## 11.3YC1 - Closed Newton-Coates Formulae

The closed Newton-Coates formulae for approximating an integral over a single interval use equally spaced points along the interval. These formulae are called "closed" because the end points $a, b$ are used in the calculation. These approximations can be derived using Lagrange interpolating functions which are covered later in the 2nd/3rd term course.

Some of the standard closed Newton-Coates formulae and their errors are given below:

**Trapezoidal Rule** (2 $f$'s)

$$\int_a^b f(x)dx = \frac{h}{2}(f(a) + f(b)) - \frac{h^3}{12}f''(\zeta), \quad h = b - a$$

**Simpson's Rule** (3 $f$'s)

$$\int_a^b f(x)dx = \frac{h}{3}(f(a) + 4f(a+h) + f(b)) - \frac{h^5}{90}f^{(4)}(\zeta), \quad h = (b-a)/2$$

**Simpson's 3/8 Rule** (4 $f$'s)

$$\int_a^b f(x)dx = \frac{3h}{8}(f(a) + 3f(a+h) + 3f(a+2h) + f(b)) - \frac{3h^5}{80}f^{(4)}(\zeta), \quad h = (b-a)/3$$

where in each case $\zeta$ lies between $a$ and $b$ and we assume that the function $f$ has as many derivatives as we require.

## Open Newton-Coates Formulae

These formulae similar to those above and are called "open" because the end points $a, b$ are not used. Some of the standard open Newton-Coates formulae and their errors are given below:

**Midpoint Rule** (1 $f$)

$$\int_a^b f(x)dx = 2hf(a+h) + \frac{h^3}{3}f''(\zeta), \quad h = (b-a)/2$$

**Anonymous** (2 $f$'s)

$$\int_a^b f(x)dx = \frac{3h}{2}(f(a+h) + f(a+2h)) + \frac{3h^3}{4}f''(\zeta), \quad h = (b-a)/3$$

**Anonymous** (3 $f$'s)

$$\int_a^b f(x)dx = \frac{4h}{3}(2f(a+h) - f(a+2h) + 2f(a+3h)) + \frac{14h^5}{45}f^{(4)}(\zeta), \quad h = (b-a)/4$$

where in each case $\zeta$ lies between $a$ and $b$, and we assume that the function $f$ has as many derivatives as we require.

### Examples

Open and Closed Newton-Coates formulae results for

$$\int_1^2 (1+x)^{-1} dx = \ln 3 - \ln 2 = 0.405465\ldots$$

| Number of $f$'s | | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| closed | approx. | | 0.41667 | 0.40556 | 0.40551 | 0.40547 |
| | abs. error | | 0.01120 | 0.00009 | 0.00004 | 0.00000 |
| open | approx. | 0.40000 | 0.40179 | 0.40539 | 0.40541 | |
| | abs. error | -0.00547 | -0.00368 | -0.00008 | -0.00005 | |

$$\int_1^3 (1+x)^{-1} dx = \ln 4 - \ln 2 = 0.693147\ldots$$

| Number of $f$'s | | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| closed | approx. | | 0.75000 | 0.69444 | 0.69375 | 0.69317 |
| | abs. error | | 0.05685 | 0.00130 | 0.00060 | 0.00003 |
| open | approx. | 0.66667 | 0.67500 | 0.69206 | 0.69238 | |
| | abs. error | -0.02648 | -0.01815 | -0.00108 | -0.00077 | |

$$\int_1^{10} (1+x)^{-1} dx = \ln 11 - \ln 2 = 1.704748\ldots$$

| Number of $f$'s | | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| closed | approx. | | 2.65909 | 1.80944 | 1.76165 | 1.71691 |
| | abs. error | | 0.95434 | 0.10469 | 0.05690 | 0.01216 |
| open | approx. | 1.38462 | 1.46250 | 1.63594 | 1.65154 | |
| | abs. error | -0.32013 | -0.24225 | -0.06881 | -0.05321 | |

## Error Analysis for Trapezoidal Rule (single interval)

The error $E$ in the Trapezoidal Rule approximation is found from

$$I = \int_a^b f(x)\,dx = \frac{h}{2}(f(a) + f(b)) + E$$

where $h = b - a$. It makes things easier if we introduce the midpoint $c = (a + b)/2$ and rewrite the above as

$$E = \int_{c-h/2}^{c+h/2} f(x)\,dx - \frac{h}{3}\left(f(c - h/2) + f(c + h/2)\right) \ .$$

First look at the integral $I$. Taylor expand $f(x)$ about $x = c$ to get

$$I = \int_{c-h/2}^{c+h/2} \left( f(c) + (x - c)f'(c) + \frac{(x - c)^2}{2} f''(c) + + \cdots \right) dx \ .$$

Using the substitution $s = x - c$ this simplifies to:

$$I = \int_{-h/2}^{h/2} \left( f(c) + sf'(c) + \frac{s^2}{2} f''(c) + \cdots \right) ds$$

and evaluates to

$$I = \left[ sf(c) + \frac{s^2}{2} f'(c) + \frac{s^3}{3!} f''(c) + \cdots \right]_{s=-h/2}^{s=h/2} \ .$$

The even powers of $s$ above cancel out at $s = \pm h/2$ leaving

$$I = \left[ sf(c) + \frac{s^3}{3!} f''(c) + \cdots \right]_{s=-h/2}^{s=h/2}$$

which evaluates to

$$I = hf(c) + \frac{h^3}{24} f''(c) + \cdots$$

The other part comes out similarly as

$$\begin{aligned} \frac{h}{2}(f(c - h/2) + f(c + h/2)) &= \frac{h}{2}\left( 2f(c) + \frac{2}{2!}\frac{h^2}{2^2} f''(c) + \cdots \right) \\ &= hf(c) + \frac{h^3}{8} f''(c) + \cdots \end{aligned}$$

Combining these Taylor expansions we get as an approximation, neglecting terms of higher order than $h^3$

$$E \approx \left( hf(c) + \frac{h^3}{24} f''(c) \right) - \left( hf(c) + \frac{h^3}{8} f''(c) \right) = -\frac{h^3}{12} f''(c)$$

a more rigorous calculation taking all orders into account gives the exact result

$$E_{TR} = -\frac{h^3}{12} f''(\zeta)$$

where $\zeta$ lies between $a$ and $b$.

## Error Analysis Results for some Composite Rules

In all cases $\zeta$ is a number lying between $a$ and $b$, but the $\zeta$ is different for each rule.

## Composite Trapezoidal Rule

Combining the results for the single Trapezoidal Rule after some algebra and use of the intermediate value theorem gives

$$E_{CTR} = -(b-a)\frac{h^2}{12}f''(\zeta)$$

## Composite Midpoint Rule

The result in this case is

$$E_{CMR} = (b-a)\frac{h^2}{6}f''(\zeta)$$

## Composite Simpson's Rule

The result in this case is

$$E_{CSR} = -(b-a)\frac{h^4}{180}f^{(4)}(\zeta)$$

## Error Analysis for Simpson's Rule (single interval)

The error $E$ in the Simpson's Rule approximation is found from

$$I = \int_a^b f(x)\,dx = \frac{h}{3}(f(a) + 4f(\frac{a+b}{2}) + f(b)) + E$$

where $h = (b-a)/2$. It makes things easier if we introduce the midpoint $c = (a+b)/2$ and rewrite the above as

$$E = \int_{c-h}^{c+h} f(x)\,dx - \frac{h}{3}\left(f(c-h) + 4f(c) + f(c+h)\right) \ .$$

First look at the integral $I$. Taylor expand $f(x)$ about $x = c$ to get

$$I = \int_{c-h}^{c+h} \left(f(c) + (x-c)f'(c) + \frac{(x-c)^2}{2}f''(c) + \frac{(x-c)^3}{3!}f^{(3)}(c) + \cdots\right)dx \ .$$

Using the substitution $s = x - c$ this simplifies to:

$$I = \int_{-h}^{h} \left(f(c) + sf'(c) + \frac{s^2}{2}f''(c) + \frac{s^3}{3!}f^{(3)}(c) + \frac{s^4}{4!}f^{(4)}(c) + \cdots\right)ds$$

and evaluates to

$$I = \left[sf(c) + \frac{s^2}{2}f'(c) + \frac{s^3}{3!}f''(c) + \frac{s^4}{4!}f^{(3)}(c) + \frac{s^5}{5!}f^{(4)}(c) + \cdots\right]_{s=-h}^{s=h} \ .$$

The even powers of $s$ above cancel out at $s = \pm h$ leaving

$$I = \left[ sf(c) + \frac{s^3}{3!} f''(c) + \frac{s^5}{5!} f^{(4)}(c) + \cdots \right]_{s=-h}^{s=h}$$

which evaluates to

$$I = 2hf(c) + \frac{h^3}{3} f''(c) + \frac{h^5}{60} f^{(4)}(c) + \cdots$$

The other part comes out similarly as

$$\begin{aligned} \frac{h}{6}(f(c-h)+ \quad 4f(c) \quad &+f(c+h)) \\ &= \quad \frac{h}{3}\left(6f(c) + \frac{2h^2}{2!} f''(c) + \frac{2h^4}{4!} f^{(4)}(c) + \cdots\right) \\ &= \quad 2hf(c) + \frac{h^3}{3} f''(c) + \frac{h^5}{36} f^{(4)}(c) + \cdots \end{aligned}$$

Finally, we terminate the Taylor expansions at terms involving $h^5$ to get

$$E = \left(2hf(c) + \frac{h^3}{3} f''(c) + \frac{h^5}{60} f^{(4)}(c)\right) - \left(2hf(c) + \frac{h^3}{3} f''(c) + \frac{h^5}{36} f^{(4)}(c)\right)$$

This comes out to

$$E = -\frac{h^5}{90} f^{(4)}(c) + O(h^7)$$

This comes out toIt can be shown that a more detailed treatment gives

$$E = -\frac{h^5}{90} f^{(4)}(\zeta)$$

where $\zeta$ lies between $a$ and $b$.

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