Tutorial 1 Solutions 11.3YC1 **J C Eilbeck, October 15, 1999**

Question 1

x_0	x_2	x_1	$f(x_0)$	$f(x_2)$	$f(x_1)$
0	$\swarrow$0.5	1	$-$	$\swarrow -$	$+$
0.5	0.75$\searrow$	1	$-$	$+\searrow$	$+$
0.5	0.625$\searrow$	0.75	$-$	$+\searrow$	$+$
0.5	0.5625$\searrow$	0.625	$-$	$+\searrow$	$+$
0.5	0.53125	0.5625	$-$	$+$	$+$

This can be interpreted as either 4 or 5 steps – either is acceptable. Root estimate is midpoint at last step: 0.53125. Error estimate is half interval size: $0.5(0.5625 - 0.5) = 1/32$.

x_0	x_2	x_1
2	2.5	3
...	...	...
2.5625	2.59375	2.625

Again, this can be interpreted as either 4 or 5 steps. Root estimate is midpoint at last step: 2.59375. Error estimate is half interval size: 1/32.

Error estimate is upper limit on absolute error.

Question 2

Initial bracket length $L_0 = |\pi - 0| = \pi$. After k steps: $L_k = \pi/2^k$. We want $L_k \le 10^{-3}$. Take logs, $\log_{10}\pi - k\log_{10}2 \le -3$ so $k \ge (\log_{10}\pi + 3)/\log_{10}2 = 11.61\ldots$. k is integer so use $k = \mathrm{ceil}(11.61) = 12$ steps. (ceil() means round up to nearest integer).

Question 3

$L_k = |a-b|/2^k \le 10^{-n}$.
Take logs: $\log_{10}|a-b| - k\log_{10}2 \le -n$ so

$$k \ge \frac{(\log_{10}|a-b| + n)}{\log_{10}2}$$

Use $k = \mathrm{ceil}\left((\log_{10}|a-b| + n)/\log_{10}2\right)$, which increases with n since $\log_{10}2 > 0$.

Question 4

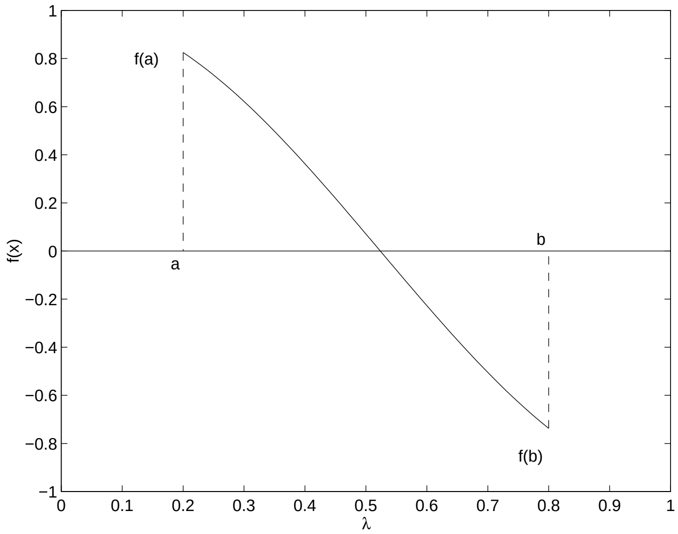
	(i)	(ii)	(iii)
(a)	4×10^{-2}	1×10^{-2}	≈ 2
(b)	3×10^{-4}	1×10^{-3}	≈ 3
(c)	4×10^{-5}	1×10^{-2}	≈ 2
(d)	3×10^{-2}	1×10^{-3}	≈ 3

Question 5

(a) $|x^* - \pi|/|\pi| \le 10^{-3}$ so $|x^* - \pi| \le \pi \times 10^{-3}$, i.e. $\pi(1 - 10^{-3}) \le x^* \le \pi(1 + 10^{-3})$.
(b), (c), (d) are similar.

Question 6

(a)



$f' < 0 \Rightarrow$ function decreases monotonically
$f(a) > 0 > f(b) \Rightarrow$ only one root between a and b.

(b) $f(a)f(b) > 0 \Rightarrow f(a), f(b) > 0$ both +ve, or $f(a), f(b) < 0$ both −ve. $f' > 0 \Rightarrow$ increases monotonically from $f(a)$ to $f(b)$. Draw both cases – no roots.

(c) $f(a)f(b) < 0 \Rightarrow f$ changes sign. No other information, so there is at least one root. Draw it.

Question 7

Show that $f(1) < 0 < f(2)$ and that $f'(x) > 0$ for all $1 \le x \le 2$. Then use the result of 6(a). Apply the bisection method starting with $[1, 2]$ and stopping when $\frac{1}{2}|a - b| \le 5 \times 10^{-3}$. The absolute error is $\le \frac{1}{2}|a - b|$ so this satisfoes the condition abs. error $\le 5 \times 10^{-3}$.

Question 8

(a) Apply bisection method to $f(x) = x^3 - 3$ with initial bracket $[1, 2]$, say. Stop when current interval $[a, b]$ satisfies $\frac{1}{2}|a - b| / \min(|a|, |b|) \le 5 \times 10^{-3}$.

(b) Apply bisection method to $f(x) = x^n - a$ (solution is $x = a^{\frac{1}{n}}$).

Question 9

The equation doesn't affect the result. Use the formula as in Questions 2 and 3. Solutions: (a) 14 (b) 15 (c) $\mathrm{ceil}(N/\log_{10} 2)$.

Question 10

(a) Depends on whether $[a, b]$ is a nontrivial bracket or not. If it is, method works. If not, the method fails at the start. In this case, if it is known that there are two roots inside $[a, b]$, it would be possible to evaluate the function at various points inside this interval, until one or more nontrivial brackets were found, then start the bisection method.

(b) $[a, b]$ is not a nontrivial bracket since $f(a)f(b) > 0$. Method fails.