. In this paper we obtain a generalization of the Frobenius–Stickelberger addition formula for the (hyperelliptic) σ-function of a genus 2 curve in the case of three vector-valued variables. The result is given explicitly in the form of a polynomial in Kleinian $\wp$-functions.

1. Introduction

In this paper we consider the sigma function $\sigma(u_1, u_2)$ associated with a curve of genus 2

$$y^2 = \lambda_0 + \lambda_1 x + \cdots + \lambda_4 x^4 + 4x^5. \tag{1}$$

The function $\sigma(u_1, u_2) = \sigma(\boldsymbol{u})$ is entire in the complex variables (u_1, u_2) and the parameters $\lambda_0, \ldots, \lambda_4$ of the curve. It is characterized by a set of fourth-order partial differential equations, the Baker equations [Bak07, pg. 49], [BEL97b, §6.1] and plays a fundamental role in the generalization of Weierstrass elliptic function theory to curves of higher genera.

The elliptic function $\sigma(u)$ depends on a scalar variable u and is associated with the cubic curve

$$y^2 = 4x^3 - g_2 x - g_3.$$

The σ-function satisfies the addition theorem

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp(v) - \wp(u), \tag{2}$$

where the Weierstrass elliptic function $\wp(u)$ is related to $\sigma(u)$ by

$$\wp(u) = -\frac{d^2}{du^2} \ln \sigma(u).$$

Equation (2) can be generalized in various ways. We can increase the genus of the curve and/or increase the number of terms to be added. When higher genera are considered, the argument of σ is taken to be a general point of the Jacobi variety of the algebraic curve of genus g. An addition formula is in fact a consequence of the theorem of the square for the theta divisor [Bar83], and as such, it holds for more general theta functions than for Jacobian ones. It is natural to develop both types of generalizations on the basis of the Kleinian theory of σ-functions [Kle88] (see also [Bak95, Bak98] and [BEL97a, BEL97b]) which represents a natural generalization of the Weierstrass elliptic functions to hyperelliptic curves of higher genera.

The first generalizations of (2) to two-variable formulae in the hyperelliptic case of genera 2 and 3 were given by Baker in [Bak95, Bak98]; a formula for arbitrary g was given by Buchstaber et al. [BEL97b, BEL97c]. In these latter papers the right hand side of the addition formula was presented as the Pfaffian of a matrix whose entries are linear in the Kleinian $\wp$ functions.

The generalization to a higher number of variables in the genus 1 case (elliptic curves) seems to have been found first by Frobenius and Stickelberger [FS77], although special cases were known earlier to Brioschi [Bri64] and Kiepert [Kie73].

The Frobenius and Stickelberger addition formula [FS77] for the elliptic σ-function is

$$\frac{\sigma(z_0+z_1+\ldots+z_n)\prod_{0\le k<l\le n}\sigma(z_k-z_l)}{\sigma^{n+1}(z_0)\ldots\sigma^{n+1}(z_n)} = \frac{1}{(-1)^{\frac{1}{2}n(n-1)}1!2!\ldots n!}\begin{vmatrix} 1 & \wp(z_0) & \wp'(z_0) & \ldots & \wp^{(n-1)}(z_0) \\ 1 & \wp(z_1) & \wp'(z_1) & \ldots & \wp^{(n-1)}(z_1) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 1 & \wp(z_n) & \wp'(z_n) & \ldots & \wp^{(n-1)}(z_n) \end{vmatrix}. \tag{3}$$

This formula is widely used in various problems of mathematics and physics. It was a key formula in the proof of the complete integrability of the elliptic Calogero-Moser system [Kri80].

The generalizations of addition theorems of this form are know as Schottky–Klein addition formulae (cf. [Bak95, pg. 430], [Fay73, eq.(43)]). In this paper we shall develop a generalization of (3) to the hyperelliptic curve of genus 2 in the case $n = 2$. Our contribution is to give an explicit version of the right-hand-side in terms of Kleinian $\wp$-functions. This result improves on the formula given in [BEL97b, 6.6.1] because it removes the denominator and reduces the number of derivatives of the σ function to three.

The principal ingredients of our treatment are the hyperelliptic $\wp$-functions of Klein [Kle88] (see also [Bak95, Bak07, BEL97a, BEL97b]) and results by Ônishi [Ôni02b, Ôni02a].

2. Ultraelliptic σ-functions

In this section we give a brief introduction to Klein's theory of σ-functions for a genus two curve. Let X be the hyperelliptic curve of genus $g = 2$ given by the formula

$$\begin{aligned} y^2 &= R(x), \\ R(x) &= \lambda_0 + \lambda_1 x + \cdots + \lambda_4 x^4 + 4x^5 \\ &= 4\prod_{i=1}^{5}(x - e_i), \quad e_i \neq e_j, \end{aligned} \tag{4}$$

where the branch points e_i and the parameters λ_i are arbitrary complex numbers.

A basis of holomorphic differentials $du_i(x, y)$, $i = 1, 2$ on the Riemann surface of the curve and the associated differentials of the second kind $dr_i(x, y)$, $i = 1, 2$, have the form

$$du_1(x,y) = \frac{dx}{y}, \quad du_2(x,y) = \frac{xdx}{y}, \tag{5}$$

$$dr_1(x,y) = \frac{12x^3 + 2\lambda_4 x^2 + \lambda_3 x}{4y}dx, \quad dr_2(x,y) = \frac{x^2 dx}{y}. \tag{6}$$

Introduce a standard homology basis of a, b cycles and period matrices

$$2\omega = \left(\oint_{a_j} du_i\right)_{i,j=1,2}, \quad 2\omega' = \left(\oint_{b_j} du_i\right)_{i,j=1,2}, \tag{7}$$

$$2\eta = \left(-\oint_{a_j} dr_i\right)_{i,j=1,2}, \quad 2\eta' = \left(-\oint_{b_j} dr_i\right)_{i,j=1,2}. \tag{8}$$

These periods satisfy the generalized Legendre relations

$$\begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix} = -\frac{i\pi}{2}\begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix},$$

where 1_2 is the 2×2 unit matrix. The period matrix $\tau = \omega'\omega^{-1}$ is symmetric and its imaginary part is positive-definite.

We define the lattice $\Lambda = \mathbb{Z}^2 \oplus 2\omega\mathbb{Z} \oplus 2\omega'\mathbb{Z}$. There exist 16 linearly independent half-periods $\mathbf{\Omega}_I$ (namely $2\mathbf{\Omega}_I \in \Lambda$), where I is a set of indices. To describe them explicitly, we pick the homology basis as shown in Fig.1. Denote $\mathfrak{A}_k$ the Abelian image of the branch point e_k,

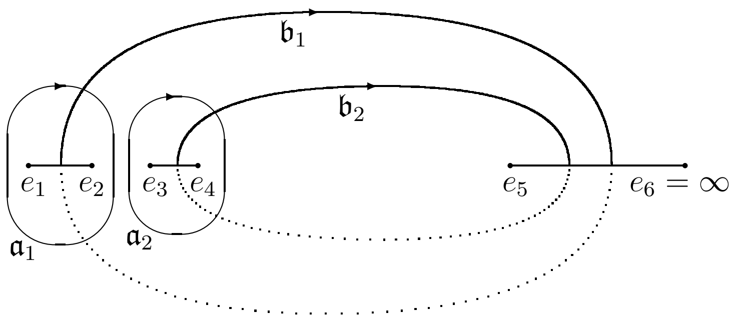

FIGURE 1. Homology basis on the Riemann surface of the curve $V(x, y)$ with real branch points $e_1 < e_2 < \ldots < e_6 = \infty$ (upper sheet). The cuts are drawn from e_{2i-1} to e_{2i}, $i = 1, 2, 3$. The b–cycles are completed on the lower sheet (dotted lines).

$k = 1, \ldots, 5$, as

$$\boldsymbol{\mathfrak{A}}_k = \int\limits_{(\infty,\infty)}^{(e_k,0)} d\boldsymbol{u}(x, y) \tag{9}$$

and the characteristic $[\boldsymbol{\mathfrak{A}}_k]$ of the point $\boldsymbol{\mathfrak{A}}_k$,

$$[\boldsymbol{\mathfrak{A}}_k] = \left[\begin{array}{c} \boldsymbol{\varepsilon}_k'^{\,t} \\ \boldsymbol{\varepsilon}_k^t \end{array} \right]; \quad \boldsymbol{\mathfrak{A}}_k = \omega \boldsymbol{\varepsilon}_k + \omega' \boldsymbol{\varepsilon}_k' \tag{10}$$

with components of the vectors $\boldsymbol{\varepsilon}_k$, $\boldsymbol{\varepsilon}_k'$ equal to 0 or 1. Then we have

$$[\boldsymbol{\mathfrak{A}}_1] = \left[\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right], \quad [\boldsymbol{\mathfrak{A}}_2] = \left[\begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right], \quad [\boldsymbol{\mathfrak{A}}_3] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right], \tag{11}$$

$$[\boldsymbol{\mathfrak{A}}_4] = \left[\begin{array}{cc} 0 & 1 \\ 1 & 1 \end{array} \right], \quad [\boldsymbol{\mathfrak{A}}_5] = \left[\begin{array}{cc} 0 & 0 \\ 1 & 1 \end{array} \right], \quad [\boldsymbol{\mathfrak{A}}_6] = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right] \tag{12}$$

for the characteristics of branch points, while the characteristic of the vector of Riemann constants with the base point $e_6 = \infty$ is

$$2\omega[\boldsymbol{K}_\infty] = [\boldsymbol{\mathfrak{A}}_2 + \boldsymbol{\mathfrak{A}}_4] = \left[\begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right]. \tag{13}$$

The 16 half-periods, subdivided into 10 even, $\boldsymbol{\Omega}_{i,j}$ and 6 odd $\boldsymbol{\Omega}_i$ are given by the formula

$$\begin{aligned}\boldsymbol{\Omega}_{i,j} &= \mathfrak{A}_i + \mathfrak{A}_j + 2\omega \boldsymbol{K}_\infty, \quad 1 \le i < j \le 5,\\ \boldsymbol{\Omega}_i &= \mathfrak{A}_i + 2\omega \boldsymbol{K}_\infty, \quad 1, \ldots, 5.\end{aligned}$$

We recall that $\mathrm{Jac}(X) = \mathbb{C}^2/\mathbb{Z}^2 \oplus 2\omega\mathbb{Z} \oplus 2\omega'\mathbb{Z}$ is the Jacobian of the curve X, and denote by ϵ a primitive eighth root of 1, $\epsilon^8 = 1$.

The θ-function with characteristic $[\delta] = \begin{bmatrix} \boldsymbol{\delta}^t \\ \boldsymbol{\delta}'^t \end{bmatrix}$ is

$$\theta[\delta](\boldsymbol{u}|\tau) = \sum_{\boldsymbol{n}\in\mathbb{Z}^2} \exp\left[2\pi i\left\{\frac{1}{2}(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\delta})^t\tau(\boldsymbol{n}+\frac{1}{2}\boldsymbol{\delta}) + (\boldsymbol{n}+\frac{1}{2}\boldsymbol{\delta})^t(\boldsymbol{u}+\frac{1}{2}\boldsymbol{\delta}')\right\}\right]$$

while the ultraelliptic σ -function is given by the formula

$$\sigma(\boldsymbol{u}) = C\exp\left(\frac{1}{2}\boldsymbol{u}^t\eta\omega^{-1}\boldsymbol{u}\right)\theta[\boldsymbol{K}_\infty](2\omega^{-1}\boldsymbol{u}|\tau). \tag{14}$$

The matrix $\eta\omega^{-1}$ is symmetric and the constant C is

$$C = \frac{\epsilon\pi}{\sqrt{\det(2\omega)}}\frac{1}{\sqrt{\prod_{1\le i<j\le 5}(e_i - e_j)}}.$$

The σ-function represents a natural generalization of the Weierstrass σ-function and has the same property of invariance under the action of the symplectic group $\mathrm{Sp}(4,\mathbb{Z})$.

The σ-function can be expanded in a neighbourhood of the origin by a series in u_1, u_2 with the first few terms given by

$$\sigma(u_1, u_2) = u_1 - \frac{1}{3}u_2^3 + \frac{1}{24}\lambda_2 u_1^3 + O(\boldsymbol{u}^5).$$

In the rational limit, $\lambda_i = 0$, the first term of the σ-expansion represents the Schur function, $u_1 - \frac{1}{3}u_2^3$.

The Lie algebra annihilating the σ-function and defining it uniquely as a power expansion with coefficients given recursively was recently found in [BL02].

Introduce the Kleinian $\wp$-functions

$$\begin{aligned}\wp_{ij}(\boldsymbol{u}) &= -\frac{\partial^2}{\partial u_i\partial u_j}\ln\sigma(\boldsymbol{u}), \quad i,j = 1,2,\\ \wp_{ijk}(\boldsymbol{u}) &= -\frac{\partial^3}{\partial u_i\partial u_j\partial u_k}\ln\sigma(\boldsymbol{u}), \quad i,j,k = 1,2.\end{aligned}$$

These Kleinian $\wp$-functions are given as rational functions on $X \times X$ [Bak07, pg. 38]. The two-index symbols $\wp_{i,j}$ are

$$\begin{aligned} \wp_{22} &= x_1 + x_2, \qquad \wp_{12} = -x_1x_2, \\ \wp_{11} &= \frac{F(x_1,x_2) - 2y_1y_2}{4\,(x_1-x_2)^2}, \end{aligned} \tag{15}$$

where $F(x_1,x_2)$ is the Kleinian 2-polar

$$F(x_1,x_2) = \sum_{r=0}^{2} x_1^r x_2^r \left[2\lambda_{2r} + \lambda_{2r+1}(x_1+x_2)\right].$$

The first two of relations (15) represent the solution of the Jacobi inversion problem. The 3-index symbols $\wp_{ijk}$ are given by

$$\begin{aligned} &\wp_{222} = \frac{y_1 - y_2}{x_1 - x_2}, \quad \wp_{122} = \frac{x_1y_2 - x_2y_1}{x_1 - x_2}, \quad \wp_{112} = -\frac{x_1^2y_2 - x_2^2y_1}{x_1 - x_2}, \\ &\wp_{111} = \frac{y_2\psi(x_1,x_2) - y_1\psi(x_2,x_1)}{4(x_1-x_2)^3}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \psi(x_1,x_2) \;=\; & 4\lambda_0 + \lambda_1(3x_1+x_2) + 2\lambda_2x_1(x_1+x_2) + \\ & \lambda_3x_1^2(x_1+3x_2) + 4\lambda_4x_1^3x_2 + 4x_1^3x_2(3x_1+x_2). \end{aligned}$$

The values of the $\wp$-functions at half-periods are as follows: $\wp_{ij}(\boldsymbol{\Omega}_k) = \infty$, $\wp_{ijk}(\boldsymbol{\Omega}_l) = \infty$ at all odd half-periods, $\wp_{ijk}(\boldsymbol{\Omega}_{m,n}) = 0$ at all even half-periods. Also

$$\begin{aligned} &\wp_{22}(\boldsymbol{\Omega}_{m,n}) = e_m + e_n, \\ &\wp_{12}(\boldsymbol{\Omega}_{m,n}) = -e_me_n, \\ &\wp_{11}(\boldsymbol{\Omega}_{m,n}) = e_me_n(e_p+e_q+e_r) + e_pe_qe_r \equiv e_{m,n}, \end{aligned} \tag{17}$$

for all $1 \le m < n \le 5$ and $p \neq q \neq r \in \{1,\ldots,5\}/\{i,j\}$.

To characterize the class of $\wp$-functions more completely, we shall give the differential relations between them. To that end we introduce the 4×4-matrix H of rank 3

$$H = \begin{pmatrix} \lambda_0 & \frac{1}{2}\lambda_1 & -2\wp_{11} & -2\wp_{12} \\ \frac{1}{2}\lambda_1 & \lambda_2 + 4\wp_{11} & \frac{1}{2}\lambda_3 + 2\wp_{12} & -2\wp_{22} \\ -2\wp_{11} & \frac{1}{2}\lambda_3 + 2\wp_{12} & \lambda_4 + 4\wp_{22} & 2 \\ -2\wp_{12} & -2\wp_{22} & 2 & 0 \end{pmatrix}.$$

The following relation is valid for arbitrary $\boldsymbol{k}, \boldsymbol{l} \in \mathbb{C}^4$

$$\boldsymbol{l}^t \pi \pi^t \boldsymbol{k} = -\frac{1}{4}\mathrm{det}\,\begin{pmatrix} H & \boldsymbol{l} \\ \boldsymbol{k}^t & 0 \end{pmatrix}, \tag{18}$$

where $\pi^t = (\wp_{222}, -\wp_{221}, \wp_{211}, -\wp_{111})$. The ideal given by (18) defines the meromorphic embedding of the Jacobi variety $\mathrm{Jac}(X)$ into the complex space $\mathbb{C}^5$; a basis of the ideal is given by three equations

$$\begin{aligned}(\boldsymbol{k}^t;\boldsymbol{l}^t) &= ((0,0,0,1);(0,0,0,1))\\ &= ((0,0,1,0);(0,0,0,1))\\ &= ((0,0,1,0);(0,0,1,0)),\end{aligned}$$

respectively.

The vector π satisfies the equation $H\pi = 0$, or in detailed form

$$-\wp_{12}\wp_{222} + \wp_{22}\wp_{221} + \wp_{211} = 0, \tag{19}$$

$$2\wp_{11}\wp_{222} + \left(\frac{1}{2}\lambda_3 + 2\wp_{12}\right)\wp_{221}$$
$$-(\lambda_4 + 4\wp_{22})\wp_{211} + 2\wp_{111} = 0, \tag{20}$$

$$\frac{1}{2}\lambda_1\wp_{222} - (\lambda_2 + 4\wp_{11})\,\wp_{221}$$
$$+(\frac{1}{2}\lambda_3 + 2\wp_{12})\wp_{211} + 2\wp_{22}\wp_{111} = 0, \tag{21}$$

$$-\lambda_0\wp_{222} + \frac{1}{2}\lambda_1\wp_{221} + 2\wp_{11}\wp_{211} - 2\wp_{12}\wp_{111} = 0, \tag{22}$$

and so the functions $\wp_{22}, \wp_{12}$ and $\wp_{11}$ are related by the equation

$$\det H = 0. \tag{23}$$

The equation (23) defines the quartic Kummer surface in $\mathbb{C}^3$ in coordinates $X = \wp_{22}$, $Y = \wp_{12}$, $Z = \wp_{11}$. Moreover, the following differential equations hold:

$$\begin{aligned}\wp_{2222} &= 6\wp_{22}^2 + \frac{1}{2}\lambda_3 + \lambda_4\wp_{22} + 4\wp_{12},\\ \wp_{2221} &= 6\wp_{22}\wp_{12} + \lambda_4\wp_{12} - 2\wp_{11},\\ \wp_{2211} &= 2\wp_{22}\wp_{11} + 4\wp_{12}^2 + \frac{1}{2}\lambda_3\wp_{12},\\ \wp_{2111} &= 6\wp_{12}\wp_{11} + \lambda_2\wp_{12} - \frac{1}{2}\lambda_1\wp_{22} - \lambda_0,\\ \wp_{1111} &= 6\wp_{11}^2 - 3\lambda_0\wp_{22} + \lambda_1\wp_{12} + \lambda_2\wp_{11} - \frac{1}{2}\lambda_0\lambda_4 + \frac{1}{8}\lambda_1\lambda_3.\end{aligned}$$

All these relations generalize known relations of the Weierstrass theory of elliptic functions to the genus two case.

To complete this introduction we give expressions of the θ-quotients in terms of Kleinian functions. Let

$$\boldsymbol{v} = (2\omega)^{-1} \sum_{k=1}^{2} \int\limits_{(\infty,\infty)}^{(x_k,y_k)} d\boldsymbol{u} - \boldsymbol{K}_\infty,$$

then

$$\frac{\theta^2[\mathfrak{A}_k](\boldsymbol{v}|\tau)}{\theta^2(\boldsymbol{v}|\tau)} = \frac{\mathrm{e}^{-\frac{\imath\pi}{2}|\mathfrak{A}_k|}}{\sqrt{2R'(e_k)R'(e_l)}} \mathcal{P}_k(\boldsymbol{u}), \tag{24}$$

$$\frac{\theta^2[\mathfrak{A}_k+\mathfrak{A}_l](\boldsymbol{v}|\tau)}{\theta^2(\boldsymbol{v}|\tau)} = \frac{\mathrm{e}^{-\frac{\imath\pi}{2}\{|\mathfrak{A}_k|+|\mathfrak{A}_l|\}}(e_k-e_l)}{\sqrt{2R'(e_k)R'(e_l)}} \mathcal{Q}_{k,l}(\boldsymbol{u}), \tag{25}$$

(26)
$$\frac{\theta[\mathfrak{A}_k](\boldsymbol{v}|\tau)\theta[\mathfrak{A}_l](\boldsymbol{v}|\tau)\theta[\mathfrak{A}_k+\mathfrak{A}_l](\boldsymbol{v}|\tau)}{\theta^3(\boldsymbol{v}|\tau)} = \frac{\mathrm{e}^{-\frac{\imath\pi}{2}\{|\mathfrak{A}_k|+|\mathfrak{A}_l|\}}(e_k-e_l)}{\sqrt{2R'(e_k)R'(e_l)}} \mathcal{R}_{k,l}(\boldsymbol{u}),$$

where

$$\begin{aligned}
\mathcal{P}_k(\boldsymbol{u}) &= e_k^2 - \wp_{22}(\boldsymbol{u})e_k - \wp_{12}(\boldsymbol{u}),\\
\mathcal{Q}_{k,l}(\boldsymbol{u}) &= \wp_{11}(\boldsymbol{u}) + \wp_{12}(\boldsymbol{u})(e_k+e_l) + \wp_{22}(\boldsymbol{u})e_ke_l + e_{k,l},\\
\mathcal{R}_{k,l}(\boldsymbol{u}) &= \wp_{112}(\boldsymbol{u}) + (e_k+e_l)\wp_{122}(\boldsymbol{u}) + e_ke_l\wp_{222}(\boldsymbol{u}),
\end{aligned}$$

$|\mathfrak{A}_k| = (-1)^{\boldsymbol{\delta}_k^t\boldsymbol{\delta}_k}$ for the characteristic $[\mathfrak{A}_k] = \begin{bmatrix}\boldsymbol{\delta}^t\\ \boldsymbol{\epsilon}^t\end{bmatrix}$ and the quantities $e_{k,l}$ are given in (17).

3. The addition rule on $\mathrm{Jac}(X) \times \mathrm{Jac}(X)$

The addition rule on $\mathrm{Jac}(X)\times\mathrm{Jac}(X)$ in terms of Kleinian $\wp$-functions was given by Baker[Bak95, Bak07]. Let

$$\begin{aligned}
\boldsymbol{u} &= \int_{(\infty,\infty)}^{(x_1,y_1)} d\boldsymbol{u} + \int_{(\infty,\infty)}^{(x_2,y_2)} d\boldsymbol{u},\\
\boldsymbol{u}' &= \int_{(\infty,\infty)}^{(x_1',y_1')} d\boldsymbol{u} + \int_{(\infty,\infty)}^{(x_2',y_2')} d\boldsymbol{u}.
\end{aligned} \tag{27}$$

Then Baker's addition formula is

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{u}')\sigma(\boldsymbol{u}-\boldsymbol{u}')}{\sigma(\boldsymbol{u})^2\sigma(\boldsymbol{u}')^2} = \wp_{11}' - \wp_{11} - \wp_{22}'\wp_{12} + \wp_{22}\wp_{12}', \tag{28}$$

where $\wp_{ij} = \wp_{ij}(\boldsymbol{u})$, and $\wp_{ij}' = \wp_{ij}(\boldsymbol{u}')$. The direct way to derive this formula is to use well known addition formulae with right hand side of the form $\theta[\boldsymbol{K}_\infty](\boldsymbol{u}+\boldsymbol{v}|\tau)\theta[\boldsymbol{K}_\infty](\boldsymbol{u}-\boldsymbol{v}|\tau)$ in combination with expressions for θ-quotients (24,25). But this direct method requires large calculations involving θ-characteristics and we were unable to succeed with this approach in more complicated cases. We describe an alternative calculation in the next section.

We shall first demonstrate our approach by re-deriving (28) using a different method. To do that we recall that the θ-divisor (θ) is a subvariety in $\mathrm{Jac}(X)$ given by the equation

$$\theta(\boldsymbol{u}|\tau) = 0, \quad \text{or equivalently} \quad \sigma(\boldsymbol{u}) = 0. \tag{29}$$

According to Riemann's vanishing theorem, points from (θ) are represented by

$$\boldsymbol{u} = \int_{(\infty,\infty)}^{(x,y)} d\boldsymbol{u} - 2\omega \boldsymbol{K}_\infty. \tag{30}$$

The co-ordinates of a point of the curve (x, y) can be given in terms of the σ-functions restricted to the θ-divisor as follows (see [Gra91, Jor92])

$$x_i = -\left.\frac{\sigma_1(\boldsymbol{u}_i)}{\sigma_2(\boldsymbol{u}_i)}\right|_{(\theta)}, \quad 2y_i = -\left.\frac{\sigma(2\boldsymbol{u}_i)}{\sigma_2(\boldsymbol{u}_i)}\right|_{(\theta)}.$$

Recall that the Weierstrass gap sequence at the branch point at infinity for the genus two curve is the complement of the sequence of non-negative integers $n_i = 2\alpha_i + 5\beta_i$ where α_i, β_i are positive integers or 0, which give the orders of poles of monomials $w_i(x, y) = x^{\alpha_i} y^{\beta_i}$ at infinity, and are the over-lined integers:

$$\overline{0},\ 1,\ \overline{2},\ 3,\ \overline{4,\ 5,\ 6,\ 7,\ \ldots}.$$

We shall use the following result, a special case of a result by Ônishi [Ôni02b, Ôni02a].

Theorem Let X be an algebraic curve of of genus 2. Then we have

$$\frac{\sigma(\boldsymbol{u}_0 + \boldsymbol{u}_1 + \cdots + \boldsymbol{u}_n) \prod_{0 \le k < l \le n} \sigma(\boldsymbol{u}_k - \boldsymbol{u}_l)}{\sigma_2^{n+1}(\boldsymbol{u}_0) \ldots \sigma_2^{n+1}(\boldsymbol{u}_n)}$$

$$= \frac{1}{2^{[n/2-1]}} \begin{vmatrix} 1 & w_1(x_0, y_0) & \ldots & w_n(x_0, y_0) \\ 1 & w_1(x_1, y_1) & \ldots & w_n(x_1, y_1) \\ \vdots & \vdots & \vdots & \vdots \\ 1 & w_1(x_n, y_n) & \ldots & w_n(x_n, y_n) \end{vmatrix}, \tag{31}$$

where $[\cdot]$ means integer part, $(x_0, y_0), \ldots, (x_n, y_n)$ is a non-special divisor on X, $\boldsymbol{u}_i$ is the Abel image

$$\boldsymbol{u}_i = \int_{(\infty,\infty)}^{(x_i,y_i)} d\mathbf{u},$$

and $\sigma_2(\boldsymbol{u}_i)$ is the value of the σ derivative restricted to the θ-divisor, $(\theta):\sigma(\boldsymbol{u}_i)=0$

$$\begin{aligned}\sigma_2(\boldsymbol{u}_i) &= \sigma_2(u_{i_1},u_{i_2})\\ &= \frac{\partial}{\partial u_{i_2}}\sigma(u_{i_1},u_{i_2})|_{\sigma(\boldsymbol{u}_i)=0}.\end{aligned}$$

The factor $1/2^{[n/2-1]}$ arises in our version of this theorem as we use a different normalization of the curve than Ônishi.

In particular, for $n=1$ we have

$$\frac{\sigma(\boldsymbol{u}_0+\boldsymbol{u}_1)\sigma(\boldsymbol{u}_0-\boldsymbol{u}_1)}{\sigma_2^2(\boldsymbol{u}_0)\sigma_2^2(\boldsymbol{u}_1)} = x_1 - x_0.$$

We now return to the derivation of the Baker addition formula. For this we introduce the notation

$$\Delta(\mathbf{x},\mathbf{y};\mathbf{x}',\mathbf{y}') = \frac{1}{2}\begin{vmatrix} 1 & x_1 & {x_1}^2 & y_1\\ 1 & x_2 & {x_2}^2 & y_2\\ 1 & {x'}_1 & {x'_1}^2 & {y'}_1\\ 1 & {x'}_2 & {x'_2}^2 & {y'}_2 \end{vmatrix},$$

where $\boldsymbol{x}=(x_1,x_2)^t, \boldsymbol{y}=(y_1,y_2)^t$, etc. Then by applying the above theorem, we obtain after simplification

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{u}')\sigma(\boldsymbol{u}-\boldsymbol{u}')}{\sigma_2^2(\boldsymbol{u})\sigma_2^2(\boldsymbol{u}')} = \frac{\Delta(\boldsymbol{x},\boldsymbol{y};\boldsymbol{x}',\boldsymbol{y}')\Delta(\boldsymbol{x},\boldsymbol{y};\boldsymbol{x}',-\boldsymbol{y}')}{V(\boldsymbol{x},\boldsymbol{x}')V(\boldsymbol{x})V(\boldsymbol{x}')},$$

where V is the Vandermonde determinant of its arguments

$$V(\boldsymbol{x}) = V(x_1,x_2) = \begin{vmatrix} 1 & x_1\\ 1 & x_2\end{vmatrix}, \tag{32}$$

$$V(\boldsymbol{x},\boldsymbol{x}') = V(x_1,x_2,x_1',x_2') = \begin{vmatrix} 1 & x_1 & {x_1}^2 & x_1^3\\ 1 & x_2 & {x_2}^2 & x_2^3\\ 1 & {x'}_1 & {x'_1}^2 & {x'}_1^3\\ 1 & {x'}_2 & {x'_2}^2 & {x'}_2^3\end{vmatrix}, \tag{33}$$

thus $V(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'')$ is 6×6 Vandermonde determinant.

After expanding these determinants, factorizing, applying the equation of the curve (4) and the relations (15), we arrive finally at the required formula (28). This is a simplified version of the calculation carried out by Baker [Bak95, pp. 331-332] and also derived by him using another method in 1907 [Bak07, pg. 100].

4. The addition rule on $\mathrm{Jac}(X) \times \mathrm{Jac}(X) \times \mathrm{Jac}(X)$

By analogy with the previous section we define

$$\Delta(\boldsymbol{x},\boldsymbol{y};\boldsymbol{x}',\boldsymbol{y}';\mathbf{x}'',\mathbf{y}'') = \frac{1}{4}\begin{vmatrix} 1 & x_1 & {x_1}^2 & y_1 & {x_1}^3 & x_1y_1 \\ 1 & x_2 & {x_2}^2 & y_2 & {x_2}^3 & x_2y_2 \\ 1 & x'_1 & {x'_1}^2 & y'_1 & {x'_1}^3 & x'_1y'_1 \\ 1 & x'_2 & {x'_2}^2 & y'_2 & {x'_2}^3 & x'_2y'_2 \\ 1 & x''_1 & {x''_1}^2 & y''_1 & {x''_1}^3 & x''_1y''_1 \\ 1 & x''_2 & {x''_2}^2 & y''_2 & {x''_2}^3 & x''_2y''_2 \end{vmatrix}.$$

If we denote

$$\boldsymbol{u}'' = \int_{(\infty,\infty)}^{(x''_1,y''_1)} d\boldsymbol{u} + \int_{(\infty,\infty)}^{(x''_2,y''_2)} d\boldsymbol{u},$$

and $\boldsymbol{u}$, $\boldsymbol{u}'$ as before (27), then changing the σ-functions by the formula (31) we obtain

$$\frac{\sigma(\boldsymbol{u}+\boldsymbol{u}'+\boldsymbol{u}'')\sigma(\boldsymbol{u}-\boldsymbol{u}')\sigma(\boldsymbol{u}'-\boldsymbol{u}'')\sigma(\boldsymbol{u}''-\boldsymbol{u})}{\sigma(\boldsymbol{u})^3\sigma(\boldsymbol{u}')^3\sigma(\boldsymbol{u}'')^3} = \Delta(\boldsymbol{x},\boldsymbol{y};\boldsymbol{x}',\boldsymbol{y}';\boldsymbol{x}'',\boldsymbol{y}'')$$

(34)

$$\times \frac{\Delta(\boldsymbol{x},\boldsymbol{y};\boldsymbol{x}',-\boldsymbol{y}')\Delta(\boldsymbol{x}',\boldsymbol{y}';\boldsymbol{x}'',-\boldsymbol{y}'')\Delta(\boldsymbol{x}'',\boldsymbol{y}'';\boldsymbol{x},-\boldsymbol{y})}{V(\boldsymbol{x},\boldsymbol{x}',\boldsymbol{x}'')V^2(\boldsymbol{x})V^2(\boldsymbol{x}')V^2(\boldsymbol{x}'')})$$

A straightforward expansion of the right hand side of the expression would result in almost 10^7 terms, beyond the capability of current algebraic computer systems and machines. We give here a brief description of the techniques used to reduce this to a manageable calculation.

It is necessary to first expand each determinant separately, then factor as far as possible each coefficient of the y_i's or product of the y_i's. To reduce the number of terms at this stage we next substitute single variables for each linear combination in x, i.e. $x_1-x_2=z_1, x_1-x'_1=z_2$, etc. At this stage the number of terms is reduced sufficiently to allow us to expand the product of the determinants as a polynomial in the z_i and y_i. Next we substitute the equation of the curve (4) to eliminate any quadratic powers of the y_i, and then substitute for any products y_1y_2 using the expression for $\wp_{11}$ in (15).

We now have three types of terms: (i) terms cubic in the y_i, for example with a factor $y_1y'_1y''_2$, (ii) terms linear in the y_i but containing a factor like $\wp_{11}$ (or $\wp'_{11}$ or $\wp''_{11}$), for example with a factor $y_1\wp'_{11}$, and (iii) terms linear in the y_i but with no $\wp_{11}$-like factors. Within these classifications we can further subdivide by considering terms with each possible choice of these factors separately. The next stage is to consider each of these subexpressions, first substituting for the z_i and then factorizing in the x_i. Finally we apply the relations (15) and (16) and recombine the results. We remark that the expression contains cubic factors z_i^{-3} which were removed by substituting the relation for $\wp_{111}$ in (16). This brings to the expression terms involving the parameters λ_i. Elimination of the λ-variables with help of relations (20-22) finally leads to the addition formula which is the principal result of this paper

$$
\begin{aligned}
&\frac{\sigma(\boldsymbol{u}+\boldsymbol{u}'+\boldsymbol{u}'')\,\sigma(\boldsymbol{u}-\boldsymbol{u}')\,\sigma(\boldsymbol{u}'-\boldsymbol{u}'')\,\sigma(\boldsymbol{u}''-\boldsymbol{u})}{\sigma(\boldsymbol{u})^3\,\sigma(\boldsymbol{u}')^3\,\sigma(\boldsymbol{u}'')^3} = \\
&\frac{1}{8}\,\wp_{112}\wp'_{122}\wp''_{222} - \frac{1}{8}\,\wp_{112}\wp'_{222}\wp''_{122} - \\
&\frac{1}{4}\left(-\wp''_{12}\wp_{22} + \wp'_{12}\wp_{22} - \wp'_{22}\wp_{12} + \wp''_{22}\wp'_{12} - 2\,\wp''_{11} + \right. \\
&\qquad \left. \wp''_{22}\wp_{12} - \wp'_{22}\wp''_{12} + 2\,\wp'_{11}\right)\wp_{111} - \\
&\frac{1}{4}\left(2\,\wp''_{22}\wp_{22}\wp'_{12} - 2\,\wp''_{12}\wp'_{22}\wp_{22} - \wp''_{22}\wp_{11} + \wp'_{22}\wp_{11} + \wp_{12}\wp'_{12} + \right. \\
&\qquad \left. \wp'_{11}\wp'_{22} - 2\,\wp''_{11}\wp'_{22} - \wp''_{11}\wp''_{22} - \wp_{12}\wp''_{12} + 2\,\wp'_{11}\wp''_{22} - {\wp'_{12}}^2 + {\wp''_{12}}^2\right)\wp_{112} + \\
&\frac{1}{4}\left(-\wp'_{11}\wp'_{22}\wp''_{22} + \wp''_{22}\wp'_{12}\wp_{12} - \wp'_{22}\wp''_{12}\wp_{12} + \wp''_{11}\wp'_{22}\wp''_{22} - 2\,\wp''_{11}\wp'_{12} - \right. \\
&\qquad \left. {\wp''_{12}}^2\wp'_{22} + 2\,\wp'_{12}\wp_{11} - 2\,\wp''_{12}\wp_{11} + {\wp'_{12}}^2\wp''_{22} + 2\,\wp'_{11}\wp''_{12}\right)\wp_{122} + \\
&\frac{1}{4}\left(\wp'_{11}\wp'_{22}\wp''_{12} - \wp'_{22}\wp''_{12}\wp_{11} - \wp''_{11}\wp''_{22}\wp'_{12} + \right. \\
&\qquad \left. {\wp''_{12}}^2\wp'_{12} - \wp''_{12}{\wp'_{12}}^2 + \wp''_{22}\wp'_{12}\wp_{11}\right)\wp_{222} + \\
&+ \text{cyclic permutations of } \wp, \wp', \wp'',
\end{aligned}
$$

where $\wp = \wp(\boldsymbol{u})$, $\wp' = \wp(\boldsymbol{u}')$, $\wp'' = \wp(\boldsymbol{u}'')$.

In assessing the validity of a computer-assisted result like this it is important to carry out checks during each stage of the calculations. One final check is to examine the behaviour of the formula above in the limit that $\boldsymbol{u} \equiv (u_1, u_2) \to 0$. (To be more precise, we take the limit of the above formula multiplied by $\sigma(\boldsymbol{u})^3$). To zeroth order in u_1, u_2 we recover the Baker formula (28) in the variables $\boldsymbol{u}', \boldsymbol{u}''$. To first order in u_1 we find the following genus two addition formula for ζ_1

$$\zeta_1(\boldsymbol{u}'+\boldsymbol{u}'') - \zeta_1(\boldsymbol{u}') - \zeta_1(\boldsymbol{u}'') = \frac{1}{2}\left(\frac{\partial}{\partial u'_1} + \frac{\partial}{\partial u''_1}\right)\log B(\boldsymbol{u}', \boldsymbol{u}'')$$

where $B(\boldsymbol{u}, \boldsymbol{v})$ is the RHS of (28)

$$B(\boldsymbol{u}, \boldsymbol{v}) = \wp_{22}(\boldsymbol{u})\wp_{12}(\boldsymbol{v}) - \wp_{22}(\boldsymbol{v})\wp_{12}(\boldsymbol{u}) + \wp_{11}(\boldsymbol{v}) - \wp_{11}(\boldsymbol{u})$$

and

$$\zeta_i(\boldsymbol{u}) = \frac{\partial}{\partial u_i} \ln \sigma(\boldsymbol{u}), \quad i = 1, 2.$$

It is straightforward to derive this addition formula by taking logarithmic derivatives of (28). An analogous formula holds for ζ_2.

Our new formula is also valid in the rational limit when the σ-function is changed to the Schur function: $\sigma(\boldsymbol{u}) \to u_1 - \frac{1}{3}u_2^3$ and thus the formula represents a non-trivial addition rule for the Schur function.

The extension of these results to higher genera, $g > 2$ and products $(\mathrm{Jac}(X))^N$, $N > 3$ will be the subject of further investigations.

Acknowledgements

VZE and EP are very thankful for NATO funding under Expert Visit Grant PST.EV.977012 that enabled VZE to visit Boston University. EP wishes to gratefully acknowledge partial funding of her research under NSF Grant DMS-0205643. VZE would also like to thank the Royal Society for financial support during a visit to Heriot-Watt University where some of this work was done. JCE and VZE also acknowledge the support of the EU under the Marie Curie LOCNET network grant.

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