Question 1.
[20 Marks Total]

(a) [**6 Marks**] Starting with the interval $[1.1, 1.2]$, use 3 steps of the Bisection method to find a smaller interval containing a root of $f(x) = \exp(x) - 2 - x$. Give the interval after 3 steps, the estimate for the root, and the maximum error. What will be the length of the final interval after a total of 10 steps of the method?

(b) [**7 Marks**] Given that the Simpson's Rule to approximate $\int_a^b f(x)\mathrm{d}x$ is

$$\int_a^b f(x)\mathrm{d}x \approx \frac{h}{3}\left(f(a) + 4f(a+h) + f(b)\right),$$

where $h = (b-a)/2$, derive the composite Simpson rule to approximate $\int_a^b f(x)\mathrm{d}x$ by dividing the region of integration into n strips of width $(b-a)/n$, with n even. Use this rule to approximate $\int_0^1 \exp(x)\mathrm{d}x$, with $n = 4$. Calculate the exact value of this integral and hence find the actual error of the approximation. If the theoretical expression for this error is

$$-\frac{(b-a)h^4}{180} f^{(iv)}(\eta),$$

find η. (Use the maximum number of digits your calculator allows).

(c) [**7 Marks**] Explain why numerical calculation of the expression

$$\sqrt{1+x} - \sqrt{1-x} \qquad (*)$$

may give rise to rounding error problems when $x \approx 0$. Derive another expression for (*) which avoids rounding error for small x and is correct to at least $O(x^3)$.

Question 2.
[20 Marks Total]

(a) [**10 Marks**] Consider the Fixed Point (FP) iteration

$$x_{k+1} = \varphi(x_k), \quad k = 0, 1, 2, \ldots$$

for finding a fixed point x_* satisfying

$$x_* = \varphi(x_*).$$

Assume that $e_{k+1} = e_k\, \varphi'(\xi_k)$, where $e_k = x_k - x_*$ and ξ_k lies between x_k and x_*. Assume further that there is an interval $I = [x_* - \delta, x_* + \delta]$, such that if $x \in I$, $|\varphi'(x)| \le C < 1$. Show that if $x_0 \in I$, then $x_k \in I$ for all k. Show further that x_k converges to x_* as $k \to \infty$.

(b) [**10 Marks**] Consider the FP iteration

$$x_{k+1} = 1 + 1/x_k, \quad k = 0, 1, \ldots$$

Carry out 6 iterations, starting at $x_0 = 1.6$.

Find the fixed point x_* of this iteration to which the x_k are converging. By calculating the ratio $|x_{k+1} - x_*|/|x_k - x_*|$, deduce that the iteration is converging linearly. Confirm these numerical estimates by calculating the value of $|\varphi'(x_*)|$. (There is no need to go into details of the theory).

Question 3.
[20 Marks Total]

(a) [**8 Marks**]

Use Newton's method to derive an iterative method for finding the root of $f(x) = a - 1/x$. Show that the iterations generated by this method satisfy the equation, with a real and positive.

$$(x_{k+1} - \frac{1}{a}) = -a(x_k - \frac{1}{a})^2, \quad k = 0, 1, \ldots$$

and hence show that if the sequence converges to $1/a$, the convergence of this scheme is superlinear of order 2.

What happens if $x_0 = 0$ or $x_0 = 2/a$?

(b) [**12 Marks**] With the aid of a diagram, explain the ideas behind the Secant method, and show that it leads to the iteration

$$x_{k+1} = \frac{x_{k-1}f(x_k) - x_k f(x_{k-1})}{f(x_k) - f(x_{k-1})}, \quad k = 0, 1, \ldots$$

Suppose the Secant method is used to find a zero of

$$p_4(x) = 1 + x + 2x^2 - 3x^4.$$

Describe how synthetic division could be used for the efficient evaluation of $p_4(x)$, at the point $x = r$, by writing

$$p_4(x) = (x - r)q_3(x) + z$$

where $q_3(x) = b_1 + b_2x + b_3x^2 + b_4x^3$ and z is independent of x. Derive the equations satisfied by the b_i and z in this case. Use these results to evaluate $p_4(x_0)$ and $p_4(x_1)$ if $x_0 = 1$ and $x_1 = 2$. Hence find x_2, the next iteration in the Secant method applied to this problem.

Question 4.
[20 Marks Total]

(a) [**8 Marks**] The Trapeziod Rule (TR) for approximating integrals is given by:

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{2}[f(a) + f(b)]$$

where $h = (b - a)$. Show by expanding the integrand and the r.h.s. in Taylor series in h around the midpoint $c = a + h/2 = b - h/2$ that

$$\begin{aligned} \int_a^b f(x)\,\mathrm{d}x &= hf(c) + Ah^3 f''(c) + O(h^5), \\ \frac{h}{2}[f(a) + f(b)] &= hf(c) + Bh^3 f''(c) + O(h^5) \end{aligned}$$

where A and B are constants, and find A and B. Hence show that the error in this rule is $-h^3 f''(c)/12 + O(h^5)$

(b) [**6 Marks**] Derive the Composite Trapeziod Rule (CTR) for approximating integrals by dividing the region of integration into n equal subintervals. Apply the CTR with $n = 200$ to the problem of evaluating $\int_0^3 f(x)\,\mathrm{d}x$, where $f(x) = 4$. What is the error in this case?

(c) [**6 Marks**] Apply the CTR with $h = 1/2$ to approximate the integral

$$\int_0^1 x^2\,\mathrm{d}x.$$

using the exact result for the integral, find the error of the CTR in this case. Estimate the error by using the formula derived above for the TR applied to each subinterval. Comment on the agreement (or otherwise) of the two forms of the error in this case.

END OF PAPER