Quasi-periodic and periodic solutions for coupled nonlinear Schrödinger equations of Manakov type

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We consider travelling periodic and quasi-periodic wave solutions of a set of coupled nonlinear Schrödinger equations. In fibre optics these equations can be used to model single mode fibres with strong birefringence, and two-mode optical fibres. Recently these equations appear as a model describing pulse-pulse interactions in wavelength-division-multiplexed channels of optical fibre transmission systems. In some cases this model reduces to the integrable Manakov system (IMS). Two phase quasi-periodic solutions for the IMS are given in terms of two dimensional Kleinian functions. The reduction of quasi-periodic solutions to elliptic functions is discussed. New solutions are found in terms of generalized Hermite polynomials, which are associated with two-gap Treibich-Verdier potentials.

Keywords: periodic solutions, quasi-periodic solutions, coupled nonlinear Schrödinger equations, Manakov system

1. Introduction

We consider a system of two coupled nonlinear Schrödinger equations

$$\begin{aligned} i\,\mathcal{U}_t + \mathcal{U}_{xx} + (\kappa\mathcal{U}\mathcal{U}^* + \chi\mathcal{V}\mathcal{V}^*)\mathcal{U} = 0, \\ i\,\mathcal{V}_t + \mathcal{V}_{xx} + (\chi\mathcal{U}\mathcal{U}^* + \rho\mathcal{V}\mathcal{V}^*)\mathcal{V} = 0, \end{aligned} \tag{1.1}$$

where κ, χ, ρ are some constants. The integrability of this system was proved by Manakov (1974) only for the case $\kappa = \chi = \rho$, which we shall refer as the *Integrable Manakov System (IMS)*.

Equations (1.1) are important for a number of physical applications when χ is positive and all remaining constants are set equal to 1. For example, for two-mode optical fibres, $\chi = 2$ (Crosignani *et al.* 1982); for propagation of two modes in fibres with strong birefringence, $\chi = \frac{2}{3}$ (Menyuk 1987) and in the general case $\frac{2}{3} \le \chi \le 2$ for elliptical eigenmodes. The special value $\chi = 1$ (IMS) corresponds to at least two possible physical cases, namely the case of a purely electrostrictive nonlinearity or, in the elliptical birefringence case, when the angle between the major and minor axes of the birefringence ellipse is approximately 35^o. The experimental observation of Manakov solitons in crystals has been reported by Kang *et al.* (1996). Recently the Manakov model has appeared in a Kerr-type approximation of photorefractive crystals (Kutusov *et al.* 1998). The pulse-pulse collision between wavelength-division-multiplexed channels of optical fibre transmission systems are described by (1.1) with $\chi = 2$, (Hasewaga and Kodama 1995; Kodama 1997; Kodama *et al.* 1996; Mollenauer *et al.* 1991). Wavelength division-multiplexing is one means of increasing the bandwidth in optical communication systems. This technique is limited by the finite bandwidth of the *Er*-doped fibre amplifiers which are now incorporated into most, if not all, such systems. An alternative to increase the bandwidth – and one which may well be used in conjunction with wavelength division-multiplexing – is polarisation division-multiplexing (Evangelides *et al.* 1992). Here, the polarisation state of the input pulses is varied from pulse to pulse in a specified way such that all pulses in a particular state can be switched into a particular channel on exit from the fibre. Relevant to this are the periodic solutions of the IMS, in which the polarisation state of the of the initial pulses varies from pulse to pulse in a specified manner. It is the properties of these solutions that we examine here.

General quasi-periodic solutions in terms of n-phase theta functions for the IMS are derived by Adams *et al.* (1993), while a series of special solutions are given in (Alfinito *et al.* 1995; Polymilis *et al.* 1998; Porubov & Parker 1999; Pulov *et al.* 1998). The authors of this present paper have already discussed quasi-periodic and periodic solutions associated with Lamé and Treibich-Verdier potentials for a non-integrable system of coupled nonlinear Schrödinger equations in terms of a special ansatz (Christiansen *et al.* 1995). We also mention the method of constructing elliptic finite-gap solutions of the stationary KdV and AKNS hierarchy, based on a theorem due to Picard, proposed by Gesztesy & Ratnaseelan (1998) and Gesztesy & Weikard (1996, 1998a, 1998b) and the method developed by Smirnov in series of publications, the review paper (Smirnov 1994) and Smirnov (1997a, 1997b). These techniques are also useful for finding solutions to the complex Ginsburgh-Landau equations (Porubov & Velarde 1999), and for periodic waves in multicomponent photorefractive crystals (Petnikova *et al.* 1999, Vysloukh *et al.* 1998).

In the present paper we investigate (1.1) restricted to a system integrable in terms of ultraelliptic functions, by introducing a special ansatz, which was recently applied by Porubov and Parker (1999) to analyse special classes of elliptic solutions of the Manakov system ($\kappa = \chi = \rho = 1$). More precisely, we seek a solution of (1.1) in the form

$$\begin{aligned}\mathcal{U}(x,t) &= q_1(x)\exp\left\{ia_1 t + iC_1\int^x \mathrm{d}x\, q_1{}^{-2}(x)\right\},\\ \mathcal{V}(x,t) &= q_2(x)\exp\left\{ia_2 t + iC_2\int^x \mathrm{d}x\, q_2{}^{-2}(x)\right\},\end{aligned} \tag{1.2}$$

where the $q_{1,2}(x)$ are real functions and a_1, a_2, C_1, C_2 are real constants. Substituting (1.2) into (1.1) we reduce the system to the equations

$$\begin{aligned}\frac{\mathrm{d}^2 q_1}{\mathrm{d}x^2} + \rho q_1^3 + \chi q_1 q_2^2 - a_1 q_1 - C_1^2 q_1{}^{-3} = 0,\\ \frac{\mathrm{d}^2 q_2}{\mathrm{d}x^2} + \kappa q_2^3 + \chi q_2 q_1^2 - a_2 q_2 - C_2^2 q_2{}^{-3} = 0.\end{aligned} \tag{1.3}$$

The system (1.3) is a natural Hamiltonian two-particle system with a Hamiltonian of the form

$$\begin{aligned} H &= \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 + \frac{1}{4}(\rho q_1^4 + 2\chi q_1^2 q_2^2 + \kappa q_2^4) \\ &\quad -\frac{1}{2}a_1 q_1^2 - \frac{1}{2}a_2 q_2^2 + \frac{1}{2}C_1^2 {q_1}^{-2} + \frac{1}{2}C_2^2 {q_2}^{-2}, \end{aligned} \tag{1.4}$$

where $p_i(x) = \mathrm{d}q_i(x)/\mathrm{d}x$, $i = 1, 2$.

These equations describe the motion of particles interacting with a quartic potential $Aq_1^4 + Bq_1^2q_2^2 + Cq_2^4$ and perturbed by an inverse squared potential. Nowadays four nontrivial cases of complete integrability are known for the nonperturbed quartic potential: (i) $A : B : C = 1 : 2 : 1$, (ii) $A : B : C = 1 : 12 : 16$, (iii) $A : B : C = 1 : 6 : 1$, and (iv) $A : B : C = 1 : 6 : 8$. Cases (i), (ii) and (iii) are separable in ellipsoidal, paraboidal and Cartesian coordinates respectively, whilst case (iv) is separable in the general sense (Ravoson *et al.* 1994). The case (ii) appears as one of the entries in the polynomial hierarchy discussed in Eilbeck *et al.* (1993). The cases (iii) and (iv) are proved to be canonically equivalent under the action of a Miura map restricted to the stationary coupled KdV systems associated with a fourth order Lax operator (Baker *et al.* 1995). Moreover all the cases (i)-(iv) allow the deformation of the potential by linear combination of inverse squares and squares with certain limitations on the coefficients (Eilbeck *et al.* 1993, Baker *et al.* 1995). There are also Lax representations known for all these cases which yield hyperelliptic algebraic curves in the cases (i) and (ii) and a 4-gonal curve in the cases (iii) and (iv). Various results concerning cases (i)-(iv) can be found in Hietarinta (1987) and Perelomov (1991).

Although each system listed yield nontrivial classes of solutions of the system (1.1), we shall discuss only the case (i) in detail. This brings us back to the IMS, but the techniques we describe can also be applied to the other cases which are not described by the IMS. The integrability of case (i), and separability in ellipsoidal coordinates was proved by Wojciechowski (1985) (see also Kostov 1989, Tondo 1995). We employ this result to integrate the system in terms of ultraelliptic functions (hyperelliptic functions of a genus two curve) and then reduce hyperelliptic functions to elliptic ones by imposing additional constraints on the parameters of the system.

The paper is organised as follows. In the first section we construct the Lax representation of the system, develop a genus two algebraic curve, which is associated with the system, and reduce the problem to solution of the Jacobi inversion problem associated with a genus two algebraic curve. In §2 we develop the integration of the system in terms of *Kleinian hyperelliptic functions*, which represent a natural generalization of Weierstrass elliptic functions to hyperelliptic curves of higher genera; recently this realization of Abelian functions was discussed in (Buchstaber *et al.* 1997a, 1997b; Eilbeck *et al.* 1999). In §2 the curve we use is a genus two curve, although general hyperelliptic curves have genus $g \geq 2$. Often the special case of a genus two hyperelliptic curve is called an ultraelliptic curve, and we since we restrict ourselves to the genus two case, use these two terms interchangeably throughout the paper. We explain in §3 the outline of the Kleinian realization of hyperelliptic functions and give the principal formulae for the case of a genus two curve. In §4 we develop a reduction of Kleinian hyperelliptic function to elliptic functions in terms of *Darboux coordinates* for the curve admitting additional involution. In this way a quasiperiodic solution in terms of elliptic functions is obtained. In the last section we construct a set of elliptic periodic solutions from spectral theory for the Schrödinger equation with an elliptic potential.

2. Lax representation

The system 1 : 2 : 1 ($\kappa = \chi = \rho = 1$) is a completely integrable Hamiltonian system

$$\begin{array}{l} \frac{\mathrm{d}^2 q_1}{\mathrm{d}x^2} + (q_1^2+q_2^2)q_1 - a_1 q_1 - C_1^2 {q_1}^{-3} = 0, \\ \frac{\mathrm{d}^2 q_2}{\mathrm{d}x^2} + (q_1^2+q_2^2)q_2 - a_2 q_2 - C_2^2 {q_2}^{-3} = 0 \end{array} \tag{2.1}$$

with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + \frac{1}{4}\left(q_1^2+q_2^2\right)^2 - \frac{1}{2}a_1 q_1^2 - \frac{1}{2}a_2 q_2^2 + \frac{1}{2}\frac{C_1^2}{q_1^2} + \frac{1}{2}\frac{C_2^2}{q_2^2}, \tag{2.2}$$

where the variables $(q_1, p_1; q_2, p_2)$ are the canonically conjugated variables with respect to the standard Poisson bracket, $\{\cdot\,;\,\cdot\}$.

This system has a the Lax representation, as a special case of the Lax representation given by Kostov (1989). This is the matrix equation

$$\begin{array}{rcl} \frac{\partial L(\lambda)}{\partial \zeta} & = & [M(\lambda), L(\lambda)], \\ L(\lambda) & = & \begin{pmatrix} V(\lambda) & U(\lambda) \\ W(\lambda) & -V(\lambda) \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 \\ Q(\lambda) & 0 \end{pmatrix}, \end{array} \tag{2.3}$$

which is equivalent to (2.1), where $U(\lambda), W(\lambda), Q(\lambda)$ have the form

$$\begin{array}{rcl} U(\lambda) & = & -a(\lambda)\left(1 + \frac{1}{2}\frac{q_1^2}{\lambda - a_1} + \frac{1}{2}\frac{q_2^2}{\lambda - a_2}\right), \\ V(\lambda) & = & -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\zeta}U(\lambda), \\ W(\lambda) & = & a(\lambda)\left(-\lambda + \frac{q_1^2}{2} + \frac{q_2^2}{2} + \frac{1}{2}\left(p_1^2 + \frac{C_1^2}{q_1^2}\right)\frac{1}{\lambda - a_1} + \right. \\ & & \left. + \frac{1}{2}\left(p_2^2 + \frac{C_2^2}{q_2^2}\right)\frac{1}{\lambda - a_2}\right), \\ Q(\lambda) & = & \lambda - q_1^2 - q_2^2, \end{array}$$

and $a(\lambda) = (\lambda - a_1)(\lambda - a_2)$.

The Lax representation yields a hyperelliptic curve $V = (\nu, \lambda)$,

$$\det\left(L(\lambda) - \frac{1}{2}\nu \mathbf{1}_2\right) = 0,$$

where $\mathbf{1}_2$ is the 2×2 unit matrix. The curve is given explicitly by

$$\begin{array}{rcl} \nu^2 & = & 4(\lambda - a_1)(\lambda - a_2)(\lambda^3 - \lambda^2(a_1 + a_2) + \lambda(a_1 a_2 - H) - G) \\ & & -C_1^2(\lambda - a_2)^2 - C_2^2(\lambda - a_1)^2, \end{array} \tag{2.4}$$

where H is the Hamiltonian (2.2), and the second independent integral of motion G, $\{H;G\}=0$ is given by

$$\begin{aligned} G &= \frac{1}{4}(p_1q_2-p_2q_1)^2+\frac{1}{2}(q_1^2+q_2^2)(a_1a_2-\frac{1}{2}a_2q_1^2-\frac{1}{2}a_1q_2^2)\\ &\quad -\frac{1}{2}p_1^2a_2-\frac{1}{2}p_2^2a_1-\frac{1}{4}\frac{(2a_2-q_2^2)C_1^2}{q_1^2}-\frac{1}{4}\frac{(2a_1-q_1^2)C_2^2}{q_2^2}. \end{aligned} \tag{2.5}$$

The parameters C_i are linked with the coordinates of the points $(a_i,\nu(a_i))$ by the formula

$$C_i^2=-\frac{\nu(a_i)^2}{(a_i-a_j)^2},\quad i,j=1,2. \tag{2.6}$$

We write the curve (2.4) in the form

$$\nu^2=4\lambda^5+\alpha_4\lambda^4+\alpha_3\lambda^3+\alpha_2\lambda^2+\alpha_1\lambda+\alpha_0, \tag{2.7}$$

where the *moduli* of the curve α_i are expressible in terms of physical parameters – level of energy H and constants a_1,a_2, C_1,C_2 as follows

$$\begin{aligned} \alpha_4 &= -8(a_1+a_2),\\ \alpha_3 &= -4H+4(a_1+a_2)^2+8a_1a_2,\\ \alpha_2 &= 4H(a_1+a_2)-4F-C_1^2-C_2^2-8a_1a_2(a_1+a_2),\\ \alpha_1 &= 4F(a_1+a_2)-4a_1a_2H+2C_1^2a_2+2C_2^2a_1+4a_1^2a_2^2,\\ \alpha_0 &= -4a_1a_2F-C_1^2a_2^2-C_2^2a_1^2. \end{aligned}$$

We define new coordinates μ_1,μ_2 as zeros of the entry $U(\lambda)$ in the Lax operator. Then

$$q_1^2=2\frac{(a_1-\mu_1)(a_1-\mu_2)}{a_1-a_2},\quad q_2^2=2\frac{(a_2-\mu_1)(a_2-\mu_2)}{a_2-a_1}. \tag{2.8}$$

The definition of μ_1,μ_2 in combination with the Lax representation gives the equations

$$\nu_i=V(\mu_i)=-\frac{1}{2}\frac{\partial}{\partial x}U(\mu_i),\quad i=1,2, \tag{2.9}$$

which can be transformed into equations of the the form†

$$u_1 = \int_{a_1}^{\mu_1}\mathrm{d}u_1+\int_{a_2}^{\mu_2}\mathrm{d}u_1, \tag{2.10}$$

$$u_2 = \int_{a_1}^{\mu_1}\mathrm{d}u_2+\int_{a_2}^{\mu_2}\mathrm{d}u_2, \tag{2.11}$$

where $\mathrm{d}u_{1,2}$ denote independent canonical holomorphic differentials

$$\mathrm{d}u_1=\frac{\mathrm{d}\lambda}{\nu},\quad \mathrm{d}u_2=\frac{\lambda\,\mathrm{d}\lambda}{\nu}. \tag{2.12}$$

and $u_1=a, u_2=2x+b$ with the constants a,b defined by the initial conditions. The integration of the problem is then reduced to the solution of the *Jacobi inversion problem* associated with the curve, which consists of the expression of the symmetric functions of $(\mu_1,\mu_2,\nu_1,\nu_2)$ as a function of two complex variables (u_1,u_2).

† In what follows we shall denote the integral bounds by the second coordinate of the curve $V=V(\nu,\lambda)$, eq. (2.4).

3. Exact solutions in terms of Kleinian hyperelliptic functions

In this section we give the trajectories of the system in terms of Kleinian hyperelliptic functions (e.g. Baker 1897; Buchstaber *et al.* 1997a), associated with an algebraic curve of genus two (2.7) which can be also written in the form

$$\nu^2 = 4\prod_{i=0}^{4}(\lambda - \lambda_i), \tag{3.1}$$

where $\lambda_i \neq \lambda_j$ are branch points. At all real branch points the closed intervals $[\lambda_{2i-1}, \lambda_{2i}], i = 0, \dots 4$ will be referred to as lacunae (Zakharov *et al.* 1980; McKean & van Moerbeke 1975). We equip the curve with a homology basis $(\mathfrak{a}_1, \mathfrak{a}_2; \mathfrak{b}_1, \mathfrak{b}_2) \in H_1(V, \mathbb{Z})$ and fix the basis in the space of holomorphic differentials as in (2.12). The associated canonical meromorphic differentials of the second kind $\mathrm{d}\boldsymbol{r}^T = (\mathrm{d}r_1, \mathrm{d}r_2)$ have the form

$$\mathrm{d}r_1 = \frac{\alpha_3\lambda + 2\alpha_4\lambda^2 + 12\lambda^3}{4\nu}\mathrm{d}\lambda, \qquad \mathrm{d}r_2 = \frac{\lambda^2}{\nu}\mathrm{d}\lambda. \tag{3.2}$$

The 2×2 matrices of their periods are

$$\begin{aligned} 2\omega &= \left(\oint_{\mathfrak{a}_k} \mathrm{d}u_l\right)_{k,l=1,2}, \quad 2\omega' = \left(\oint_{\mathfrak{b}_k} \mathrm{d}u_l\right)_{k,l=1,2}, \\ 2\eta &= \left(\oint_{\mathfrak{a}_k} \mathrm{d}r_l\right)_{k,l=1,2}, \quad 2\eta' = \left(\oint_{\mathfrak{b}_k} \mathrm{d}r_l\right)_{k,l=1,2}, \end{aligned}$$

which satisfy the equations

$$\omega'\omega^T - \omega\omega'^T = 0, \quad \eta'\omega^T - \eta\omega'^T = -\frac{\mathrm{i}\pi}{2}\mathbf{1}_2, \quad \eta'\eta^T - \eta\eta'^T = 0,$$

which generalize the Legendre relations between complete elliptic integrals to the case $g = 2$.

The fundamental σ function in this case is a natural generalization of the Weierstrass elliptic σ function and is defined as follows

$$\begin{aligned} \sigma(\boldsymbol{u}) &= \frac{\pi}{\sqrt{\det(2\omega)}} \frac{\epsilon}{\sqrt[4]{\prod_{1\le i<j\le 5}(\lambda_i - \lambda_j)}} \\ &\quad \times \exp\left\{\boldsymbol{u}^T\eta(2\omega)^{-1}\boldsymbol{u}\right\}\theta[\varepsilon]((2\omega)^{-1}\boldsymbol{u}|\omega'\omega^{-1}), \end{aligned}$$

where $\epsilon^8 = 1$, and $\theta[\varepsilon](\boldsymbol{v}|\tau)$ is the θ function with an odd characteristics $[\varepsilon] = \left[\begin{array}{cc} \varepsilon_1 & \varepsilon_2 \\ \varepsilon_1' & \varepsilon_2' \end{array}\right]$, $4(\varepsilon_1\varepsilon_1' + \varepsilon_2\varepsilon_2') = 1 \mod 2$, which is the characteristics of the vector of Riemann constants, and the θ function is defined by its Fourier series

$$\theta[\varepsilon](\boldsymbol{v}|\tau) = \sum_{\boldsymbol{m}\in\mathbb{Z}^2} \exp i\pi\left\{(\boldsymbol{m}+\boldsymbol{\varepsilon})^T\tau(\boldsymbol{m}+\boldsymbol{\varepsilon}) + 2(\boldsymbol{v}+\boldsymbol{\varepsilon}')^T\tau(\boldsymbol{m}+\boldsymbol{\varepsilon})\right\}.$$

Alternatively, the σ function can be defined by its expansion near $\boldsymbol{u} = 0$

$$\sigma(\boldsymbol{u}) = u_1 + \frac{1}{24}\alpha_2 u_1^3 - \frac{1}{3}u_2^3 + o(\boldsymbol{u}^5) \tag{3.3}$$

and further terms can be computed with the help of a bilinear differential equation (Baker 1907).

The σ-function possesses the following periodicity property: put

$$\boldsymbol{E}(\boldsymbol{m},\boldsymbol{m}') = \eta\boldsymbol{m} + \eta'\boldsymbol{m}', \quad \text{and} \quad \boldsymbol{\Omega}(\boldsymbol{m},\boldsymbol{m}') = \omega\boldsymbol{m} + \omega'\boldsymbol{m}',$$

where $\boldsymbol{m},\boldsymbol{m}' \in \mathbb{Z}^n$, then

$$\sigma[\varepsilon](\boldsymbol{z} + 2\boldsymbol{\Omega}(\boldsymbol{m},\boldsymbol{m}'),\omega,\omega') = \exp\{2\boldsymbol{E}^T(\boldsymbol{m},\boldsymbol{m}')(\boldsymbol{z} + \boldsymbol{\Omega}(\boldsymbol{m},\boldsymbol{m}'))\} \\ \times \exp\{-\pi i \boldsymbol{m}^T\boldsymbol{m}' - 2\pi i \boldsymbol{\varepsilon}^T\boldsymbol{m}'\}\sigma[\varepsilon](\boldsymbol{z},\omega,\omega').$$

As a modular function the Kleinian σ-function is invariant under the transformation of the symplectic group, which represents an important characteristic feature.

We introduce the Kleinian hyperelliptic functions as the logarithmic derivatives

$$\zeta_i(\boldsymbol{u}) = \frac{\partial}{\partial u_i}\ln\sigma(\boldsymbol{u}), \quad \wp_{ij}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j}\ln\ \sigma(\boldsymbol{u}), i,j = 1,2,$$

with $\wp_{12} = \wp_{21}$. The multi-index symbols $\wp_{i,j,k}$ etc. are defined as logarithmic derivatives with respect to the corresponding variables u_i, u_j, u_k.

The principal result of the theory is the formula of Klein, which reads in the case of genus two as follows. Let

$$\boldsymbol{u} = \int_\infty^{\mu_1} \mathrm{d}\mathbf{u} + \int_\infty^{\mu_2} \mathrm{d}\mathbf{u}$$

be an arbitrary vector in $\mathbb{C}^2$, and (μ_1,λ_1), (μ_2,λ_2) be arbitrary points on the curve. Then the following formula is valid

$$\sum_{k,l=1}^{2} \wp_{kl}\left(\int_\infty^{\mu} \mathrm{d}\mathbf{u} + \boldsymbol{u}\right)\mu^{k-1}\mu_i^{l-1} = \frac{F(\mu,\mu_i) - 2\nu\nu_i}{4(\mu-\mu_i)^2}, \quad i = 1,2, \tag{3.4}$$

where

$$F(\mu_1,\mu_2) = \sum_{r=0}^{2} \mu_1^r\mu_2^r[2\alpha_{2r} + \alpha_{2r+1}(\mu_1+\mu_2)]. \tag{3.5}$$

The analogous formulae for the hyperelliptic ζ-functions are

$$\zeta_1\left(\int_\infty^{\mu} \mathrm{d}\mathbf{u} + \boldsymbol{u}\right) = \int_\infty^{\mu} \mathrm{d}r_1 + \int_\infty^{\mu_1} \mathrm{d}r_1 + \int_\infty^{\mu_2} \mathrm{d}r_1 - \frac{1}{2}\wp_{222}(\boldsymbol{u}) - \\ \frac{\nu(\mu - \wp_{22}(\boldsymbol{u})) - \mu\,\wp_{122}(\boldsymbol{u}) - \wp_{112}(\boldsymbol{u})}{2\mathcal{P}(\lambda,\boldsymbol{u})}, \tag{3.6}$$

and

$$\zeta_2\left(\int_\infty^{\mu} \mathrm{d}\mathbf{u} + \boldsymbol{u}\right) = \int_\infty^{\mu} \mathrm{d}r_2 + \int_\infty^{\mu_1} \mathrm{d}r_2 + \int_\infty^{\mu_2} \mathrm{d}r_2 - \tag{3.7}$$

$$\frac{\nu - \mu\,\wp_{222}(\boldsymbol{u}) - \wp_{122}(\boldsymbol{u})}{2\mathcal{P}(\lambda,\boldsymbol{u})}, \tag{3.8}$$

where

$$\mathcal{P}(\lambda, \boldsymbol{u}) = \lambda^2 - \wp_{22}(\boldsymbol{u})\lambda - \wp_{12}(\boldsymbol{u}). \tag{3.9}$$

By expanding these equalities at $\mu = \infty$ we obtain a complete set of the relations for the hyperelliptic functions.

The first group of the relations represents the solution of the Jacobi inversion problem in the form

$$\mathcal{P}(\lambda, \boldsymbol{u}) = 0, \tag{3.10}$$

that is, the pair (μ_1, μ_2) is a pair of roots of (3.10). Thus we get

$$\wp_{22}(\boldsymbol{u}) = \mu_1 + \mu_2,\ \wp_{12}(\boldsymbol{u}) = -\mu_1\mu_2. \tag{3.11}$$

The corresponding ν_i is expressed as

$$\nu_i = \wp_{222}(\boldsymbol{u})\mu_i + \wp_{122}(\boldsymbol{u}), \quad i = 1, 2. \tag{3.12}$$

The functions $\wp_{22}, \wp_{12}$ are called basis functions. The function $\wp_{11}(\boldsymbol{u})$ is expressed as a symmetric function of μ_1, μ_2 and ν_1, ν_2 from (3.4)

$$\wp_{11}(\boldsymbol{u}) = \frac{F(\mu_1, \mu_2) - 2\nu_1\nu_2}{4(\mu_1 - \mu_2)^2}, \tag{3.13}$$

where $F(\mu_1, \mu_2)$ is given in (3.5).

The next group of relations, which can be derived by an expansion of (3.4), are the pairwise products of the $\wp_{ijk}$ functions expressed in terms of $\wp_{22}, \wp_{12}, \wp_{11}$ and constants α_s of the defining equation (3.1). We give here only the basis equations

$$\wp_{222}^2 = 4\wp_{22}^3 + 4\wp_{12}\wp_{22} + \alpha_4\wp_{22}^2 + 4\wp_{11} + \alpha_3\wp_{22} + \alpha_2, \tag{3.14}$$

$$\wp_{222}\wp_{122} = 4\wp_{12}\wp_{22}^2 + 2\wp_{12}^2 - 2\wp_{11}\wp_{22} + \alpha_4\wp_{12}\wp_{22} + \frac{1}{2}\alpha_3\wp_{12} + \frac{1}{2}\alpha_1. \tag{3.15}$$

The next group of the equations, which is derived as the result of expanding the equalities (3.4), are the expressions of four index symbols $\wp_{ijkl}$ as quadrics in $\wp_{ij}$ (again we give the basis functions only)

$$\wp_{2222} = 6\wp_{22}^2 + \frac{1}{2}\alpha_3 + \alpha_4\wp_{22} + 4\wp_{12}, \tag{3.16}$$

$$\wp_{1222} = 6\wp_{22}\wp_{12} + \alpha_4\wp_{12} - 2\wp_{11}, \tag{3.17}$$

These equations can be identified with completely integrable partial differential equations and dynamical systems, which can be solved in terms of Abelian functions of a hyperelliptic curve of genus two. In particular, these equations represent the KdV hierarchy with "times" $(t_1, t_2) = (u_2, u_1) = (x, t)$,

$$\mathcal{X}_{k+1}[\mathsf{U}] = \mathcal{R}\mathcal{X}_k[\mathsf{U}], \tag{3.18}$$

where $\mathcal{R} = \partial_x^2 - \mathsf{U} + c - \frac{1}{2}\mathsf{U}_x\partial^{-1}$, $c = \frac{1}{12}\alpha_4$ is the Lenard recursion operator. The first two equations from the hierarchy are

$$\mathsf{U}_{t_1} = \mathsf{U}_x, \quad \mathsf{U}_{t_2} = \frac{1}{2}(\mathsf{U}_{xxx} - 6\mathsf{U}_x\mathsf{U}), \tag{3.19}$$

the second equation is the KdV equation, which is obtained from (3.16) as the result of differentiation by $x = u_2$ and setting $\mathsf{U} = 2\wp_{22} + \frac{1}{6}\alpha_4$. The equation (3.16) plays the role of the stationary equation in the hierarchy and is obtained as the result of the action of the recursion operator. The relations (3.14) and (3.15) are solved with respect to α_2 and α_1 respectively and represent in this context the levels of integrals of motion.

Let us introduce finally the *Baker-Akhiezer* function, which in the framework of the formalism developed is expressible in terms of the Kleinian σ-function as follows

$$\Psi(\lambda, \boldsymbol{u}) = \frac{\sigma\left(\int_\infty^\lambda \mathrm{d}\boldsymbol{u} - \mathbf{u}\right)}{\sigma(\boldsymbol{u})}\exp\left\{\int_\infty^\lambda \mathrm{d}\mathbf{r}^T\boldsymbol{u}\right\}, \tag{3.20}$$

where λ is arbitrary and $\boldsymbol{u}$ is the Abel image of an arbitrary point $(\nu_1, \mu_1) \times (\nu_2, \mu_2) \in V \times V$. It is straightforward to show by direct calculation, using the relations (3.16) and (3.14), that $\Psi(\lambda, \boldsymbol{u})$ satisfy the Schrödinger equation

$$\left\{\frac{\partial^2}{\partial {u_2}^2} - 2\wp_{22}(\boldsymbol{u})\right\}\Psi(\lambda, \boldsymbol{u}) = \left(\lambda + \frac{1}{4}\alpha_4\right)\Psi(\lambda, \boldsymbol{u}) \tag{3.21}$$

for all (ν, μ).

Now we are in a position to write the solution of the system in terms of Kleinian σ-functions and identify the constants in terms of the moduli of the curve. Using (3.11),(2.8) the solutions of (2.1) have the following form in terms of the Kleinian functions $\wp_{22}(\boldsymbol{u}), \wp_{12}(\boldsymbol{u})$

$$\begin{aligned} q_1^2(x) &= 2\frac{a_1^2 - \wp_{22}(\boldsymbol{u})a_1 - \wp_{12}(\boldsymbol{u})}{a_1 - a_2}, \\ q_2^2(x) &= 2\frac{a_2^2 - \wp_{22}(\boldsymbol{u})a_2 - \wp_{12}(\boldsymbol{u})}{a_2 - a_1}, \end{aligned} \tag{3.22}$$

where the vector $\boldsymbol{u}^T = (a, 2x + b)$. Finally, the solutions of the IMS reads in this case

$$\begin{aligned} \mathcal{U}(x,t) &= \sqrt{2\frac{\mathcal{P}(a_1, \boldsymbol{u})}{a_1 - a_2}}\exp\left\{ia_1 t - \frac{1}{2}\nu(a_i)\int\limits_{\cdot}^{x} \frac{\mathrm{d}x}{\mathcal{P}(a_1, \boldsymbol{u})}\right\}, \\ \mathcal{V}(x,t) &= \sqrt{2\frac{\mathcal{P}(a_2, \boldsymbol{u})}{a_2 - a_1}}\exp\left\{ia_2 t - \frac{1}{2}\nu(a_2)\int\limits_{\cdot}^{x} \frac{\mathrm{d}x}{\mathcal{P}(a_1, \boldsymbol{u})}\right\}. \end{aligned} \tag{3.23}$$

The solutions $q_i(x)$ of (2.1) are linked as follows with the Baker-Akhiezer function. It follows from the definition of the Baker-Akhiezer function and an application of the formulae given above to the hyperelliptic ζ-function (3.8), that

$$\frac{\partial\Psi(\lambda;\boldsymbol{u})}{\partial u_2}=\frac{\nu+\partial\mathcal{P}(\lambda;\boldsymbol{u})/\partial u_2}{2\mathcal{P}(\lambda;\boldsymbol{u})}\Psi(\lambda;\boldsymbol{u}).$$

By integrating this equality under the assumption, that $u_1=$ const., we obtain

$$\Psi(\lambda;\boldsymbol{u})=\mathcal{C}\sqrt{\mathcal{P}(\lambda;\boldsymbol{u})}\exp\left\{\frac{1}{2}\nu\int_{.}^{u_2}\frac{\mathrm{d}u_2}{\mathcal{P}(\lambda;\boldsymbol{u})}\right\}, \tag{3.24}$$

where $\mathcal{C}$ is constant with respect to the variable u_2. The substitution of this Baker-Akhiezer function into the Schrödinger equation (3.21) and comparison with the dynamical equations of the system 1:2:1 leads to the conclusion that

$$\Psi(a_1,x)=\mathcal{U}(x,0),\qquad \Psi(a_2,x)=\mathcal{V}(x,0), \tag{3.25}$$

where $\mathcal{U}(x,t)$ and $\mathcal{V}(x,t)$ are given in (3.23). This formulae clarify the origin of the ansatz (1.2).

4. Periodic solutions expressed in terms of elliptic functions of different moduli

In this section, we consider the reduction by Jacobi (see e.g. Krazer 1903)) of hyperelliptic integrals to elliptic ones, when the hyperelliptic curve V has the form

$$w^2=z(z-1)(z-\alpha)(z-\beta)(z-\alpha\beta). \tag{4.1}$$

The curve (4.1) covers two-sheetedly two tori

$$\begin{aligned}\pi_\pm : V=(w,z)\to E_\pm=(\eta_\pm,\xi_\pm),\\ \eta_\pm^2=\xi_\pm(1-\xi_\pm)(1-k_\pm^2\xi_\pm)\end{aligned} \tag{4.2}$$

with Jacobi moduli

$$k_\pm^2=-\frac{(\sqrt{\alpha}\mp\sqrt{\beta})^2}{(1-\alpha)(1-\beta)}, \tag{4.3}$$

The covers $\pi_\pm$ are described by the formulae

$$\eta_\pm \;=\; -\sqrt{(1-\alpha)(1-\beta)}\frac{z\mp\sqrt{\alpha\beta}}{(z-\alpha)^2(z-\beta)^2}w, \tag{4.4}$$

$$\xi \;=\; \xi_\pm=\frac{(1-\alpha)(1-\beta)z}{(z-\alpha)(z-\beta)}. \tag{4.5}$$

The following formula is valid for the reduction of holomorphic hyperelliptic differentials to the elliptic ones:

$$\frac{d\xi_\pm}{\eta_\pm}=-\sqrt{(1-\alpha)(1-\beta)}(z\mp\sqrt{\alpha\beta})\frac{\mathrm{d}z}{w}. \tag{4.6}$$

Suppose that the spectral curve (2.7) admits the symmetry of (4.1) and apply the reduction case discussed to the problem. Then the equations of the Jacobi inversion problem (2.11) can be rewritten in the form

$$\sqrt{(1-\beta)(1-\alpha)}\sum_{i=1}^{2}\int_{z_0}^{z_i}(z-\sqrt{\alpha\beta})\frac{\mathrm{d}z}{w} = 2u_+\,, \tag{4.7}$$

$$\sqrt{(1-\beta)(1-\alpha)}\sum_{i=1}^{2}\int_{x_0}^{z_i}(z+\sqrt{\alpha\beta})\frac{\mathrm{d}z}{w} = 2u_-\,. \tag{4.8}$$

with $(\nu_i,\mu_i) = (2w_i, z_i)$ and

$$u_\pm = -\sqrt{(1-\alpha)(1-\beta)}(u_2 \mp \sqrt{\alpha\beta}u_1). \tag{4.9}$$

Reducing the hyperelliptic integrals in (4.7,4.8) to elliptic ones according to (4.4,4.5).

$$\int_0^{\sqrt{\xi(\mu_1)}}\frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k_\pm^2x^2)}}+\int_0^{\sqrt{\xi(\mu_2)}}\frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-k_\pm^2x^2)}} = u_\pm\,,$$

one can further express the symmetric functions of μ_1,μ_2,ν_1,ν_2 on $V\times V$ in terms of elliptic functions of tori $E_\pm$. To this end we introduce the *Darboux coordinates* (see Hudson 1905, p.105)

$$\begin{aligned} X_1 &= \mathrm{sn}(u_+,k_+)\mathrm{sn}(u_-,k_-),\\ X_2 &= \mathrm{cn}(u_+,k_+)\mathrm{cn}(u_-,k_-),\\ X_3 &= \mathrm{dn}(u_+,k_+)\mathrm{dn}(u_-,k_-), \end{aligned} \tag{4.10}$$

where $\mathrm{sn}(u_\pm,k_\pm), \mathrm{cn}(u_\pm,k_\pm), \mathrm{dn}\,(u_\pm,k_\pm)$ are standard Jacobi elliptic functions.

We apply further the addition theorem for Jacobi elliptic functions,

$$\begin{aligned} \mathrm{sn}(u_1+u_2,k) &= \frac{s_1^2-s_2^2}{s_1c_2d_2-s_2c_1d_1},\\ \mathrm{cn}\,(u_1+u_2,k) &= \frac{s_1c_1d_2-s_2c_2d_1}{s_1c_2d_2-s_2c_1d_1},\\ \mathrm{dn}(u_1+u_2,k) &= \frac{s_1d_1c_2-s_2d_2s_1}{s_1c_2d_2-s_2c_1d_1}, \end{aligned}$$

where we have denoted $s_i = \mathrm{sn}(u_i,k), c_i = \mathrm{cn}(u_i,k), d_i = \mathrm{dn}(u_i,k), i = 1,2$ and use formulae (3.11,3.13) for the Kleinian hyperelliptic functions. Then straightforward calculations lead to the formulae

$$X_1 = -\frac{(1-\alpha)(1-\beta)(\alpha\beta+\wp_{12})}{(\alpha+\beta)(\wp_{12}-\alpha\beta)+\alpha\beta\wp_{22}+\wp_{11}}, \tag{4.11}$$

$$X_2 = -\frac{(1+\alpha\beta)(\alpha\beta-\wp_{12})-\alpha\beta\wp_{22}-\wp_{11}}{(\alpha+\beta)(\wp_{12}-\alpha\beta)+\alpha\beta\wp_{22}+\wp_{11}}, \tag{4.12}$$

$$X_3 = -\frac{\alpha\beta\wp_{22}-\wp_{11}}{(\alpha+\beta)(\wp_{12}-\alpha\beta)+\alpha\beta\wp_{22}+\wp_{11}}.$$

The formulae (4.12) can be inverted as follows

$$\wp_{11} = (B-1)\frac{A(X_2+X_3)-B(X_3+1)}{X_1+X_2-1}, \tag{4.13}$$

$$\wp_{12} = (B-1)\frac{1+X_1-X_2}{X_1+X_2-1}, \tag{4.14}$$

$$\wp_{22} = \frac{A(X_2-X_3)+B(X_3-1)}{X_1+X_2-1}, \tag{4.15}$$

where $A=\alpha+\beta$, $B=1+\alpha\beta$.

We can use these results to present a few solutions in terms of elliptic functions of the initial problem, which are quasi-periodic in x. Using (4.14) and (4.15) for solutions of the (2.1) in the form (3.22) we have

$$\begin{aligned} q_1^2(x) &= \frac{2}{a_1-a_2}\left(a_1^2 - \frac{A(X_2-X_3)+B(X_3-1)}{X_1+X_2-1}a_1 \right. \\ &\qquad \left. -(B-1)\frac{1+X_1-X_2}{X_1+X_2-1}\right), \\ q_2^2(x) &= \frac{2}{a_2-a_1}\left(a_2^2 - \frac{A(X_2-X_3)+B(X_3-1)}{X_1+X_2-1}a_2 \right. \\ &\qquad \left. -(B-1)\frac{1+X_1-X_2}{X_1+X_2-1}\right), \end{aligned}$$

where

$$u_\pm = -2\sqrt{(1-\alpha)(1-\beta)}(x\mp c) \tag{4.16}$$

and c is a constant depending on the initial conditions. The only compatibility condition, which appears as the result of comparing the general curve coming from the Lax representation with the reduction case considered in this section, is

$$a_1+a_2 = \frac{1}{2}(1+\alpha)(1+\beta).$$

The levels of the integrals of motion H and G, denoted by $\mathcal{H}$ and $\mathcal{G}$ respectively, are

$$\begin{aligned} \mathcal{H} &= a_1^2+a_2^2+4a_1a_2-2\alpha\beta-(1+\alpha\beta)(\alpha+\beta) \\ \mathcal{G} &= (a_1+a_2)^3-\frac{1}{4}(C_1^2+C_2^2)+[2\alpha\beta+(1+\alpha\beta)(\alpha+\beta)](a_1+a_2) \\ &\quad -\alpha\beta(1+\alpha)(1+\beta). \end{aligned}$$

We also remark, that the quasi periodic solution derived is associated with the Jacobi reduction case in which the ultraelliptic integrals are reduced to elliptic ones by means of a second order substitution. This means in the language of two-dimensional θ-functions, that the associated period matrix is equivalent to a matrix with off-diagonal element $\tau_{12}=\frac{1}{2}$. This reduction was considered in various places (see e.g. Belokolos *et al.* 1994, Enolskii and Salerno 1996). Solutions of this type for the nonlinear Schrödinger equation ($\sigma=0$) were recently obtained by Chow (1995).

The analogous technique can be used out for the other well-documented case of reduction, when $\tau_{12} = 1/N$ and the $N = 3, 4, \ldots$. In general this reduction can be carried out for covers of arbitrary degree within the Weierstrass-Poincaré reduction theory (see e.g. Belokolos *et al.* 1994; Krazer 1903).

5. Elliptic periodic solutions

In this section we develop a method (see also Eilbeck and Enolskii 1994; Enolskii and Kostov 1994; Kostov 1989) which allows us to construct periodic solutions of (2.1) in a straightforward way based on the application of spectral theory for the Schrödinger equation with elliptic potentials (Airault *et al.* 1977; McKean & van Moerbeke 1975). We start with the formula (3.16) and the equation for the Baker function $\Psi(\lambda; \boldsymbol{u})$.

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\Psi(\lambda, \boldsymbol{u}) - \mathsf{U}(\boldsymbol{u})\Psi(x, \boldsymbol{u}) = \left(\lambda + \frac{\alpha_4}{4}\right)\Psi(\lambda, \boldsymbol{u}), \tag{5.1}$$

where we identify the potential as

$$\mathsf{U}(\boldsymbol{u}) = 2\wp_{22}(\boldsymbol{u}) + \frac{1}{6}\alpha_4.$$

We assume, without loss of generality, that the associated curve has the property $\alpha_4 = 0$. To make this assumption applicable to the initial curve of the system (2.1), derived from the Lax representation, we make a shift of the spectral parameter,

$$\lambda \longrightarrow \lambda + \Delta, \qquad \Delta = \frac{2}{5}a_1 + \frac{2}{5}a_2. \tag{5.2}$$

Suppose, that U is a two-gap Lamé or two gap Treibich-Verdier potential, i.e.

$$\mathsf{U}(\boldsymbol{u}) \equiv \mathsf{U}(x) = 2\sum_{i=1}^{N}\wp(x - x_i), \tag{5.3}$$

where $\wp(x)$ is the standard Weierstrass elliptic function with periods $2\omega, 2\omega'$, and the numbers x_i take values from the set $\{0, \omega_1 = \omega, \omega_2 = \omega + \omega', \omega_3 = \omega'\}$. It is known, that the set of such potentials is exhausted by six potentials (Treibich & Verdier 1990)

$$\begin{aligned}
\mathsf{U}_3(x) &= 6\wp(x), && (5.4)\\
\mathsf{U}_4(x) &= 6\wp(x) + 2\wp(x + \omega_i), \quad i = 1, 2, 3, && (5.5)\\
\mathsf{U}_5(x) &= 6\wp(x) + 2\wp(x + \omega_i) + 2\wp(x + \omega_j), \quad i \neq j = 1, 2, 3, && (5.6)\\
\mathsf{U}_6(x) &= 6\wp(x) + 6\wp(x + \omega_i), \quad i = 1, 2, 3, &&\\
\mathsf{U}_8(x) &= 6\wp(x) + 2\sum_{i=1}^{3}\wp(x + \omega_i), &&\\
\mathsf{U}_{12}(x) &= 6\wp(x) + 6\sum_{i=1}^{3}\wp(x + \omega_i), &&
\end{aligned}$$

where the subscript indicates the number of $2\wp$ functions involved and display the degree of the cover of the associated genus two curve over the elliptic curve. Because the last three potentials can be obtained from the first three by Gauss transform, we shall denote the first three as *basis potentials.* The potential (5.4) is two gap Lamé potential, which is associated with a three sheeted cover of the elliptic curve; the potentials (5.5,5.6) are Treibich-Verdier potentials (Treibich & Verdier 1990; Verdier 1990) associated with four and five sheeted covers correspondingly.

To display the class of periodic solutions of system (2.1) we introduce the *generalized Hermite polynomial* $\mathcal{F}(x,\lambda)$ by the formula

$$\mathcal{F}(x,\lambda)=\lambda^2-\pi_{22}(x)\lambda-\pi_{12}(x) \tag{5.7}$$

with $\pi_{22}(x)$ and $\pi_{12}(x)$ given as follows

$$\begin{aligned}\pi_{22}(x) &= \sum_{j=1}^{N}\wp(x-x_j)+\frac{1}{3}\sum_{j=1}^{5}\lambda_j,\\ \pi_{12}(x) &= -3\sum_{i<j}\wp(x-x_i)\wp(x-x_j)-\frac{Ng_2}{8}-\frac{1}{6}\sum_{i<j}\lambda_i\lambda_j+\frac{1}{6}\left(\sum_{j=1}^{5}\lambda_j^2\right),\end{aligned}$$

where x_i are half-periods and N is the degree of the cover (see for example Enolskii and Kostov 1994). The introduction of this formula is based on the possibility of computing the symmetric function $\mu_1\mu_2$ in terms of differential polynomial of the first one with the help of the equation (3.16), which serves in this context as a "trace formula" (Zakharov *et al.* 1980).

The solutions of the system (2.1) are then

$$q_1^2(x)=2\frac{\mathcal{F}(x,a_1-\Delta)}{a_1-a_2},\quad q_2^2(x)=2\frac{\mathcal{F}(x,a_2-\Delta)}{a_2-a_1}. \tag{5.8}$$

The final formula in terms of Hermite polynomials for the elliptic periodic solutions of the system (1.1) then reads

$$\begin{aligned}\mathcal{U}(x,t)&=\sqrt{2\frac{\mathcal{F}(x,a_1-\Delta)}{a_1-a_2}}\exp\left\{ia_1t-\frac{1}{2}\nu(a_1-\Delta)\int_{\cdot}^{x}\frac{\mathrm{d}x}{\mathcal{F}(x,a_1-\Delta)}\right\},\\ \mathcal{V}(x,t)&=\sqrt{2\frac{\mathcal{F}(x,a_2-\Delta)}{a_2-a_1}}\exp\left\{ia_2t-\frac{1}{2}\nu(a_2-\Delta)\int_{\cdot}^{x}\frac{\mathrm{d}x}{\mathcal{F}(x,a_2-\Delta)}\right\},\end{aligned} \tag{5.9}$$

where we have used (5.8) and (2.6).

It is important to remark that if the potential is known, then the associated algebraic curve of genus two can be described with the help of the Novikov equation (Novikov 1974). Let us consider the two-gap potential normalized by its expansion near the singular point

$$\mathsf{U}(x)=\frac{6}{x^2}+ax^2+bx^4+cx^6+dx^8+O(x^{10}), \tag{5.10}$$

where a, b, c, d are constants. Then the algebraic curve associated with this potential has the form (Belokolos & Enolskii 1989)

$$\begin{aligned}\nu^2 &= \lambda^5 - \frac{5\cdot 7}{2}a\lambda^3 + \frac{3^2\cdot 7}{2}b\lambda^2 \\ &+ \left(\frac{3^4\cdot 7}{8}a^2 + \frac{3^3\cdot 11}{4}c\right)\lambda - \frac{3^4\cdot 17}{4}ab + \frac{3^2\cdot 11\cdot 13}{2}d. \end{aligned} \tag{5.11}$$

We shall consider below examples of genus two curves, which are associated with the two gap elliptic potentials (5.4), (5.5) and (5.6).

Consider the potential U_3 and construct the associated curve (5.11)

$$\nu^2 = (\lambda^2 - 3g_2)(\lambda + 3e_1)(\lambda + 3e_2)(\lambda + 3e_3), \tag{5.12}$$

The Hermite polynomial $\mathcal{F}_3(\wp(x), \lambda)$ (Whittaker & Watson 1986) associated with the Lamé potential (5.4), which is already normalized as in (5.10), has the form

$$\mathcal{F}_3(\wp(x), \lambda) = \lambda^2 - 3\wp(x)\lambda + 9\wp^2(x) - \frac{9}{4}g_2. \tag{5.13}$$

Then the finite and real solution of the system (2.1) is given by the formula (5.8) with the Hermite polynomial depending on the argument $x + \omega'$ (the shift in ω' provides the holomorphicity of the solution). The solution is real under the choice of the arbitrary constants $a_{1,2}$ in such way, that the constants $a_{1,2} - \Delta$ lie in *different* lacunae. According to (2.6) the constants C_i are then given by

$$C_i^2 = -\frac{4\nu^2(a_i - \Delta)}{(a_i - a_j)^2}, \tag{5.14}$$

where Δ is the shift (5.2), ν is the coordinate of the curve (5.12), and the levels of the integrals H and G have the following form

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_0 + \frac{21}{4}g_2, \\ \mathcal{G} &= \mathcal{G}_0 - \frac{27}{4}g_3 - \frac{21}{20}g_2\left(a_1 + a_2\right),\end{aligned}$$

where

$$\begin{aligned}\mathcal{H}_0 &= \frac{1}{25}\left(a_1 + a_2\right)^3, \\ \mathcal{G}_0 &= \frac{1}{25}\left(a_1 + a_2\right)^3 - \frac{1}{4}C_1^2 - \frac{1}{4}C_2^2.\end{aligned} \tag{5.15}$$

These results are in complete agreement with solutions obtained in Porubov & Parker (1999) by introducing an ansatz of the form

$$q_i(x) = \sqrt{A_i\wp(x)^2 + B_i\wp(x) + C_i}, \quad i = 1, 2$$

with the constants A_i, B_i, C_i which are defined from the compatibility condition of the ansatz with the equations of motion. In what follows we shall consider solutions of the form

$$q_i(x) = \sqrt{\mathcal{Q}_i(\wp(x))}, \quad i = 1, 2,$$

where $\mathcal{Q}_i$ are rational functions of $\wp(x)$.

With this aim, we consider the following Treibich-Verdier potential

$$\mathsf{U}_4(x) = 6\wp(x) + 2\wp(x+\omega_1) - 2e_1, \tag{5.16}$$

associated with a four sheeted cover. The potential is normalized according to (5.10). The associated spectral curve is of the form

$$\nu^2 = 4(\lambda + 6e_1)\prod_{k=1}^{4}(\lambda - \lambda_k), \tag{5.17}$$

$$\begin{aligned} \lambda_{1,2} &= e_3 + 2e_2 \pm 2\sqrt{(5e_3+7e_2)(2e_3+e_2)}, \\ \lambda_{3,4} &= e_2 + 2e_3 \pm 2\sqrt{(5e_2+7e_3)(2e_2+e_3)}. \end{aligned} \tag{5.18}$$

The Hermite polynomial associated with this curve is given by the formula

$$\begin{aligned} \mathcal{F}_4(x,\lambda) = {}& \lambda^2 - (3\wp(x) + \wp(x+\omega_1) - e_1)\lambda \\ & +9\wp(x)(\wp(x) + \wp(x+\omega) - e_1) - 3e_1\wp(x+\omega_1) \\ & +\frac{9}{4}g_2 - 51e_1^2. \end{aligned} \tag{5.19}$$

The finite real solution of (2.1) results from the substitution of this Hermite polynomial $\mathcal{F}_4(x+\omega', \lambda)$ into (5.9), depending on an argument shifted by an imaginary half period, into (5.8). To fix the reality of the solution we shall fix the parameters $a_i - \Delta$ in the permitted zones. The levels of the integrals H and G have the following form

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_0 + \frac{7}{2}g_2 + 105e_1^2, \\ \mathcal{G} &= \mathcal{G}_0 + \left(\frac{7}{10}g_2 - 21e_1^2\right)(a_1+a_2) - 63e_1g_2 - \frac{171}{2}g_3 + 126e_1^3, \end{aligned}$$

where $\mathcal{H}_0$ and $\mathcal{G}_0$ are given in (5.15) and the constants C_i are computed by the formula (5.14) in which ν represents the coordinate of the curve (5.17).

Consider further the Treibich Verdier potential

$$\mathsf{U}_5(x) = 6\wp(x) + 2\wp(x+\omega_2) + 2\wp(x+\omega_3) + 2e_1, \tag{5.20}$$

associated with a five sheeted cover. The potential is normalized according to (5.10). The associated spectral curve is of the form

$$\begin{aligned} \nu^2 = {}& (\lambda + 6e_2 - 3e_3)(\lambda + 6e_3 - 3e_2) \times \\ & \times \left[\lambda^3 + 3e_1\lambda^2 - (29e_2^2 - 22e_2e_3 + 29e_3^2)\lambda + 159(e_2^3+e_3^3) - 51e_2e_3(e_2+e_3)\right] \end{aligned} \tag{5.21}$$

The associated Hermite polynomials are given by the formula

$$\begin{aligned} \mathcal{F}_5(x,\lambda) = {}& \lambda^2 - (3\wp(x) + \wp(x+\omega_2)) + \wp(x+\omega_3) + e_1)\lambda \\ & +9\wp(x)(\wp(x) + \wp(x+\omega_2) + \wp(x+\omega_3)) + 3\wp(x+\omega_2)\wp(x+\omega_3) + \\ & +3e_1(3\wp(x) + \wp(x+\omega_2)) + \wp(x+\omega_3)) - \frac{39}{2}g_2 + 54e_1^2. \end{aligned}$$

The solution of the system results from the substitution of these expressions into (5.8) as before, but this solution exhibits blow up (a pole at $x = 0$).

The levels of the integrals H and G have the following form

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_0 + \frac{161}{4} g_2 - 105 e_1^2, \\ \mathcal{G} &= \mathcal{G}_0 + \left(21 e_1^2 - \frac{161}{20} g_2\right)(a_1 + a_2) - \frac{405}{4} e_1 g_2 - \frac{63}{2} g_3 - 279 e_1^3,\end{aligned}$$

where $\mathcal{H}_0$ and $\mathcal{G}_0$ are given by (5.15) and the constants C_i are computed by the formula (5.14) in which ν represents the coordinate of the curve (5.21).

We remark that, following Airault *et al.* (1977), all elliptic potentials of the Schrödinger equations and their isospectral transformation under the action of the KdV flow have the form

$$\mathsf{U}(x) = 2 \sum_{i=1}^{N} \wp(x - x_i(t)), \tag{5.22}$$

The number N is a positive integer $N > 2$ (the number of "particles") and the numbers $\boldsymbol{x} = (x_1(t), \dots, x_N(t))$ belongs to the locus $\mathcal{L}_N$, i.e., the geometrical position of the points given by the equations

$$\mathcal{L}_N = \left\{ (\boldsymbol{x}); \sum_{i \neq j} \wp'(x_i(t) - x_j(t)) = 0,\ j = 1, \dots N \right\}. \tag{5.23}$$

If the evolution of the particles x_i over the locus is given by the equations

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = 6 \sum_{j \neq i} \wp(x_i(t) - x_j(t)),$$

then the potential (5.22) is an elliptic solution of the KdV equation. Hence the elliptic potentials which were discussed can serve as input for the isospectral deformation along the locus. Moreover these elliptic potential do not exhaust the whole variety of elliptic potentials. We can mention here the elliptic potentials of Smirnov (1989, 1994) for which the shifts x_i are not half-periods. Including these potentials in the study can enlarge the classes of elliptic solutions of the system (1.1)

6. Conclusions

In this paper we have described a family of elliptic solutions of the coupled nonlinear Schrödinger equations, using a Lax pair method and the general method of reduction of Abelian functions to elliptic functions. Our approach is systematic in the sense that special solutions (periodic, solitons, etc.) are obtained in a unified way. We also emphasise, that the solutions described in this paper can be extended up to the orbit of the symmetries group of the IMS enumerated by Alfinito *et al.* (1995).

Although we consider only the family of elliptic solutions associated with the integrable case 1 : 2 : 1 of quartic potential, the approach developed here can be applied to other integrable cases listed in the introduction, and will be published elsewhere.

In fiber optics applications, these periodic and quasi-periodic waves should be of interest in optical transmission systems.

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