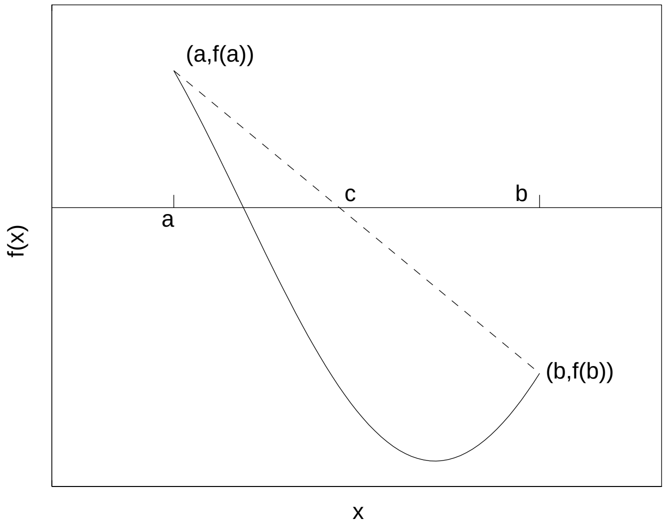
Tutorial 2 Solutions 11.3YC1 **J C Eilbeck, November 12, 1999**

Question 1



In the figure, $[a, b]$ is the given non-trivial bracket, and the straight line is the line joining $(a, f(a))$ to $(b, f(b))$. The point $(c, 0)$ is the point where this straight line crosses the x-axis. To find c, either find equation of line, or use a simple geometry argument based on similar triangles.

In the coordinate geometry approach, the equation of $y(x)$ is

$$y(x) = \frac{(f(b) - f(a)(x - a)}{(b - a)} + f(a)$$

so $y(c) = 0 \Rightarrow$

$$c = a - \frac{f(a)(b - a)}{f(b) - f(a)} = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

In the geometric derivation, ratios are $f(a)/f(b) = (c - a)/(c - b) \Rightarrow$

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

(a) In the Bisection algorithm, $a = x_0, b = x_1, c = x_2$, so replace the line $x_2 = \frac{1}{2}(x_1 + x_0)$ by $x_2 = (x_0 f(x_1) - x_1 f(x_0))/(f(x_1) - f(x_0))$.

(b)

x_0	x_2	x_1	$f(x_0)$	$f(x_2)$	$f(x_1)$
0	0.594198	1	$-$	$+$	$+$
0	0.530681	0.594198	$-$	$+$	$+$
0	0.524263	0.530681	$-$	$+$	$+$
0	0.523661	0.524263	$-$	$+$	$+$
0	0.523605	0.523661	$-$		$+$

This can be interpreted as either 4 or 5 steps – either is acceptable. Root estimate is final $x_2 = 0.523605$. Exact solution is $\sin^{-1} 0.5 = 0.523598\ldots$, so this estimate has absolute error of about 6×10^{-6}.

Note that in the Regula Falsi method the interval does not necessarily go to zero, so there are no rigorous error bounds on the result as in the Bisection method.

Question 2

(a) Newton: $x_{k+1} = x_k - f(x_k)/f'(x_k) = x_k - (x_k^3 - 2x_k^2 - 5)/(3x_k^2 - 4 * x_k)$.

k	x_k	
0	2.7	
1	2.69069557...	
2	2.690647449...	agrees to 5 s.f.
3	2.69064744802...	agrees to 9 s.f.

(b) $x_3 = 0.7390851332\ldots$ (agrees with x_2 to 9 sig. figs.)
(Both parts done in Matlab).

Question 3

Newton for $f(x) = 1/x - a = 0$: $x_{k+1} = x_k - f(x_k)/f'(x_k) = x_k - (1/x_k - a)/(-1/x_k^2) = 2x_k - ax_k^2$, $k = 0, 1, \ldots$. Now if $x_0 = 10/a$, then $x_1 = 2 \times 10/a - a(10/a)^2 = 20/a - 100/a = -80/a < 0$. It is shown in the notes that for those sequences generated by $x_{k+1} = 2x_k - ax_k^2, \quad (a > 0)$, $x_k < 0 \Rightarrow x_{k+1} < 0 \Rightarrow x_{k+2} < 0$, etc. Since we have just shown $x_1 < 0$, this means $x_k < 0$ for all $k \geq 1$ and hence convergence to the required positive result is not possible.

Question 4

(a)

$$\lim_{j\to\infty} \left| \frac{p_{j+1} - 0}{p_j - 0} \right| = \lim_{j\to\infty} \frac{j}{j+1} = 1, \quad \text{so sublinear.}$$

(b) Sublinear, (c) Sublinear.

(d)

$$\lim_{j\to\infty} \left| \frac{p_{j+1} - 0}{p_j - 0} \right| = \lim_{j\to\infty} \frac{e^{-(j+1)}}{e^{-j}} = 1/e < 1, \quad \text{so linear.}$$

(e)

$$\lim_{j\to\infty} \frac{e^{-(j+1)^2}}{e^{-j^2}} = 0, \quad \text{so superlinear.}$$

Question 5

(a) Let $e_k = L_0 2^{-k}$. Then

$$\lim_{k\to\infty} \left| \frac{x_{k+1}}{e_k} \right| = \lim_{k\to\infty} \frac{L_0 2^{-(k+1)}}{L_0 2^{-k}} = \frac{1}{2}.$$

Question 6

Put $x_{k+1} = \phi(x_k)$, $\phi(x) = x^3$. Test for *cubic* convergence:

$$\lim_{k\to\infty} \frac{|x_{j+1} - 0|}{|x_j - 0|^3} = \lim_{k\to\infty} \frac{|x_{k+1}|}{|x_k|^3} = \lim_{k\to\infty} \frac{|x_k^3|}{|x_k|^3} = 1 \neq 0.$$

Question 6

Assume $f(x)$ has root of multiplicity m at x^*. Write this as $f(x) = (x - x^*)^m g(x)$ where $g(x^*) \neq 0$.

Define

$$\begin{aligned}\mu(x) &= f(x)/f'(x) = \frac{(x-x^*)^m g(x)}{m(x-x^*)^{m-1}g(x) + (x-x^*)^m g'(x)} \\ &= \frac{(x-x^*)g(x)}{mg(x) + (x-x^*)g'(x)}.\end{aligned}$$

Now

$$\mu(x^*) = \frac{0 \times g(x^*)}{g(x^*) + 0} = 0$$

so x^* is a root of $\mu(x) = 0$. Newton's method applied to $\mu(x) = 0$ gives $x_{k+1} = x_k - \mu(x_k)/\mu'(x_k) \equiv \phi(x_k)$. We need to show that $\phi(x^*) = x^*$, $\phi'(x^*) = 0$ for quadratic convergence. Now

$$\phi(x^*) = x^* - \mu(x^*)/\mu'(x^*) = x^* - 0/\mu'(x^*) = x^*$$

since

$$\begin{aligned}\mu'(x) &= \frac{d}{dx}\left[\frac{(x-x^*)^m g(x)}{m(x-x^*)^{m-1}g(x) + (x-x^*)^m g'(x)}\right] \\ &= \frac{g(x) + (x-x^*)g'(x)}{mg(x) + (x-x^*)g'(x)} - \frac{(x-x^*)g(x)(mg'(x) + g'(x) + (x-x^*)g''(x))}{(\qquad)^2} \\ &\Rightarrow \mu'(x^*) = \frac{1}{m}\end{aligned}$$

Finally

$$\phi'(x) = 1 - \frac{\mu'}{\mu} + \frac{\mu''\mu}{(\mu')^2} = \frac{\mu''(x)\mu(x)}{(\mu'(x))^2}$$

so

$$\phi'(x^*) = \frac{\mu''(x^*) \times 0}{(\mu'(x^*))^2} = 0 \quad (\mu'(x^*) \neq 0).$$

Using the theorem given in the lecture notes on the convergence of fixed point iteration gives the required result.

(Note, we should show also that $\mu''(x^*) \neq 0$ to demonstrate that the scheme is quadratically convergent but not cubically convergent. This is quite complicated. Interpret the question to mean "converges *at least* quadratically".)