11.3YC1 (Introductory Numerical Analysis)
December 1998, Duration 2 Hours.
Attempt All Four Questions. Total Mark on Paper is 80.
Approved Calculators may be Used.

Question 1.
[20 Marks Total]

(a) [**6 Marks**] Starting with the interval $[0.5, 0.6]$, use 3 steps of the Bisection method to find a smaller interval containing a root of $f(x) = \exp(x) - 2\cos(x)$. What will be the length of the final interval after 10 steps of the method?

(b) [**7 Marks**] The Mid-point formula and its error for approximate quadrature is

$$\int_a^b f(x)\,dx = 2hf(x_0) + \frac{h^3}{3}f''(\xi)$$

where ξ lies between a and b, $h = (b-a)/2$, and $x_0 = a + h$. Show that this formula without the error term is exact for all functions of the form $f(x) = mx + c$. If $a = 0, b = 1, f(x) = e^x$, find the exact and approximate values of $\int_a^b f(x)\,dx$ and hence find ξ in this case.

(c) [**7 Marks**] Explain why numerical calculation of the expression

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad (*)$$

may give rise to rounding error problems when $b \gg 0$, $a \approx 0$, $c \approx 0$. Show that the expression

$$\frac{-2c}{b + \sqrt{b^2 - 4ac}}$$

is equal to (*), and explain why using this for numerical calculations in the case $b \gg 0$, $a \approx 0$, $c \approx 0$ will avoid these problems.

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Question 2.
[20 Marks Total]

(a) [**8 Marks**] The Regula Falsi method, given two points $(x_k, f(x_k))$ and $(x_{k+1}, f(x_{k+1}))$, with $f(x_k)f(x_{k+1}) < 0$, calculates the next approximation to the root x_{k+2} by finding the intersection of the line joining these two points with the x-axis. By drawing a diagram or otherwise, find the equation of x_{k+2}, and explain briefly how the method chooses the non-trivial bracket for the next iteration of the scheme.

(b) [**5 Marks**] Apply the Regula Falsi method for 3 steps starting with $x_0 = 0$, $x_1 = 1$, to find approximations to the root of

$$f(x) = \frac{1}{(x+1)} - \frac{3}{4}.$$

(c) [**7 Marks**] Assuming that the value of the left-hand end of the bracket in the problem in the previous section is always 0, show that the iterations in the Regula Falsi method in this case simplify to

$$x_{k+1} = \frac{1}{4}(1 + x_k), \quad k = 2, 3, \ldots$$

Show that the fixed point is $x^* = 1/3$ and that the iteration will converge linearly to this point.

Question 3.
[20 Marks Total]

(a) [**8 Marks**] **Steffensen's Method** is a modified version of Aitken's acceleration. It is used to find an approximate solution to the Fixed Point (FP) equation $x = g(x)$ and works as follows:

1. Take the latest guess x_0 to the FP and calculate x_1, x_2 by two iterations of the FP iteration $x_{k+1} = g(x_k)$.
2. Calculate a better guess using Aitken's method
$$\hat{x}_0 = x_0 - (x_1 - x_0)^2/(x_2 - 2x_1 + x_0)$$
3. Set $x_0 = \hat{x}_0$ and go to 1.

Carry out two steps of this method to find an approximate solution to the FP problem $x = 1 + (\sin(x))^2$, starting with $x_0 = 1.5$.

(b) [**12 Marks**] Suppose a function $f(x)$ has a root p of multiplicity m, i.e. $f(x) = (x-p)^m q(x)$, with $q(p) \neq 0$. Show that the function $\mu(x) = f(x)/f'(x)$ also has a root at p but that it has multiplicity 1, i.e. $\mu(x)$ can be written as $\mu(x) = (x-p)r(x)$ with $r(p) \neq 0$. Show that Newton's method applied to $\mu(x)$ gives the iteration

$$x_{k+1} = x_k - \frac{f(x_k)f'(x_k)}{[f'(x_k)]^2 - f(x_k)f''(x_k)}$$

Carry out one step of this method with $f(x) = \exp(x) - 1 - x$, $x_0 = 0.1$.

Question 4.
[20 Marks Total]

(a) **[12 Marks]** An open Newton-Cotes Rule for approximating integrals is given by:

$$\int_a^b f(x)dx \approx (3h/2)[f(a+h)+f(b-h)]$$

where $h = (b-a)/3$. Show by expanding the integrand and the r.h.s. in Taylor series in h around the midpoint $c = a + 3h/2 = b - 3h/2$ that

$$\begin{aligned} \int_a^b f(x)dx &= 3hf(c) + Ah^3 f''(c) + O(h^4), \\ (3h/2)[f(a+h)+f(b-h)] &= 3hf(c) + Bh^3 f''(c) + O(h^4) \end{aligned}$$

where A and B are constants, and find A and B. Hence show that the error in this rule is $3h^3 f''(c)/4 + O(h^4)$

(b) **[8 Marks]** Prove that the relative error in using k-digit chopping to convert a real number y into a machine number $fl(y)$ (using base 10 throughout) is $\leq 10^{1-k}$.

END OF PAPER