Solutions to 11.3YC1 (Introductory Numerical Analysis)
December Exam 1999

Question 1.
[20 Marks Total]

(a) **[6 Marks]** The bisection method gives

k	x_0	x_1	$f(x_0)$	$f(x_1)$	x_2	$f(x_2)$
1.1	1.2	-0.095833976	0.120116923	1.15	0.008192910	
1.1	1.15	-0.095833976	0.008192910	1.125	-0.044783151	
1.125	1.15	-0.044783151	0.008192910	1.1375	-0.018538793	

final interval is $[1.1375, 1.15]$, estimate for root is $0.5(1.1375 + 1.15) = 1.14375$, maximum error is $0.5(1.15 - 1.1375) = 0.00625$.

After 10 steps the interval will be of length $0.1/2^{10} = 0.00009765625$

(b) **[7 Marks]** Integrating each strip using SR

$$\int_{a+kh}^{a+(k+2)h} f(x)\,\mathrm{d}x \approx \frac{h}{3}\left(f(a+kh) + 4f(a+(k+1)h) + f(a+(k+2)h)\right), \quad k = 0, 2, 4, \ldots, (b-a)/h - 2$$

gives

$$\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &\approx \frac{h}{3}\left(\left(f(a) + 4f(a+h) + f(a+2h)\right) + \right. \\
&\quad + \left(f(a+2h) + 4f(a+3h) + f(a+4h)\right) + \ldots + \\
&\quad \left. + \left(f(b-2h) + 4f(b-h) + f(b)\right)\right) \\
&= \frac{h}{3}\left(f(a) + 4f(a+h) + 2f(a+2h) + 4f(a+3h) + 2f(a+4h) + \ldots + \right. \\
&\quad \left. + (2f(b-2h) + 4f(b-h) + f(b)\right)
\end{aligned}$$

With $f(x) = \exp(x)$, $a = 0, b = 1, h = 1/4$, we have

$$\begin{aligned}
I_{CSR} &= h/3(f(0) + 4f(0.25) + 2f(0.5) + 4f(0.75) + f(1)) = 1.718318843 \\
I_{\text{exact}} &= \exp(1) - 1 = 1.718281828 \\
\text{error} &= I_{\text{exact}} - CSR = -0.000037015
\end{aligned}$$

and solving

$$-\frac{(b-a)h^4}{180} f^{(iv)}(\eta) = -0.000037015$$

for η gives

$$\eta = \ln(-180 \times \text{error}/h^4) = 0.533947$$

(c) **[7 Marks]** Either

$$\begin{aligned}
\sqrt{1+x} - \sqrt{1-x} &= \frac{(\sqrt{1+x} - \sqrt{1-x})(\sqrt{1+x} + \sqrt{1-x})}{(\sqrt{1+x} + \sqrt{1-x})} \\
&= \frac{1 + x - (1-x)}{\sqrt{1+x} + \sqrt{1-x}} \\
&= \frac{2x}{\sqrt{1+x} + \sqrt{1-x}}
\end{aligned}$$

which is well behaved at $x \approx 0$, or

$$\begin{aligned}\sqrt{1+x} - \sqrt{1-x} &= \left(1 + \frac{1}{2}x + \frac{1}{2}(-\frac{1}{2})\frac{x^2}{2!} + \ldots\right) \\ &\quad - \left(1 - \frac{1}{2}x + \frac{1}{2}(-\frac{1}{2})\frac{(-x)^2}{2!} + \ldots\right) \\ &= x + \frac{1}{8}x^3 + O(x^5)\end{aligned}$$

Question 2.
[20 Marks Total]

(a) [10 Marks]

Note that $x_k \in I_\delta$ is the same as saying $|e_k| \leq \delta$ since $|e_k| = |x_k - x_*|$. Assume first that $x_k \in I_\delta$, it follows that

$$|e_{k+1}| = |e_k||\varphi'(\xi_k)| \leq \delta\,|\varphi'(\xi_k)| < \delta$$

since ξ_k lies between x_k and x_*, i. e. $\xi_k \in I_\delta$.
So if $x_k \in I_\delta$ then $|e_k| \leq \delta$ and $|e_{k+1}| \leq \delta$, i. e. $|x_k - x_*| < \delta$, so $x_{k+1} \in I_\delta$.

Hence $x_k \in I_\delta \Rightarrow x_{k+1} \in I_\delta$, and if $x_0 \in I_\delta$, then $x_1 \in I_\delta, x_2 \in I_\delta, \ldots$, as required (and also all the $\xi_k \in I$).

Further we have

$$\begin{aligned}|e_k| &= |e_{k-1}||\varphi'(\xi_{k-1})| \\ &\leq C|e_{k-1}| \\ &\leq C|e_{k-2}||\varphi'(\xi_{k-2})| \\ &\leq C^2|e_{k-2}| \leq C^3|e_{k-3}| \leq \ldots C^k|e_0|\end{aligned}$$

Recall that $0 \leq C < 1$. So

$$\lim_{k\to\infty}|e_k| = |e_0|\lim_{k\to\infty}C^k = |e_0| \times 0 = 0$$

hence $e_k \to 0$ as $k \to \infty$ ($e_k = x_k - x_*$) and equivalently $x_k \to x_*$ as $k \to \infty$. So the sequence $\{x_k\}$ converges to x_* under the given assumptions.

(b) [10 Marks] The FP iteration gives

k	x_k
0	1.6
1	1.625
2	1.615385
3	1.619048
4	1.617647
5	1.618182
6	1.617978

To find the fixed point(s), solve $x = 1 + 1/x \to x^2 - x - 1 = 0 \to x = \frac{1}{2}(1 \pm \sqrt{5})$. Clearly in this case the iteration is converging to the positive root, i.e. $x_* = \frac{1}{2}(1+\sqrt{5}) = 1.618034$. Calculating $|x_k - x_*|$ and $|x_{k+1} - x_*|/|x_k - x_*|$ we get

k	$\lvert x_k - x_*\rvert$	$\lvert x_k - x_*\rvert / \lvert x_{k-1} - x_*\rvert$
0	0.018034	
1	0.006966	0.3863
2	0.002649	0.3803
3	0.001014	0.38278
4	0.000387	0.38167
5	0.000148	0.3824
6	0.0000565	0.3818

$|\varphi'(x_*)| = -1/x_*^2 = 0.381966\ldots$, which agrees with these calculations, since we expect from the theory that

$$\lim_{k\to\infty} \frac{|x_{k+1} - x_*|}{|x_k - x_*|} = \varphi'(x_*).$$

Question 3.
[20 Marks Total]

(a) [**8 Marks**] Newton's method is

$$x_{k+1} = x_k - f(x_k)/f'(x_k), \quad k = 0, 1, \ldots$$

Putting $f(x) = a - 1/x$, we get

$$\begin{aligned} x_{k+1} &= x_k - \frac{a - 1/x_k}{1/x_k^2} \\ &= x_k - (ax_k^2 - x_k) = x_k(2 - ax_k) \end{aligned}$$

So

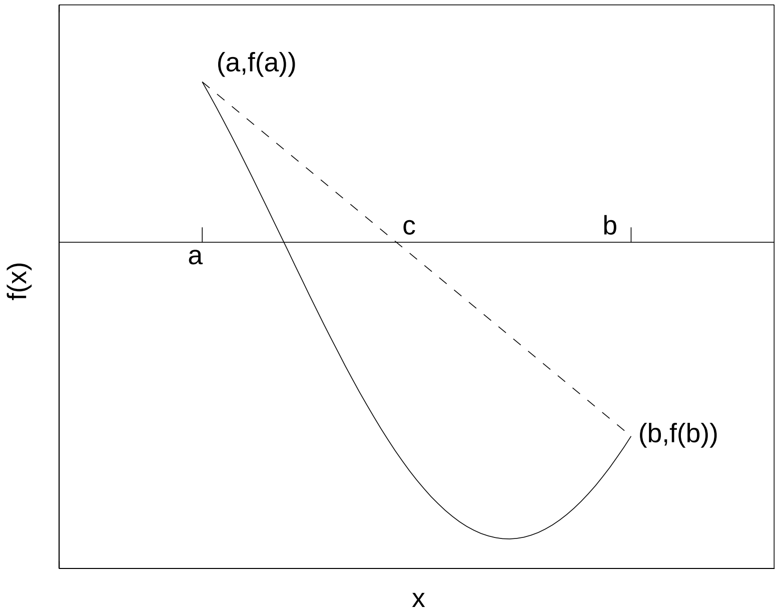
$$\begin{aligned} (x_{k+1} - \frac{1}{a}) &= x_k(2 - ax_k) - \frac{1}{a} = -a(x_k^2 - \frac{2x_k}{a} + \frac{1}{a^2}) \\ &= -a(x_k - \frac{1}{a})^2 \end{aligned}$$

As required.

Since

$$\frac{|x_{k+1} - \frac{1}{a}|}{|x_k - \frac{1}{a}|^2} = |a|$$

then by definition, if x_k converges to $1/a$, is is superlinear convergence of order 2.



(b) [12 Marks]

In the figure, $x_k = a, x_{k+1} = b$ are the two latest values, and the straight line is the line joining $(x_k, f(x_k))$ to $(x_{k+1}, f(x_{k+1}))$. The point $(c, 0) = (x_{k+2}, 0)$ is the point where this straight line crosses the x-axis. To find x_{k+2}, either find equation of line, or use a simple geometry argument based on similar triangles.

In the geometric derivation, ratios are $f(x_k)/f(x_{k+1}) = (x_{k+2} - x_k)/(x_{k+2} - x_{k+1})$, and solving for x_{k+2} we get

$$x_{k+2} = \frac{x_k f(x_{k+1}) - x_{k+1} f(x_k)}{f(x_{k+1}) - f(x_k)} \ , \ k = 0, 1, \ldots$$

If we have $p_4(x)$ in the given form, clearly $p_4(r) = z$. Writing out both sides in full gives

$$\begin{aligned} 1 + x + 2x^2 - 3x^4 &= (x - r)(b_1 + b_2 x + b_3 x^2 + b_4 x^3) + z \\ &= b_4 x^4 + (b_3 - r b_4) x^3 + (b_2 - r b_3) x^2 + (b_1 - r b_2) x + (z - r b_1) \end{aligned}$$

So $b_4 = -3$, $b_3 - r b_4 = 0 \to b_3 = r b_4$, $b_2 - r b_3 = 2 \to b_2 = r b_3 + 2$, $b_1 - r b_2 = 1 \to b_1 = r b_2 + 1$, $z - r b_1 = 1 \to z = r b_1 + 1$.

When $r = x_0 = 1$, $b_4, b_3, b_2, b_1, z = -3, -3, -1, 0, 1$, so $f(x_0) = 1$.
When $r = x_1 = 2$, $b_4, b_3, b_2, b_1, z = -3, -6, -10, -19, -37$, so $f(x_1) = -37$.
Inserting these numbers into the Secant formula gives

$$x_2 = 39/38 = 1.026316\ldots$$

Question 4.
[20 Marks Total]

(a) [8 Marks]

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x &= \int_a^b [f(c) + (x - c) f'(c) + (x - c)^2 f''(c)/2 + \\ &+ \quad (x - c)^3 f'''(c)/3! + O(x - c)^4]\,\mathrm{d}x \\ &= \Big[x f(c) + (x - c)^2 f'(c)/2 + (x - c)^3 f''(c)/3! + \end{aligned}$$

$$
\begin{aligned}
&+ \quad (x-c)^4 f'''(c)/4! + O(x-c)^5\Big]_a^b \\
&= \quad hf(c) + \frac{1}{24}h^3 f''(c) + O(h^5),
\end{aligned}
$$

So $A = 1/24$. Similarly

$$
\begin{aligned}
\frac{h}{2}[f(a)+f(b)] &= \frac{h}{2}\Bigg[f(c) - \frac{h}{2}f'(c) + \left(\frac{h}{2}\right)^2 f''(c) - \left(\frac{h}{2}\right)^3 f'''(c) + \left(\frac{h}{2}\right)^4 f^{(iv)}(c) + O(h)^5 + \\
&\quad f(c) + \frac{h}{2}f'(c) + \left(\frac{h}{2}\right)^2 f''(c) + \left(\frac{h}{2}\right)^3 f'''(c) + \left(\frac{h}{2}\right)^4 f^{(iv)}(c) + O(h)^5\Bigg] \\
&= hf(c) + \frac{1}{8}h^3 f''(c) + O(h^5)
\end{aligned}
$$

So $B = 1/8$. Hence the error in the TR is $(\frac{1}{24} - \frac{1}{8})h^3 f''(c) = -\frac{1}{12}h^3 f''(c)$.

(b) [6 Marks]

$$
\begin{aligned}
\int_a^b f(x)\,\mathrm{d}x &\approx \frac{h}{2}[f(a)+f(a+h)] + \frac{h}{2}[f(a+h)+f(a+2h)] + \ldots + \\
&\quad + \frac{h}{2}[f(b-h)+f(b)] \\
&= h\left[\frac{1}{2}f(a) + f(a+h) + f(a+2h) + \ldots + f(b-h) + \frac{1}{2}f(b)\right]
\end{aligned}
$$

When $f(x) = 4, a = 0, b = 3, n = 200$, we have

$$
\int_a^b f(x)\,\mathrm{d}x \approx \frac{3}{200}4(\frac{1}{2} + 1 + 1 + \ldots + 1 + \frac{1}{2}) = 12
$$

This is exact since in the error formula for each strip, $f'' = 0$.

(c) [6 Marks]

CTR gives $\frac{1}{2}[0 + \frac{1}{4} + \frac{1}{2}] = \frac{3}{8}$. Hence error is $\frac{1}{3} - \frac{3}{8} = -\frac{1}{24}$. Error estimate for each interval of CTR is $-h^3 f''(c)/12$ where c is the midpoint of the interval. In this case $f''(x) = 2$ everywhere, so error estimate is $2 \times -(\frac{1}{2})^3 \frac{2}{12} = -\frac{1}{24}$. Estimate is exact for the same reason as above.