Tutorial 3 Solutions 11.3YC1 **J C Eilbeck, November 12, 1999**

Question 1

The function $f(x)$ has a zero of multiplicity m if
$0 = f(x^*) = f'(x^*) = \ldots = f^{(m-1)}(x^*) \neq f^{(m)}(x^*)$.
In the first case, $f(0) = 0, f'(0) = -\sin 0 = 0, f''(0) = -\cos 0 = -1 \neq 0$, so $m = 2$.
2nd case: $f(0) = e^0 - 1 = 0$, etc.

Question 2

Modified Newton gives (results here from Matlab, details will depend on whatever calculator you use)

$$x_{k+1} = x_k - mf(x_k)/f'(x_k), \quad k = 0, 1, \ldots$$

$m = 2$ in 1st case $\ldots x_3 = -1.8322\ldots \times 10^{-13}$
$m = 3$ in 2nd case $\ldots x_3 = -8.4468\ldots \times 10^{-13}$

Question 3

Aitken formula:

$$\hat{x}_{k+1} = x_k - (\Delta x_k)^2/\Delta^2 x_k, \quad k = 0, 1, \ldots$$

k	x_k	Δx_k	$\Delta^2 x_k$	$\hat{x}_k$
0	0.5	0.377583	-0.616154	0.731385
1	0.877583	-0.238571	0.402244	0.736086
2	0.639012	0.163673	-0.271580	0.737653
3	0.802685	-0.107907		
4	0.694778			

etc. $\hat{x}_k$ is changing very little compared to x_k so convergence appears to be faster.

Question 4

In both cases the limit is $x^* = 0$.

(a)

k	x_k	Δx_k	$\Delta^2 x_k$	$\hat{x}_k$
1	1.	-0.5000000	0.3333333	0.2499999
2	0.5000000	-0.1666667	0.833334e-1	0.1666668
3	0.3333333	-0.833333e-1	0.333333e-1	0.1249999
4	0.2500000	-0.500000e-1	0.166667e-1	0.1000003
5	0.2000000	-0.333333e-1	0.95237e-2	0.833322e-1
6	0.1666667	-0.238096e-1	0.59525e-2	0.7142989e-1
7	0.1428571	-0.178571e-1	0.39682e-2	0.6249926e-1
8	0.1250000	-0.138889e-1	0.27778e-2	0.5555602e-1
9	0.1111111	-0.111111e-1	0.20202e-2	0.5000007e-1
10	0.1000000	-0.909091e-2	0.151513e-2	0.4545376e-1
11	0.9090909e-1	-0.757576e-2		
12	0.8333333e-1			

(b)

k	x_k	Δx_k	$\Delta^2 x_k$	$\hat{x}_k$
1	1.	-0.7500000	0.6111111	0.795454e-1
2	0.2500000	-0.1388889	0.902778e-1	0.363248e-1
3	0.1111111	-0.4861110e-1		
4	0.6250000e-1			

Question 5

Newton for $f(x) = e^{-x} + 3x = 0$ is

$$x_{k+1} = x_k - (e^{x_k} + 3x_k)/(e^{x_k} + 3), \quad k = 0, 1, \ldots$$

Secant is

$$x_{k+1} = x_k - (e^{x_k} + 3x_k)(x_k - x_{k-1})/(f(x_k) - f(x_{k-1})), \quad k = 0, 1, \ldots$$

k	*Newton* x_k	*Secant* x_k	*Newton* $\lvert x_k - x_{k-1}\rvert/\lvert x_k\rvert$	*Secant* $\lvert x_k - x_{k-1}\rvert/\lvert x_k\rvert$
0	0.5	0.5	–	–
1	-0.38011599	0.6	2.3	–
2	-0.589609	-0.369457	3.6e-1	1.6
3	-0.618401	-0.532739	4.7e-2	4.4e-1
4	-0.6190609	-0.606519	1.1e-3	1.4e-1
5	-0.6190612	-0.618271	5.7e-2	1.9e-2
6		-0.619053		1.3e-3
		etc.		etc.

The secant method is slower in this case.

Question 6

(a)

Newton using synthetic division gives

k	x_k	$\lvert x_k - x_{k-1}\rvert$
0	2.7	–
1	2.69069557	9.3e-3
2	2.69064744	4.8e-5 (stop)

1st root $\approx 2.69064744 = r_1$.
Output of synthetic division gives

$$\frac{x^3 - 2x^2 - 5}{x - r_1} = x^2 + 0.69064744x + 1.85828879$$

We can't solve this quadratic by Newton's method because it has complex roots, but the quadratic roots formula gives $-0.3453237 \pm i1.318727$.

(b)

$x_0 = -2.8, x_2 = -2.87938524 \approx$ root $1 = r_1$. Result of synthetic division at this root gives deflated polynomial $x^2 + 0.120614758x - 0.3472963 = 0$, with roots -0.6527036 and 0.5320888.

Question 7

Find a, b, c from $p(x_j) = f(x_j)$, $j = k-2, k-1, k$, i.e.

$$\begin{pmatrix} x_{k-2}^2 & x_{k-2} & 1 \\ x_{k-1}^2 & x_{k-1} & 1 \\ x_k^2 & x_k & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} f(x_{k-2}) \\ f(x_{k-1}) \\ f(x_k) \end{pmatrix}$$

This is possible by hand, but messy (easy in Matlab).

It is easier to look for α, β, γ in the equivalent polynomial $p(x) = \alpha(x - x_k)^2 + \beta(x - x_k) + \gamma$. So $p(x_k) = f(x_k) \Rightarrow \gamma = f(x_k)$, $p(x_k - 1) = f(x_k - 1) \Rightarrow \alpha(x_{k-1} - x_k)^2 + \beta(x_{k-1} - x_k) = f(x_{k_1}) - f(x_k)$, etc.

Now if we have a, b, c in $p(x) = ax^2 + bx + c$, we can find the roots of $p(x) = 0$, i.e.

$$x_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

then choose the root closest to x_k, i.e. the one with $|x_+ - x_k|, |x_- - x_k| =$ minimum. Sometimes this method may fail, for example if the roots turn out to be complex.