A Fixed Point Iteration Theorem

If $\varphi(x)$ and its derivatives are continuous, and $|\varphi'(x_*)| < 1$, $\varphi(x_*) = x_*$, then there is an interval $I_\delta = [x_* - \delta, x_* + \delta]$, $\quad \delta > 0$, such that the iteration $x_{k+1} = \varphi(x_k)$ converges to x_* for every $x_0 \in I_\delta$. Further, if $\varphi'(x_*) \neq 0$, then the convergence is linear with ratio $|\varphi'(x_*)|$. Alternatively, if $0 = \varphi'(x_*) = \varphi''(x_*) = \ldots = \varphi^{(p-1)}(x_*)$, and $\varphi^{(p)}(x_*) \neq 0$, then the convergence is of order p.

Proof

(Similar to the proof for convergence of the Newton-Raphson method).

(a)

Let $e_k = x_k - x_*$ for $k = 0, 1, \ldots$ ($e_k \equiv$ "error").
So

$$\begin{aligned}
e_{k+1} &= x_{k+1} - x_* = \varphi(x_k) - \varphi(x_*) \text{ by definition of } x_* \text{ and } \{x_k\}. \\
&= \varphi(x_*) + (x_k - x_*)\varphi'(\xi_k) - \varphi(x_*) \text{ by Taylor expansion for some } \xi_k \\
&\quad \text{ between } x_k \text{ and } x_*. \\
&= e_k \varphi'(\xi_k)
\end{aligned}$$

(b)

By hypothesis $|\varphi'(x_*)| < 1$ and $\varphi'(x)$ is continuous, so there is an interval $I_\delta = [x_* - \delta, x_* + \delta]$, (i.e. close to x_*) where $x \in I_\delta \Rightarrow |\varphi'(x)| \leq C < 1$, for some constant C such that $|\varphi'(x_*)| < C < 1$. (See Figure).

(c)

Note that $x \in I_\delta$ is the same as saying $|e_k| \leq \delta$ since $|e_k| = |x_k - x_*|$.
(a) and (b) give us

$$|e_{k+1}| = |e_k||\varphi'(\xi_k)| \leq \delta \varphi'(\xi_k) < \delta$$

since ξ_k lies between x_k and x_*, i.e. $\xi_k \in I_\delta$.
So if $x_k \in I_\delta$ then $|e_k| \leq \delta$ and $|e_{k+1}| \leq \delta$, i.e. $|x_k - x_*| < \delta$, so $x_{k+1} \in I_\delta$.
So finally $x_k \in I_\delta \Rightarrow x_{k+1} \in I_\delta$, and if $x_0 \in I_\delta$, then so is $x_1 \in I_\delta, x_2 \in I_\delta, \ldots$.
Therefore all $x_k \in I_\delta$ (and so are all the ξ_k).

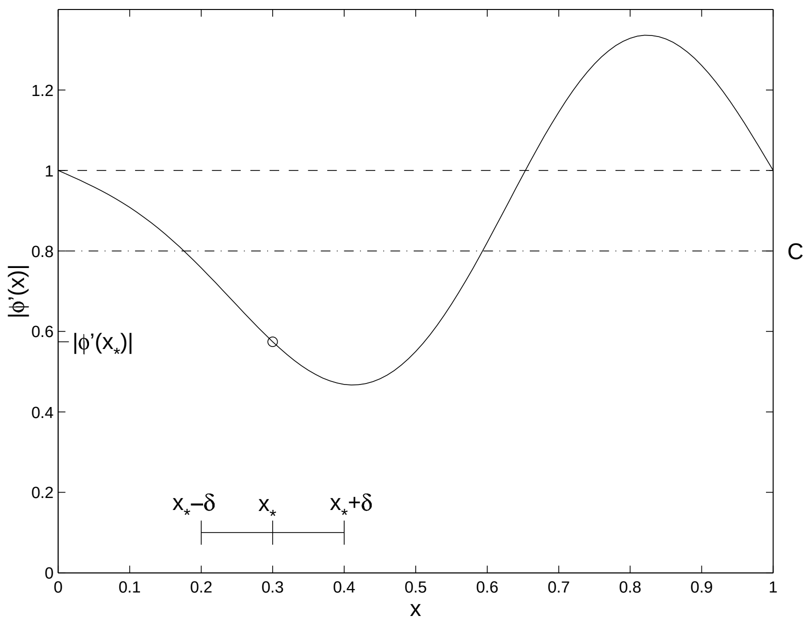

Figure 1: Illustration of I_δ

(d)

$$\begin{aligned} |e_k| &= |e_{k-1}||\varphi'(\xi_{k-1})| \text{ from (a)} \\ &\leq C|e_{k-1}| \text{ from (b)} \\ &\leq C|e_{k-2}||\varphi'(\xi_{k-2})| \\ &\leq C^2|e_{k-2}| \leq C^3|e_{k-3}| \leq \dots C^k|e_0| \end{aligned}$$

Recall that $0 \leq C < 1$. So

$$\lim_{k\to\infty} |e_k| = |e_0| \lim_{k\to\infty} C^k = |e_0| \times 0 = 0$$

hence $e_k \longrightarrow 0$ as $k \longrightarrow \infty$ ($e_k = x_k - x_*$) and equivalently $x_k \longrightarrow x_*$ as $k \longrightarrow \infty$. So the sequence $\{x_k\}$ converges to x_* under the assumptions of the theorem.

(e)

Check linear convergence.

$$\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = \lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|}$$

$$
\begin{aligned}
&= \lim_{k\to\infty} |\varphi'(\xi_k)| \qquad \text{from (a)} \\
&= |\varphi'(x_*)|
\end{aligned}
$$

since $\xi_k \to x_*$ as $x_k \to x_*$ and ξ_k is trapped between x_k and x_*. Also $|\varphi'(x_*)| < 1$ by assumptions of theorem, so convergence is *linear* if $|\varphi'(x_*)| \neq 0$ and *superlinear* if $|\varphi'(x_*)| = 0$.

(f)

(The case $0 = \varphi'(x_*) = \varphi''(x_*) = \ldots = \varphi^{(p-1)}(x_*)$)

Taylor expand again (similar to (a))

$$
\begin{aligned}
e_{k+1} &= x_{k+1} - x_* = \varphi(x_k) - \varphi(x_*) \\
&= \varphi(x_*) + (x_k - x_*)\varphi'(x_*) + \frac{1}{2!}(x_k - x_*)^2\varphi''(x_*) \\
&\quad + \ldots + \frac{1}{(p-1)!}(x_k - x_*)^{(p-1)}\varphi^{(p-1)}(x_*) + \frac{1}{p!}(x_k - x_*)^p\varphi^{(p)}(\eta_k) \\
&\quad - \varphi(x_*)
\end{aligned}
$$

for some η_k between x_k and x_*. Since $0 = \varphi'(x_*) = \varphi''(x_*) = \ldots = \varphi^{(p-1)}(x_*)$, we have that

$$
e_{k+1} = \frac{1}{p!}(x_k - x_*)^p\,\varphi^{(p)}(\eta_k) = \frac{1}{p!}e_k^p\,\varphi^{(p)}(\eta_k)
$$

Now test for pth order convergence.

$$
\begin{aligned}
\lim_{k\to\infty} \frac{|e_{k+1}|}{|e_k|^p} &= \lim_{k\to\infty} \left| \frac{1}{p!}\varphi^{(p)}(\eta_k) \right| \\
&= \frac{1}{p!}|\varphi^{(p)}(x_*)| \neq 0
\end{aligned}
$$

by hypothesis and since $\eta_k \to x_*$.

Hence convergence is pth order as required.