The devil is in the tails: actuarial mathematics and the subprime mortgage crisis

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January 4, 2010

Abstract

In the aftermath of the 2007-2008 financial crisis, there has been criticism of mathematics and the mathematical models used by the finance industry. We answer these criticisms through a discussion of some of the actuarial models used in the pricing of credit derivatives. As an example, we focus in particular on the Gaussian copula model and its drawbacks. To put this discussion into its proper context, we give a synopsis of the financial crisis and a brief introduction to some of the common credit derivatives and highlight the difficulties in valuing some of them.

We also take a closer look at the risk management issues in part of insurance industry that came to light during the financial crisis. As a backdrop to this, we recount the events that took place at American International Group during the financial crisis. Finally, through our paper we hope to bring to the attention of a broad actuarial readership some "lessons (to be) learned" or "events not to be forgotten".

1 Introduction

"Recipe for disaster: the formula that killed Wall Street". That was the title of a web-article Salmon (2009) that appeared in Wired Magazine on February 2009. It was shortly followed by a Financial Times article Jones (2009) called "Of couples and copulas: the formula that felled Wall St". Both articles were written about an actuarial model called the Li model which is used in credit risk management. The impression gained is that an actuary developed a mathematical model which subsequently caused the downfall of Wall Street banks.

Both articles attempt to explain the limitations of the model, and its role in the 2007-2008 financial crisis ("the Crisis"). While the earlier article Salmon (2009) acknowledges that the deficiencies of the model have been known for sometime, the later Financial Times article Jones (2009) asks why no-one noticed the model's Achilles' heel.

For some of us, the implication that a mathematical model shoulders much of the blame for the difficulties on Wall Street and that few people were aware of its limitations are untenable. Indeed, we aim to demonstrate that such criticism is entirely unjustified.

Yet these criticisms of one particular model, with their unwarranted focus on the man who introduced the model to the credit derivative world, fly within a barrage of accusations directed at financial mathematics and mathematicians. A typical example is to be found in the New

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York Times of September 12, 2009: “Wall Street’s Math Wizards Forgot a Few Variables”; see Lohr (2009). Many more have been published. These accusations come not only from newspaper articles such as those cited above, but even from government-instigated reports into the Crisis. Turner (2009) has a section entitled “Misplaced reliance on sophisticated maths”. An interesting reply to the Turner Review came from Professor Sir David Wallace, Chair of the Council for the Mathematical Sciences, who on behalf of several professors of mathematics in the UK states that: “Another aspect on which we would welcome dialogue concerns the reference to *a misplaced reliance on sophisticated maths* and the possible interpretation that mathematics *per se* has a negative effect in the city. You can imagine that we strongly disagree with this interpretation! But of course the purpose of mathematical and statistical models must be better understood. In particular we believe that the FSA [Financial Services Authority] and the research community share an objective to enhance public appreciation of uncertainties in modelling future behaviour”; see Wallace (2009).

We believe that there *should* be a reliance on sophisticated mathematics. There has been too often a problem of misplaced reliance on unsophisticated mathematics or, in the words of L.C.G. Rogers, “The problem is not that mathematics was *used* by the banking industry, the problem was that it was *abused* by the banking industry. Quants were instructed to build models which fitted the market prices. Now if the market prices were way out of line, the calibrated models would just faithfully reproduce those wacky values, and *the bad prices get reinforced by an overlay of scientific respectability!*”; see Rogers (2009). For an excellent article (written in German) taking a more in-depth look at the importance of mathematics for finance and its role in the current crisis, see Föllmer (2009). The main contributions from mathematics to economics and finance are summarized in Föllmer (2009) as follows:

- understanding and clarifying models used in economics;
- making heuristic methods mathematically precise;
- highlighting model conditions and restrictions on applicability;
- working out numerous explicit examples;
- leading the way for stress-testing and robustness properties, and
- offering a relevant and challenging field of research on its own.

We cannot answer every accusation directed at financial mathematics. Instead, we look at the Li model, also called the Gaussian copula model, and use it as a proxy for mathematics applied badly in finance. It should be abundantly clear that it is not mathematics that caused the Crisis. At worst, a misuse of mathematics, and we mean mathematics in a broad sense and not just one formula, partly contributed to the Crisis.

The Gaussian copula model has been embraced enthusiastically by industry for its simplicity. While a simple model is to be preferred to a complex one, especially in a financial world which can only be partially and imperfectly described by mathematics, we believe that the model is too simple. It does not capture the main features of what it is attempting to model. Yet it was, and still is, applied to the credit derivatives which played a major part in the Crisis. We devote a large part of this article to explaining the Gaussian copula model and examining its shortcomings.

We also rebutt the claim that few people saw the flaws underlying several of the quantitative techniques used in the pricing and risk management of credit derivatives. On the contrary, many academics and practitioners were aware of them and on numerous occasions exposed these flaws.

As the fields of insurance and finance increasingly overlap, it is maybe not surprising that one casualty of the Crisis was an insurance company, American Insurance Group ("AIG"). With insurance companies selling credit default swaps, which have insurance-like features, and catastrophe bonds and mortality bonds, which are a way of selling insurance risk in the financial market, it is an opportune time to examine what caused the near-collapse of AIG. We ask what lessons other insurance companies and those involved in running them, such as actuaries and other risk professionals, can learn from the AIG story.

It is also a good time to pause and think about our roles and responsibilities in the finance industry. Are the practitioners truly aware of the assumptions, whether implicit or explicit, in the mathematics they use? If not, then they have a duty to inform themselves. It is also the duty of the academics who are publishing articles not only to make their assumptions explicit but also, upon use, to communicate their assumptions more forcefully to the end-user.

Before we delve into the above, we begin by outlining the Crisis.

2 The roots of the subprime mortgage crisis

The Crisis was complex and of global proportions. There will undoubtedly be a multitude of articles and books penned about it for years to come. Among currently available, more academic, excellent analyses are Brunnermeier (2009), Crouhy et al. (2008) and Hellwig (2009). We also highly recommend The Economist (2008). As our focus is on some of the mathematical and actuarial issues which arose from the Crisis, we relate only the story of the Crisis which is relevant for this article.

The root of the Crisis was the transfer of the risk of mortgage default from mortgage lenders to the financial market at large: banks, hedge funds, insurance companies. The transfer was effected by a process called securitization. The practical mechanics of this process can be complicated, as institutions seek to reduce costs and tax-implications. However, the essence of what is done is as follows.

A bank pools together mortgages which have been taken out by residential home-owners and commerical property organizations. The pool of mortgages is transferred to an off-balance-sheet trust called a special-purpose vehicle ("SPV"). While sponsored by the bank, the SPV is bankruptcy-remote from it. This means that a default by the bank does not result in a default by the SPV. The SPV issues coupon-bearing financial securities called mortgage-backed securities. The mortgage repayments made by the home-owners and commerical property organizations are directed towards the SPV, rather than being received by the bank which granted the mortgages. After deducting expenses, the SPV uses the mortgage repayments to pay the coupons on the mortgage-backed securities. Typically, the buyers of the mortgage-backed securities are organizations such as banks, insurance companies and hedge funds. This process allowed banks to move from an "originate to hold" model, where they held the mortgages they made on their books, to an "originate to distribute" model, where they essentially sold on the mortgages.

Not only mortgages can be securitized, but also other assets such as auto loans, student loans and credit card receivables. A security issued on fixed-income assets is called a collateralized debt obligation ("CDO"), and if the underlying assets of the CDO consist of loans then it is called a collateralized loan obligation. However, the underlying assets do not have to be fixed-income assets and the general term for a security issued on any asset is an asset-backed security.

There is nothing inherently wrong with the securitization process. It is a transfer of risk from one party to another, in this case the risk of mortgage default. It should increase the efficiency of financial markets as it allows those who are happy to take on the risk of mortgage default to buy it. Moreover, as banks must hold capital against the loans on their books, selling most of the pool of mortgages allows them to free up capital. The view on the benefits of securtization to overall financial stability in 2006 is summarized in the following quote from one of the IMF's Global Financial Stability Reports in that year: "There is a growing recognition that the dispersion of credit risk by banks to a broader and more diverse group of investors, rather than warehousing such risk on their balance sheets, has helped make the banking and overall financial system more resilient. ... The improved resilience may be seen in fewer bank failures and more consistent credit provision. *Consequently, the commercial banks, ..., may be less vulnerable today to credit or economic shocks*"; see IMF (2006, Chapter II). Indeed, this was the prevailing view until late 2006. Yet the process of transferring one type of risk creates other types of risks.

As it turned out, the main additional risk in securitization was moral hazard. A lengthy discussion of the role of moral hazard in the Crisis can be found in Hellwig (2009). For securitized products, sources of moral hazard included:

- the failure of some originators of securitized products to retain any of the riskiest part of the CDO. We examine this point in the next paragraph;
- the credit rating agencies had a conflict of interest in that they were advising customers on how to best securitize products and then credit rating those same products. SEC (2008) gives a flavor of the practices in the three main credit rating agencies leading up to the Crisis;
- the chain of financial intermediation from the originators to the buyers of some securitized products may have been too long, resulting in opaqueness, a loss of information and an increased scope for moral hazard (see also Subsection 6.2), and
- some financial institutions may have deemed themselves "too big too fail", with a corresponding disregard for the level of risk they were exposed to and a belief on their part that the government would not allow them to fail since they were systematically too important. Wolf (2008) has a delightful phrase for this: "privatising gains and socialising losses". See also anecdotal evidence from Haldane (2009b, page 12).

If a bank is not exposed to the risk of mortgage default, then it has no incentive to control and maintain the quality of the loans it makes. To protect against this, the theory was that the banks should retain the riskiest part of the mortgage pool. In practice, this did not always happen, which led to a reduction in lending standards; see Keys et al. (2008). This possibility was foreseen some fifteen years before the Crisis with remarkable prescience by Stiglitz, as he points out in Stiglitz (2008). Because of its prime importance in the current discussion of the Crisis, but also as it reflects indirectly on the possibility of bank-assurance products, we repeat some of its key statements, written in 1992: "...has the growth in securitization been a result of more efficient transactions technologies, or an unfounded reduction in concern about the importance of screening loan applicants? ... we should at least entertain the possibility that it is the latter rather than the former... At the very least, the banks have demonstrated an ignorance of two very basic aspects of risk: (a) the importance of *correlation*,... (b) the possibility of *price declines*."

As the quality of the mortgages granted declined, the risk characteristics of the underlying pool of mortgages changed. In particular, the risk of mortgage default increased. It appears that many market participants either did not realize this was happening or did not think that it was significant. In February 2007, an increase in subprime mortgage defaults was noted, and the Crisis started unfolding. There were many factors which contributed strongly to the Crisis, such as fair-value accounting, systemic interdependence, a move by banks to financing their assets with shorter maturity instruments, which left them vulnerable to liquidity drying-up, and other factors, such as ratings agencies and an excessive emphasis on revenue and growth by financial institutions. However, the reader should look elsewhere for an explanation of their impact, such as in the references mentioned at the start of this section.

3 Securitization

Securitization is the process of pooling together financial assets, such as mortgages and auto loans, and redirecting their cashflows to support coupon payments on CDOs. Here we describe CDOs in more detail.

We have described the creation of a CDO in the previous section. However, what we did not mention is that commonly CDOs are split into tranches. The tranches have different risk and return characteristics which make them attractive to different investors. Suppose that a CDO is split into three tranches. Typically, these are called the senior, mezzanine and equity tranches. Payments from the underlying assets are directed through the CDO tranches, in order of priority. There is a legal document associated with the CDO which sets out the priority of payments. After expenses, the first priority is to pay the coupons for the senior tranche, followed by the mezzanine tranche and finally the equity tranche. The contractual terms governing the priority of payments is called the payment waterfall. A schematic of a tranched CDO is shown in Figure 1. If defaults

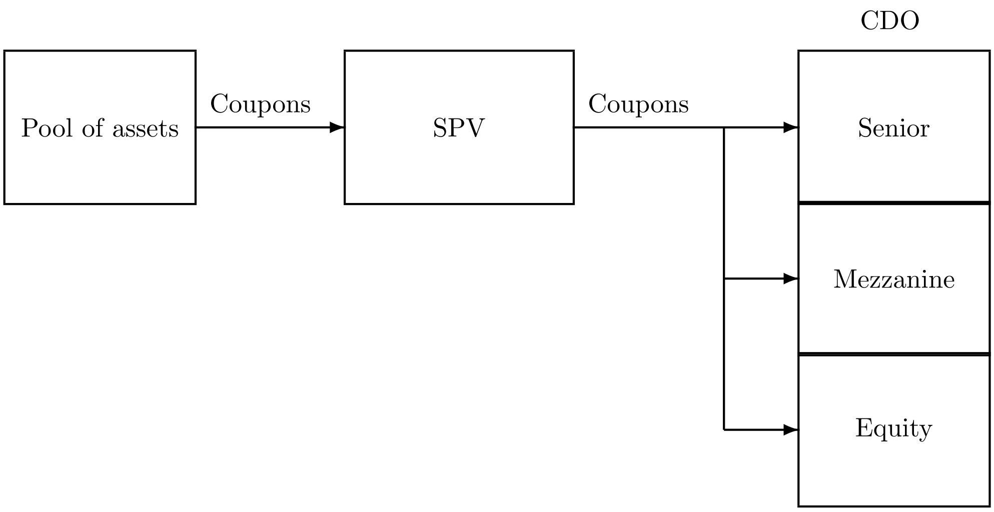


Figure 1: Diagram showing the tranching of a Collateralized Debt Obligation into three tranches: senior (highest priority), mezzanine and equity (lowest priority).

occur in the underlying assets, for example some bonds in the underlying portfolio default, then that loss is borne first by the equity tranche holders. The coupons received by the equity tranche holders are reduced. If enough defaults occur, then the equity tranche holders no longer receive any coupons and any further losses are borne by the mezzanine tranche holders. Once the mezzanine tranche holders are no longer receiving coupons, the senior tranche holders bear any further losses.

The tranching of the CDO allows the senior tranche to receive a higher credit rating than the mezzanine tranche. This allows investors who may not normally invest in the underlying assets to invest indirectly in them, through the CDO. For example, suppose the underlying pool of assets has an aggregate credit rating of BBB. Before tranching, the credit rating of the CDO would also be BBB. However, with judicious tranching, the senior tranche can achieve a AAA credit rating. This is because it is exposed to a much reduced risk of default from the underlying assets, since any losses arising from default in the underlying portfolio are borne first by the equity tranche holders and then the mezzanine tranche holders. Usually, the mezzanine tranche is BBB-rated and the equity tranche is not credit rated.

The SPV aims to maximize the size of the senior tranche, subject to it attaining a AAA credit-rating. The maximization of the size of the senior tranche may mean that it is just within the boundary of what constitutes a AAA-rated investment. Typically, the senior tranche is worth around 80% of the nominal value of the underlying portfolio of assets. This means that 20% of the underlying portfolio must default before the holders of the senior tranche of the CDO have their coupon payments reduced. Similarly, the SPV maximizes the size of the mezzanine tranche, subject to it attaining a BBB credit-rating. Typically, the mezzanine tranche is worth in the region of 15% of the nominal value of the underlying portfolio of assets. This means that 5% of the underlying portfolio must default before the holders of the mezzanine tranche of the CDO have their coupon payments reduced. The remaining part of the CDO is allocated to the equity tranche, which is unrated and is worth the remaining 5% nominal value of the underlying portfolio of assets. As the equity tranche has the lowest priority in payments, any defaults in the underlying portfolio of assets reduce the coupon payments of the equity tranche holders.

The key to valuing CDOs is modeling the defaults in the underlying portfolios. It is clear from the description above that the coupon payments received by the holders of the CDO tranches depend directly on the defaults occurring in the underlying portfolio of assets. As Duffie (2008) points out, the modeling of default correlation is currently the weakest link in the risk measurement and pricing of CDOs. There are several methods of approaching the valuation of a CDO, a few of which we mention briefly in Section 7, but first we clear the stage and allow the Gaussian copula to enter.

4 The Gaussian copula model

On March 27 1999, the second author gave a talk at the Columbia-JAFEE Conference on the Mathematics of Finance at Columbia University, New York. Its title was "Insurance Analytics: Actuarial Tools in Financial Risk-Management" and it was based on a 1998 RiskLab report that he co-authored with Alexander McNeil and Daniel Straumann; see Embrechts et al. (2002). The main emphasis of the report was on explaining to the world of risk management the various risk management pitfalls surrounding the notion of linear correlation. The concept of copula, by now omnipresent, was only mentioned in passing in Embrechts et al. (2002). However, its appearance in Embrechts et al. (2002) started an avalanche of copula-driven research; see Genest et al. (2009). During the coffee break, David Li walked up to the second author, saying that he had started using copula-type ideas and techniques, but now wanted to apply them to newly invented credit derivatives like CDOs. The well-known paper Li (2000) was published one year later. In it is outlined a copula-based approach to modeling the defaults in the underlying pool. Suppose we wish to value a CDO which has d bonds in the underlying portfolio. As we mentioned in the previous section, we can do this if we can find the joint default distribution of the d bonds. Denote by T_i the time until default of the ith bond, for $i = 1, \ldots, d$. How can we determine the distribution of the joint default time, $\mathbb{P}[T_1 \leq t_1, \ldots, T_d \leq t_d]$? If we can do this, then we have a way to value the CDO.

4.1 A brief introduction to copulas

Using copulas allows us to separate the individual behaviour of the marginal distributions from their joint dependency on each other. We focus only on the copula theory that is necessary for this article. An introduction to copulas can be found in Nelsen (2006) and a source of some of the more important references on the theory of copulas can be found in Embrechts (2009).

Consider two random variables X and Y defined on some common probability space. For example, the random variables X and Y could represent the times until default of two companies. What if we wish to specify the joint distribution of X and Y, that is to specify the distribution function ("df") $H(x, y) := \mathbb{P}[X \leq x, Y \leq y]$? If we know the individual dfs of X and Y then we can do this using a copula. A copula specifies a dependency structure between X and Y, that is how X and Y behave jointly.

More formally, a copula is defined as follows.

Definition 4.1. A d-dimensional copula $C : [0, 1]^d \to [0, 1]$ is a df with standard uniform marginal distributions.

An example of a copula is the independence copula $C^{\perp}$, defined in two-dimensions as

$$C^{\perp}(u, v) := uv, \qquad \forall u, v \in [0, 1].$$

It can be easily checked that $C^{\perp}$ satisfies Definition 4.1. We can choose from a variety of copulas to determine the joint distribution. Which copula we choose depends on what type of dependency structure we want. The next theorem tells us how the joint distribution is formed from the copula and the marginal dfs. It is the easy part of Sklar's Theorem and the proof can be found in Schweizer and Sklar (1983, Theorem 6.2.4).

Theorem 4.2. *Let C be a copula and $F_1, \ldots, F_d$ be univariate dfs. Defining*

$$H(x_1, \ldots, x_d) := C\left(F_1(x_1), \ldots, F_d(x_d)\right), \qquad \forall (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

the function H is a joint df with margins $F_1, \ldots, F_d$.

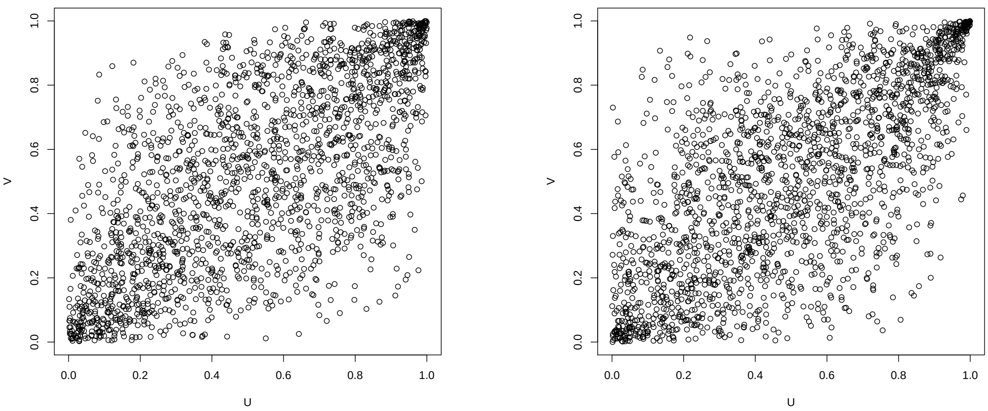
4.2 Two illustrative copulas

We look more closely at two particular copulas: the Gaussian copula and the Gumbel copula. For notational reasons, we restrict ourselves to the bivariate $d = 2$ case. The Gaussian copula is often used to model the dependency structures in credit defaults. We aim to compare it with the Gumbel copula for illustrative purposes. As before, let X and Y be random variables with dfs F and G, respectively.

First consider the bivariate Gaussian copula C^{gau}_{ρ}. This copula does not have a simple closed form but can be expressed as an integral. Denoting by Φ the univariate standard normal df, the bivariate Gaussian copula C^{gau}_{ρ} is

$$C^{\text{gau}}_{\rho}(u, v) := \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \frac{1}{2\pi\left(1-\rho^2\right)^{1/2}} \exp\left\{-\frac{s^2 - 2\rho st + t^2}{2\left(1-\rho^2\right)}\right\} \mathrm{d}s\,\mathrm{d}t, \tag{4.1}$$

for all $u, v \in [0, 1]$, $|\rho| < 1$. The parameter ρ determines the degree of dependency in the Gaussian copula. For example, setting $\rho = 0$ makes the marginal distributions independent so that $C^{\text{gau}}_0 = C^{\perp}$. As the Gaussian copula is a df, we can plot its distribution. Figure 2(a) shows a random sample of the df of C^{gau}_{ρ} with $\rho := 0.7$.



(a) Gaussian copula C_ρ^{gau} with $\rho := 0.7$.

(b) Gumbel copula C_θ^{gum} with $\theta := 2$.

Figure 2: Figures showing 2000 sample points from the copulas named under each figure.

Applying Theorem 4.2 with the bivariate Gaussian copula C_ρ^{gau}, the joint df H of the random variables X and Y is

$$H(x, y) := C_\rho^{\text{gau}}(F(x), G(y)), \qquad \forall (x, y) \in \mathbb{R}^2.$$

The Gaussian copula arises quite naturally. In fact, it can be recovered from the multivariate normal distribution. This is a consequence of the converse of Theorem 4.2, which is given next. This is the second, less trivial part of Sklar's Theorem and the proof can be found in Schweizer and Sklar (1983, Theorem 6.2.4).

Theorem 4.3. *Let H be a joint df with margins $F_1, \ldots, F_d$. Then there exists a copula $C : [0,1]^d \to [0,1]$ such that, for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$,*

$$H(x_1, \ldots, x_d) := C\left(F_1(x_1), \ldots, F_d(x_d)\right), \qquad \forall (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

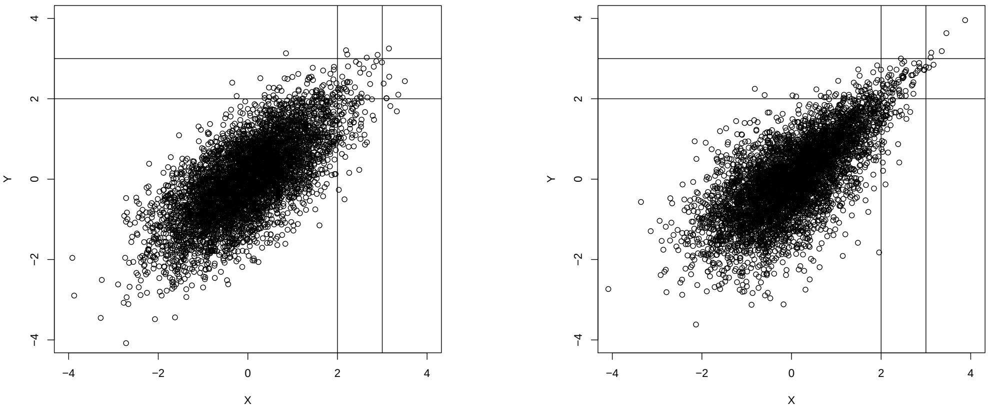
If the margins are continuous then C is unique. Otherwise C is uniquely determined on $Ran(F_1) \times \cdots \times Ran(F_d)$, where $Ran(F_i)$ denotes the range of the df F_i.

To show how the Gaussian copula arises, suppose that $\boldsymbol{Z} = (Z_1, Z_2)$ is a two-dimensional random vector which is multivariate normally distributed with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{\Sigma} = \left(\begin{smallmatrix} 1 & \rho \\ \rho & 1 \end{smallmatrix}\right)$. We write $\boldsymbol{Z} \sim N_2(\mathbf{0}, \boldsymbol{\Sigma})$ and denote the df of $\boldsymbol{Z}$ by $\boldsymbol{\Phi}_2$. We know that margins of any multivariate normally distributed random vector are univariate normally distributed. Thus $Z_1, Z_2 \sim N(0, 1)$ and the df of both Z_1 and Z_2 is Φ. The Gaussian copula C_ρ^{gau} appears by applying Theorem 4.3 to the joint normal df $\boldsymbol{\Phi}_2$ and the marginal normal dfs Φ to obtain

$$\boldsymbol{\Phi}_2(x, y) = C_\rho^{\text{gau}}(\Phi(x), \Phi(y)), \qquad \forall x, y \in \mathbb{R}.$$

From this we see that a multivariate normally distributed distribution can be obtained by combining univariate normal distributions with a Gaussian copula. Figure 3(a) shows a simulation of the joint df of X and Y when both are normally distributed with mean 0 and standard deviation 1 and with dependency structure given by the Gaussian copula C_ρ^{gau} with $\rho := 0.7$. This is exactly the bivariate normal distribution with a linear correlation between X and Y of 0.7.

Of course, we do not have to assume that the marginals are univariate normal distributions. For instance, Figure 2(a) shows a df which has standard uniform marginals with the Gaussian copula C_ρ^{gau} with $\rho := 0.7$.



(a) Gaussian copula C_ρ^{gau} with $\rho := 0.7$.

(b) Gumbel copula C_θ^{gum} with $\theta := 2$.

Figure 3: Figures showing 5000 sample points from the random vector (X, Y) which has standard normally distributed margins and dependency structure as given by the copula named under each figure. To the right of the vertical line $x = 2$ and above the horizontal line $y = 2$, in the Gaussian copula figure there are 43 sample points. The corresponding number for the Gumbel copula figure is 70. To the right of the vertical line $x = 3$ and above the horizontal line $y = 3$, in the Gaussian copula figure there is 1 sample point. The corresponding number for the Gumbel copula figure is 5.

The second copula we consider is the bivariate Gumbel copula C_θ^{gum} which has the general form

$$C_\theta^{\text{gum}}(u, v) = \exp\left\{ -\left((-\ln u)^\theta + (-\ln v)^\theta \right)^{\frac{1}{\theta}} \right\}, \qquad 1 \leq \theta < \infty, \qquad \forall u, v \in [0, 1].$$

The parameter θ has an interpretation in terms of a dependence measure called Kendall's rank correlation. Like linear correlation, Kendall's rank correlation is a measure of dependency between X and Y. While linear correlation measures how far Y is from being of the form $aX + b$, for some constants $a \in \mathbb{R} \setminus \{0\}, b \in \mathbb{R}$, Kendall's rank correlation measures the tendency of X to increase with Y. To calculate it, we take another pair of random variables $(\tilde{X}, \tilde{Y})$ which have the same df as (X, Y) but are independent of (X, Y). Kendall's rank correlation is defined as

$$\rho_\tau(X, Y) := \mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) > 0] - \mathbb{P}[(X - \tilde{X})(Y - \tilde{Y}) < 0].$$

A positive value of Kendall's rank correlation indicates that X and Y are more likely to increase or decrease in unison, while a negative value indicates that it is more likely that one decreases while the other increases. For the Gumbel copula, Kendall's rank correlation is $\rho_\tau^{\text{gum}}(X, Y) = 1 - \frac{1}{\theta}$.

Figure 2(b) shows a sample of 2000 points from the Gumbel copula C_θ^{gum} with $\theta := 2$. Using the Gumbel copula and fixing $\theta \in [1, \infty)$, the joint df of X and Y is

$$H(x, y) = C_\theta^{\text{gum}}(F(x), G(y)) = \exp\left\{ -\left((-\ln(F(x)))^\theta + (-\ln(G(y)))^\theta \right)^{\frac{1}{\theta}} \right\}, \qquad \forall x, y \in \mathbb{R}.$$

Figure 3(b) shows a simulation of the joint df of X and Y when both are normally distributed with mean 0 and standard deviation 1 and with dependency structure given by the Gumbel copula C_θ^{gum} with $\theta := 2$. The linear correlation between X and Y is approximately 0.7. Thus while we see that the two plots in Figure 3 have quite different structures - Figure 3(a) has an elliptical shape while Figure 3(b) has a teardrop shape - they have approximately the same linear correlation. This illustrates the fact that the knowledge of linear correlation and the marginal dfs does not uniquely determine the joint df of two random variables. This is also true for Kendall's rank correlation: as a random vector (X, Y) with continuous margins and dependency structure given by the bivariate Gaussian copula C_ρ^{gau} has Kendall's rank correlation $\rho_\tau(X, Y) = \frac{2}{\pi}\arcsin(\rho)$ (see McNeil et al. (2005, Theorem 5.36)), we find that the two plots in Figure 3 have approximately the same Kendall's rank correlation of 0.5. In summary, a scalar measure of dependency together with the marginal dfs does not uniquely determine the joint df.

This is especially important to keep in mind in risk management when we are interested in the risk of extreme events. By their very nature, extreme events are infrequent and so data on them is scarce. However, from a risk management perspective, we must make an attempt to model their occurrence, especially their *joint* occurence.

As we see from the two plots in Figure 3, which have identical marginal dfs and almost identical linear correlation and Kendall's rank correlation, these measures of dependency do not tell us anything about the likelihood of extreme events. In Figure 3, it is the choice we make for the copula which is critically important for determining the likelihood of extreme events. For example, consider the extreme event that both $X > 2$ and $Y > 2$. In Figure 3(a), which assumes the Gaussian copula, there are 43 sample points which satisfy this, whereas in Figure 3(b), which assumes the Gumbel copula, there are 70 such sample points. Next consider the extreme event that both $X > 3$ and $Y > 3$. In Figure 3(a) there is 1 sample point which satisfies this, whereas in Figure 3(b) there are 5 such sample points. Under the assumption that the dependency structure of the random variables is given by the Gaussian copula, extreme events are much less likely to occur than under the Gumbel copula.

4.3 The Gaussian copula approach to CDO pricing

At the start of this section, we introduced the default times (T_i) of the d bonds in the underlying portfolio of some CDO. Using F_i to denote the df of default time T_i, for $i = 1, \ldots, d$, the Li copula approach is to define the joint default time as

$$\mathbb{P}[T_1 \leq t_1, \ldots, T_d \leq t_d] := C(F_1(t_1), \ldots, F_d(t_d)), \qquad \forall (t_1, \ldots, t_d) \in [0, \infty)^d, \tag{4.2}$$

where C is a copula function. The term "Li model" or "Li formula" has become synonymous with the use of the Gaussian copula in (4.2). While Li (2000) did use the Gaussian copula as an example, it would be more accurate if these terms referred to (4.2) in its full generality, rather than just one particular instance of it. However, we use these terms as they are widely understood, that is to mean the use of the Gaussian copula in (4.2).

In practice, the Li model is generally used within a one-factor or multi-factor framework. We describe the one-factor Gaussian copula approach. Suppose the d bonds in the underlying portfolio of the CDO have been issued by d companies. Denote the asset value of company i by Z_i. Under the one-factor framework, it is assumed that

$$Z_i = \sqrt{\rho}\, Z + \sqrt{1-\rho}\, \epsilon_i, \quad \text{for } i = 1, \ldots, d,$$

where $\rho \in (0, 1)$ and $Z, \epsilon_1, \ldots, \epsilon_d$ are independent, standard normally distributed random variables. The random variable Z represents a market factor which is common to all the companies, while the random variable ϵ_i is the factor specific to company i, for each $i = 1, \dots, d$. Under this assumption, the transpose of the vector $(Z_1, \dots, Z_d)$ is multivariate normally distributed with mean zero and with a covariance matrix whose off-diagonal elements are each equal to ρ. In this framework, we interpret ρ as the correlation between the asset values of each pair of companies.

The idea is that default by company i occurs if the asset value Z_i falls below some threshold value. The default time T_i is related to the one-factor structure by the relationship $Z_i = \Phi^{-1}(F_i(T_i))$. With this relationship, the joint df of the default times is given by (4.2), with $C := C_\rho^{\mathrm{gau}}$, where ρ is the correlation between the asset values (Z_i). Once we have chosen the marginal dfs (F_i), we have fully specified the one-factor Li model.

Often, the marginal dfs are assumed to be exponentially distributed. In that case, the mean of each default time T_i can be estimated from the market, for instance from historical default information or the market prices of defaultable bonds. Using these exponential marginal dfs and the market prices of CDO tranches, investors can calculate the implied asset correlation ρ for each tranche. The implied asset correlation ρ is the asset correlation value which makes the market price of the tranche agree with the one-factor Gaussian copula model. However, as we also mention in Subsection 5.2, this results in asset correlation values which differ across tranches.

4.4 Credit default swaps and synthetic CDOs

The Li model can be used not only to value the CDOs we described in Section 3, but also another type of credit derivative called a credit defaut swap ("CDS"). A CDS is a contract which transfers the credit risk of a reference entity, such as a bond or loan, from the buyer of the CDS to the seller. The buyer of the CDS pays the seller a regular premium. If a credit event occurs, for example the reference entity becomes bankrupt or undergoes debt restructuring, then the seller of the CDS makes an agreed payoff to the buyer. What constitutes a credit event, the payoff amount and how the payoff is made is set out in the legal documentation accompanying the CDS.

There are two categories of CDSs: a single-name CDS, which protects against credit events of a single reference entity, and a multi-name CDS, which protects against credit events in a pool of reference entities. In the market, a CDS is quoted in terms of a spread. The spread is the premium payable by the buyer to the seller which makes the present value of the contract equal to zero. Roughly, a higher spread indicates a higher credit risk.

The market for CDSs is large. The Bank of International Settlements Quarterly Review of June 2009 gives the value of the notional amount of outstanding CDSs as US$42,000 billion as at December 31 2008, of which roughly two-thirds were single-name CDSs. Even after calculating the net exposure, this still corresponds to an amount above US$3,000 billion.

As CDSs grew in popularity, the banks which sold them ended up with many single-name CDSs on their books. The banks grouped together many single-name CDSs and used them as the underlying portfolio of a type of CDO called a synthetic CDO. In contrast, the cash CDOs we described in Section 3 have more traditional assets like loans or bonds in the underlying portfolio. As an indication of the market size for these instruments just prior to the Crisis, the Securities Industry and Financial Markets Association gives the value of cash CDOs issued globally in 2007 as US$340 billion and the corresponding value for synthetic CDOs as US$48 billion. It is also important to point out that products like CDOs and CDSs are currently not traded in officially regulated markets, but are traded over-the-counter ("OTC"). The global OTC derivative market is of a staggering size, with a nominal, outstanding value at the end of 2008 of US$592,000 billion; see BIS (2009). To put this amount into perspective, the total GDP for the world in 2008 was about US$61,000 billion.

Just like any CDO, a synthetic CDO can be tranched and the tranches sold to investors. The buyers of the tranches receive a regular premium and, additionally, the buyers of the equity tranche receives an upfront fee. This upfront fee can be of the order 20%-50% of the nominal value of the underlying portfolio.

There also exists synthetic CDO market indices, such as the Dow Jones' CDX family and the International Index Company's iTraxx family, which are actively traded as contracts paying a specified premium. These standardized market indices mean that there is a market-determined price for the tranches, which is expressed in terms of a spread for each tranche, in addition to an upfront fee for the equity tranche.

5 The drawbacks of the copula-based model in credit risk

The main use of the Gaussian copula model was originally for pricing credit derivatives. However, as credit derivative markets have grown in size, the need for a model for pricing has diminished. Instead, the market determines the price. However, the model is still used to determine a benchmark price and also has a significant role in hedging tranches of CDOs; see Finger (2009). Moreover, it is still widely used for pricing synthetic CDOs.

The model has some major advantages, which have for many people in industry outweighed its rather significant disadvantages, a story that we have most unfortunately been hearing far too often in risk management. Think of examples like the Black-Scholes-Merton model, or the widespread use of Value-at-Risk ("VaR") as a measure for calculating risk capital. All of these concepts have properties which need to be well understood by industry, especially when markets of the size encountered in credit risk are built upon them.

But first to the perceived advantages of the Gaussian copula model. These are that it is simple to understand, it enables fast computations and it is very easy to calibrate since only the pairwise correlation ρ needs to be estimated. Clearly, the easy calibration by only one parameter relies on the tenuous assumption that all the assets in the underlying portfolio have pairwise the same correlation. The advantages of the model meant that it was quickly adopted by industry. For instance, by the end of 2004, the three main rating agencies - Fitch Ratings, Moody's and Standard & Poor's - had incorporated the model into their rating toolkit. Moreover, it is still considered an industry standard.

Simplicity and ease of use typically comes at a price. For the Gaussian copula model, there are three main drawbacks:

- insufficient modeling of default clustering in the underlying portfolio;
- if we calculate a correlation figure for each tranche of a CDO, we would expect these figures to be the same. This is because we expect the correlation to be a function of the underlying portfolio and not of the tranches. However, under the Gaussian copula model, the tranche correlation figures are not identical, and
- no modeling of the economic factors causing defaults weakens the ability to do stress-testing, especially on a company-wide basis.

We examine each of these issues in turn.

5.1 Inadequate modeling of default clustering

One of the main disadvantages of the model is that it does not adequately model the occurrence of defaults in the underlying portfolio of corporate bonds. In times of crisis, corporate defaults occur in clusters, so that if one company defaults then it is likely that other companies also default within a short time period. Under the Gaussian copula model, company defaults become independent as their size of default increases. Mathematically, we can illustrate this using the idea of tail dependence. A tail dependence measure gives the strength of dependence in the tails of a bivariate distribution. We borrow heavily from McNeil et al. (2005) in the following exposition. Since dfs have lower tails (the left part of the df) and upper tails (the right part), we can define a tail dependence measure for each one. Here, we consider only the upper tail dependence measure. Recall that the generalized inverse of a df F is defined by $F^{\leftarrow}(y) := \inf\{x \in \mathbb{R} : F(x) \geq y\}$. In particular, if F is continuous and strictly increasing, then $F^{\leftarrow}$ equals the ordinary inverse F^{-1} of F.

Definition 5.1. Let X and Y be random variables with dfs F and G, respectively. The coefficient of upper tail dependence of X and Y is

$$\lambda_u := \lambda_u(X, Y) := \lim_{q \to 1^-} \mathbf{P}\left(Y > G^{\leftarrow}(q) \,|\, X > F^{\leftarrow}(q)\right),$$

provided a limit $\lambda_u \in [0, 1]$ exists. If $\lambda_u \in (0, 1]$ then X and Y are said to show upper tail dependence. If $\lambda_u = 0$ then X and Y are said to be asymptotically independent in the upper tail.

It is important to realize that λ_u depends only on the copula C and not on the marginal dfs F and G; see McNeil et al. (2005, page 209).

Suppose X and Y have a joint df with Gaussian copula C_ρ^{gau}. As long as $\rho < 1$, it turns out that the coefficient of upper tail dependence of X and Y equals zero; see McNeil et al. (2005, Example 5.32). This means that if we go far enough into the upper tail of the joint distribution of X and Y, extreme events appear to occur independently.

Recall that the dependence structure in the Li model is given by the Gaussian copula. The asymptotic independence of extreme events for the Gaussian copula carries over to asymptotic independence for default times in the Li model. If we seek to model defaults which cluster together, so that they exhibit dependence, the property of asymptotic independence is not desirable. This undesirable property of the Gaussian copula is pointed out in Embrechts et al. (2002) and was explicitly mentioned in the talk referred to at the beginning of Section 4. A first mathematical proof is to be found in Sibuya (1960).

Compare the coefficient of upper tail dependence of the Gaussian copula with that of the Gumbel copula. For X and Y with joint df given by C_θ^{gum}, the coefficient of upper tail dependence is given by $\lambda_u^{\mathrm{gum}} := 2 - 2^{\frac{1}{\theta}}$. As long as $\theta > 1$, then the Gumbel copula shows upper tail dependence and may hence be more suited to modeling defaults in corporate bonds.

In practice, as we do not take asymptotic limits, we wonder if the independence of the Gaussian copula in the extremes only occurs in theory and is insignificant in practice. The answer is categorically no. As we pointed out in Subsection 4.2 in relation to Figure 3, the effects of the tail independence of the Gaussian copula are seen not only in the limit. Of course, this is not a proof and we direct the reader to a more detailed discussion on this point in McNeil et al. (2005, page 212).

The Gumbel copula is not the only copula that shows upper tail dependence and we have chosen it simply for illustrative purposes. However, it demonstrates that alternatives to the Gaussian copula do exist, as was pointed out in the academic literature on numerous occasions. For example, see Frey et al. (2001) and Rogge and Schönbucher (2003).

The failure of the Gaussian copula to capture dependence in the tail is similar to the failure of the Black-Scholes-Merton model to capture the heavy-tailed aspect of the distribution of equity returns. Both the Gaussian copula and the Black-Scholes-Merton model are based on the normal distribution. Both are easy to understand and result in models with fast computation times. Yet both fail to adequately model the occurrence of extreme events.

We believe that it is imperative that the financial world considers what the model they use implies about frequency and severity of extreme events. For managing risk, it is imprudent to ignore the very real possibility of extreme events. It is unwise to rely without thought on a model based on the normal distribution to tell you how often these extreme events occur. We are not suggesting that models based on the normal distribution should be discarded. Instead, they should be used in conjunction with several different models, some of which should adequately capture extreme events, and all of whose advantages and limitations are understood by those using them and interpreting the results. Extreme Value Theory offers tools and techniques which can help in better understanding the problems and difficulties faced when trying to understand, for instance, joint extremes, market spillovers and systemic risk; see Coles (2001), Embrechts et al. (2008) and Resnick (2007) for a start.

5.2 Inconsistent implied correlation in tranches and an early warning

The one-factor Gaussian copula model is frequently used in practice for delta-hedging of the equity tranche of the synthetic CDO indices. Attracted by the high upfront fee, investors like hedge funds sell the equity tranche of a synthetic CDO. To reduce the impact of changes in the spreads of the underlying portfolio, they can delta-hedge the equity tranche by buying a certain amount of the mezzanine tranche of the same index. The idea is that small losses in the equity tranche are offset by small gains in the mezzanine tranche and vice versa. They buy the mezzanine tranche rather than the entire index because it is cheaper. Assuming the delta-hedge works as envisaged, the investor gains the high upfront fee and the regular premium payable on the equity tranche they sold, less the regular premium payable on the mezzanine tranche they bought.

First, an implied correlation is calculated for each tranche. This is the correlation which makes the market price of the tranche agree with the one-factor Gaussian copula model. Using the implied correlations, the delta for each tranche can be calculated. The delta measures the sensitivity of the tranche to uniform changes in the spreads in the underlying portfolio. Intuitively, we would expect that the implied correlation should be the same for each tranche, since it is a property of the underlying portfolio. However, the one-factor Gaussian copula model gives a different implied correlation for each tranche. Moreover, the implied correlations do not move uniformly together since the implied correlation for the equity tranche can increase more than the mezzanine tranche.

Even worse, sometimes it is not possible to calculate an implied correlation for a tranche using the one-factor Gaussian copula model. Kherraz (2006) gives a theoretical example of this and Finger (2009) gives the number of times that there has failed to be an implied correlation in the marketplace. These are all serious drawbacks of the one-factor Gaussian copula model, which were brought to the attention of market participants in a dramatic fashion in 2005. Discussions of these drawbacks can be found in Duffie (2008) and, particularly in relation to the events of May 2005 which we outline next, in Finger (2005) and Kherraz (2006).

In 2005, both Ford and General Motors were in financial troubles which threatened their credit ratings. On May 4 2005, an American billionaire Kirk Kerkorian invested US$870 million in General Motors. In spite of this, on May 5 2005, both Ford and General Motors were downgraded. Coming one day after Kerkorian's massive investment, the downgrade was not expected by the market. In the ensuing market turmoil, the mezzanine tranches moved in the opposite direction to what the delta-hedgers expected. Rather than the delta-hedge reducing their losses, it increased them.

The losses were substantial enough to warrant a front-page article Whitehouse (2005) on the Wall Street Journal which, like its successors Jones (2009) and Salmon (2009) more than three years later, went into some detail about the limitations of the model's uses. For us, this is sufficient evidence that people, both in industry and in academia, were well aware of the model's inadequacy facing complicated credit derivatives.

The broader lesson to take away is that of model uncertainty. This is the uncertainty about the choice of model. Naturally, as models are not perfect reflections of reality, we expect them to be wrong in varying degrees. However, we can attempt to measure our uncertainty about the choice of model. Cont (2006) proposes a framework to quantitatively measure model uncertainty which, while written in the context of derivative pricing, is of wider interest. In the context of hedging strategies, an empirical study of these using different models can be found in Cont and Kan (2008). Their study shows that hedging strategies are subject to substantial model risk.

5.3 Ability to do stress-testing

The use of a copula reduces the ability to test for systemic economic factors. A copula does not model economic reality but is a mathematical structure which fits historical data. This is a clear flaw from a risk-management point of view. At this point, we find it imperative to stress some points once more (they were mentioned on numerous occasions by the second author to the risk management community). First, copula technology is inherently static since there is no natural definition for stochastic processes. Hence any model based on this tool will typically fail to capture the dynamic events in fast-changing markets, of which the subprime crisis is a key example. Of course, model parameters can be made time dependent, but this will not do the trick when you really need the full power of the model, that is when extreme market conditions reign. Copula technology *is* useful for stress-testing: many companies would have shied away from buying the magical AAA-rated senior tranches of a CDO if they had stress-tested the pricing beyond the Gaussian copula model, for instance by using a Gumbel, Clayton or t-copula model. And finally, a comment on the term "calibration": too often we have seen that word appear as a substitute for bad or insufficient statistical modeling. A major contributor to the financial crisis was the totally insufficient macroeconomic modeling and stress-testing of the North American housing market. Many people believed that house prices could only go up and those risk managers who questioned that "wisdom" were pushed out with a desultory "you do not understand".

Copula technology is highly useful for stress-testing fairly static portfolios where marginal loss information is readily available, as is often the case in multi-line non-life insurance. The technology typically fails in highly dynamic and complex markets, of which the credit risk market is an example. More importantly, from a risk management viewpoint, it fails miserably exactly when one needs it.

6 The difficulties in valuing CDOs

6.1 Sensitivity of the mezzanine tranche to default correlation

Leaving aside the issue of modeling the joint default times, the problem of valuing the separate tranches in a CDO is a delicate one. In particular, the mezzanine tranche of a CDO is very sensitive to the correlation between defaults. We illustrate this with the following simple example.

Suppose that we wish to find the expected losses of a CDO of maturity 1 year which has 125 bonds in the underlying portfolio. Each bond pays a coupon of one unit which is re-distributed to the tranche-holders. For simplicity, we assume that if a bond defaults, then nothing is recovered.

We value the first three (most risky) CDO tranches, which we call the equity, mezzanine and senior tranches. The equity tranche is exposed to the first 3 defaults in the underlying portfolio of bonds. The mezzanine tranche is exposed to the next 3 defaults and the senior tranche is exposed to the subsequent 3 defaults in the underlying portfolio of bonds. Therefore, 6 defaults must occur in the underlying portfolio before further defaults affect the coupon payments to the senior tranche.

Instead of modeling the default times (T_i), we make the simple assumption that each of the underlying bonds has a fixed probability of defaulting within a year. We assume that the correlation between each pair of default events is identical. We calculate the expected loss on each tranche at the end of the year as follows

$$\text{Expected loss on equity tranche} = \sum_{k=1}^{3} k\mathbb{P}\left[k \text{ bonds default by the end of the year}\right],$$

$$\text{Expected loss on mezzanine tranche} = \sum_{k=1}^{3} k\mathbb{P}\left[k+3 \text{ bonds default by the end of the year}\right],$$

$$\text{Expected loss on senior tranche} = \sum_{k=1}^{3} k\mathbb{P}\left[k+6 \text{ bonds default by the end of the year}\right].$$

In Figures 4(a)-4(d), we show for various probabilities of default how the expected losses on each tranche vary as we change the pairwise correlation between the default events.

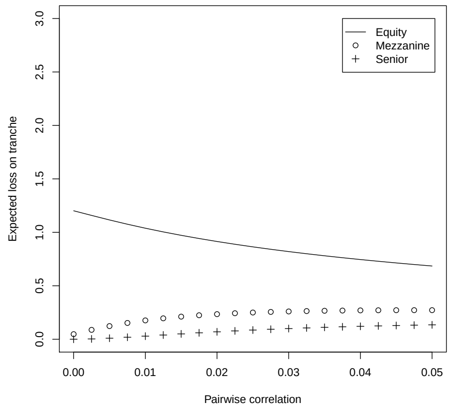
For each plot, we see that the expected loss on the equity tranche decreases as the pairwise correlation increases. The reason is that as correlation increases, it is more likely that either many bonds default or many bonds do not default. Since any defaults cause losses on the equity tranche, the increase in probability that many defaults do not occur tends to decrease the expected loss on the equity tranche. Conversely, the expected loss on the senior tranche increases as the pairwise correlation increases. More than 6 bonds must default before the senior tranche suffers a loss. An increase in correlation makes it more likely that many bonds will default and this causes the expected loss on the senior tranche to increase. However, for the mezzanine tranche there is no clear relationship emerging. In Figure 4(a), the expected loss on the mezzanine tranche increases as the pairwise correlation increases. But in Figure 4(d) the opposite happens. This simple example illustrates the sensitivity of the mezzanine tranche. This point is also highlighted by McNeil et al. (2005, Figure 9.3).

We restate the remark in Duffie (2008) that the modeling of default correlation is currently the weakest link in the risk measurement and pricing of CDOs. Given this weakness and the sensitivity of the mezzanine tranche to the default correlation, it is clear that there is a lot of uncertainty in the valuation of CDOs. Linking this uncertainty to the astronomical volumes of CDOs in the marketplace, it is not surprising that the credit crisis had to erupt eventually.

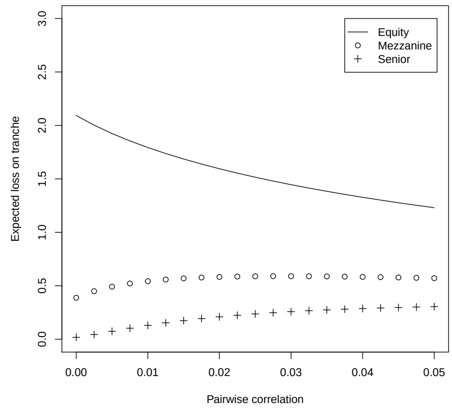
6.2 Squaring the difficulty: CDO-squared

Now suppose we wish to value a credit derivative called a CDO-squared. This is a CDO where the underlying portfolio itself consists of CDO tranches. Moreover, these tranches are typically the mezzanine tranches. This is because the mezzanine tranches are difficult to sell: they are too risky for many investors, since they are often BBB-rated, yet they are not risky enough for other investors, like hedge funds.

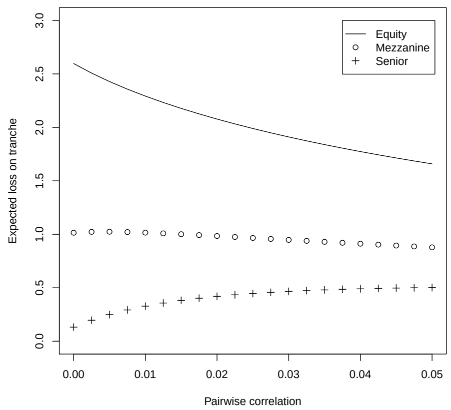
Like any CDO, the CDO-squared can be tranched. However, the valuation of the CDO-squared and its tranches are fraught with complexity. As we saw in our simple example above,



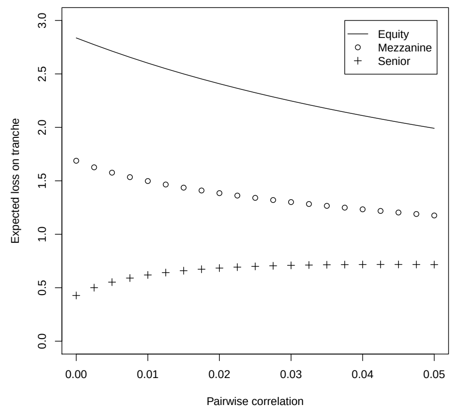
(a) individual default probability of 1%.



(b) individual default probability of 2%.



(c) individual default probability of 3%.



(d) individual default probability of 4%.

Figure 4: Expected loss on a CDO with 125 underlying names, each with identical pairwise correlation, as a function of the pairwise correlation value. The line shows the expected loss on the equity tranche ($0-3$ units of exposure). The circles show the expected loss on the mezzanine tranche ($3-6$ units of exposure) and the crosses show the expected loss on the senior tranche ($6-9$ units of exposure).

valuing each mezzanine tranche in the underlying portfolio is difficult. Valuing the tranches of the CDO-squared, which has between 100 and 200 mezzanine tranches in the underlying portfolio, is much more difficult. If we assume that there are 150 mezzanine tranches in the underlying portfolio of the CDO-squared, and each mezzanine tranche is based on a portfolio of 150 bonds, then this means modeling 22,500 bonds. It is also quite likely that some of these bonds are the same, since it is likely given the large numbers involved that some of the mezzanine tranches have the same bonds in their underlying portfolio. Given these problems, it is questionable whether a CDO-squared can be valued with any reasonable degree of accuracy.

Moreover, doing due diligence on such on a CDO-squared is not feasible, as Haldane (2009a) points out. The contracts governing each of the mezzanine tranches in the underlying portfolio are around 150 pages long. Assuming that there are 150 mezzanine tranches in the underlying portfolio, this means that there are 22,500 pages to read, not including the contract governing the CDO-squared itself. On top of that, a typical computer program mapping the cashflow of just one CDO-like structure can be thousands of lines of computer code long (often in an Excel environment), which has the attendant possibility of programming errors creeping in.

For the purposes of risk management, determining the systemic factors which the CDO-squared is exposed to would be impossible, given the number of financial instruments on which a CDO-squared is based. This means that the validity of scenario testing on such instruments is doubtful.

Even ignoring the valuation difficulties, the economic value of instruments such as CDOs-squared are questionable. As Hellwig (2009, page 153) argues in relation to mortgage-backed securities, which we recall are a type of CDO, if the securitization of mortgage-backed securities had been properly handled then there should be no significant benefits from additional diversification through a mortgage-backed security-squared. Such benefits could be gained by investors putting multiple mortgage-backed securities into their own portfolio. Furthermore, he points out that the scope for moral hazard was increased as the chain of financial intermediation increased, from the mortgage originators to the buyers of the mortgage-backed security-squared. This was also mentioned by Stiglitz (2008).

If the problems with valuing and managing the risk involved in CDOs-squared seem insurmountable, it should give the reader pause for thought that instruments called CDO-cubed exist. These are again CDOs which are based on the mezzanine tranches of CDO-squareds.

It is clear that at this level, credit risk management did reach a level of perversity which questions seriously any socio-economic benefit of such products and puts to shame the whole quant profession. In the end, from a product development point-of-view, total opaqueness reigned. The real question is not about a particular model used or misused in the pricing of such products but much more about the market structures which allowed such nonsensical products to be launched in such volumes. Already around 2005, it was noticed that overall risk capital as measured by VaR was down. They key question some risk managers asked was "But where is all the credit risk hiding?" By now, unfortunately we know!

7 Alternative approaches to valuing CDOs

Reading the articles in the Financial Times and Wired Magazine, one would think that the Gaussian copula model was the only method used to value credit derivatives. This is far from the truth. While this model is widely used, there are many alternatives to it which are also used in industry. In fact, there are entire books written on models for credit derivatives, such as Bielecki and Rutkowski (2004), Bluhm and Overbeck (2007), O'Kane (2008) and Schönbucher (2003).

Broadly, there are two main classes of models used in credit risk modeling: structural models and hazard rate models. The structural approach, sometimes called the firm-value approach, models default via the dynamics of the value of the firm. This is based on the Merton (1974) approach, which models default via the relationship of the value of the firm's assets to its liabilities at the end of a given time period. The general idea is that default occurs if the asset value is less than the firm's liabilities. The one-factor Gaussian copula model is an example of a structural model. Other examples of structural models used in industry are the CreditMetrics model, publicized by JP Morgan in 1997, and the KMV model, first developed by the company KMV and now owned by Moody's.

Hazard rate models, also commonly called reduced-form models, attempt to model the infinitesimal chance of default. In these models, the default is some exogenous process which does not depend on the firm. An industry example of a reduced-form model is CreditRisk+, which was proposed by Credit Suisse Financial Products in 1997.

We do not go into details about these or, indeed, alternative models. Instead, as a starting point we direct the interested reader to the books cited at the start of this section.

It is fair to say that in the wake of the Crisis, the approaches used by the various market participants to value CDOs can be broadly summarized as follows:

- for synthetic (corporate credit) CDOs, the notion of the base correlation curve of the Gaussian copula is used. Since the Crisis, there has been an evolution to simpler models and simpler structures;
- for cash CDOs and asset-backed securities, a more detailed modeling of the cashflow waterfall together with Monte Carlo modeling of the underlying asset pools is used;
- rating agencies models for structured assets have become much simpler, that is concentrating on fewer scenarios with extreme stress shocks, and
- the regulators put a lot of importance on stress-testing and the Holy Grail still remains liquidity risk.

Whereas we applaud the consensus on simple, economically relevant products, we are less convinced that simple models will be part of the answer to this Crisis. Even for fairly straightforward credit products, rather advanced quantitative techniques are needed. We need better models and for people to understand the assumptions and limitations of the models they use. The call is not for "less mathematics" but rather for "a better understanding of the necessary mathematics involved".

8 A failure of risk management: AIG

CDSs have also attracted a lot of attention with respect to the financial crisis, particularly in association with the insurance company AIG. In September 2008, AIG was on the verge of bankruptcy due to cashflow problems stemming from its CDS portfolio, before being saved by the US government. We explain below how AIG came close to bankruptcy and draw some relevant lessons from their risk management failures.

8.1 The AIG story

The sad story of AIG, a company of around 100,000 employees brought to its knees by a small subsidiary of 400 employees, is an example of a failure of risk management, both at the division and the group level. AIG almost went bankrupt because it ran out of cash. We do not concern ourselves here with regulation, but focus on the risk management side of the AIG story. A summary of the AIG bailout by Sjostrom (2009) is well-worth reading, and this is the basis of what we write below about AIG. We have supplemented this with other sources, mainly from AIG regulatory filings and statements submitted to a US Senate Committee hearing on AIG.

AIG is a holding company which, through its subsidiaries, is engaged in a broad range of insurance and insurance-related activities in more than 130 countries. Half of its revenues come from its US operations. As at December 31 2007, AIG had assets of US$1,000 billion dollars. This is just under half the Gross Domestic Product of France.

Despite the insurance business being a heavily regulated business, in September 2008 AIG was on the verge of bankruptcy due to cashflow problems. These cashflow problems came not from its insurance business, but from its CDS portfolio. AIG operated its CDS business through subsidiaries called AIG Financial Products Corp and AIG Trading Group, Inc and their respective subsidiaries. Collectively, these subsidiaries are referred to as AIGFP. As the parent company, AIG fully guaranteed any liabilities arising from AIGFP doing its regular business. For the most part, AIGFP sold protection on super-senior tranches of CDOs, where the underlying portfolio consisted of loans, debt securities, asset-backed securities and mortgage-backed securities. Super-senior tranches rank above AAA-rated tranches in the CDO tranche hierarchy, so that the super-senior tranche of a CDO is less risky than the AAA-rated tranche.

AIGFP believed that the money it earned from the CDSs were a free lunch because their risk models indicated that the underlying securities would never go into default. AIG (2006) states that "the likelihood of any payment obligation by AIGFP under each transaction is remote, even in severe recessionary market scenarios". The New York Times quotes the head of AIGFP as saying in August 2007 that "it is hard for us, without being flippant, to even see a scenario within any kind of realm of reason that would see us losing one dollar in any of those transactions."; see Morgenson (2008). Indeed, as at March 5 2009, according to AIG's primary regulator, there had been no credit losses on the CDSs sold on super-senior tranches of CDOs; see Polakoff (2009). By credit losses, we mean the losses caused by defaults on the super-senior tranches that the CDSs were written on. Despite this, by writing the CDSs, AIGFP and hence also AIG, exposed themselves to other risks which entailed potentially large financial obligations.

The buyer of a CDS is exposed to the credit risk of the seller. If the reference asset defaults then there is no guarantee that the seller can make the agreed payoff. Similarly, the seller is exposed to the credit risk of the buyer: the buyer may fail to make the regular premium payments. To reduce this risk, the counterparties to the CDS contract may be required to post collateral. The industry standard documentation which governs CDSs is produced by the International Swaps and Derivatives Association ("ISDA"). There are four parts to each ISDA contract, the main part being the ISDA Master Agreement. Another of these parts is the Credit Support Annex, which regulates the collateral payments. Collateral payments may be required due to changes in the market value of the reference asset or changes in the credit rating of the counterparties. Further, the Credit Support Annex is an optional part of the ISDA contract.

As at December 31 2007, the net notional amount of CDSs sold by AIGFP was US$527 billion. The majority of these CDSs were sold before 2006. Some US$379 billion of these CDSs were sold to provide mostly European banks with regulatory capital relief, rather than for risk transfer. We call these "regulatory capital CDSs". These CDSs were written on assets like corporate loans and prime residential mortgages, which were held by European banks. By buying a CDS from AIGFP, the banks transferred the credit risk of the loans to AIGFP. Up to 2005, as AIG was AAA-rated and fully guaranteed its subsidiary AIGFP, the European banks were permitted under their banking regulations to reduce the amount of regulatory capital to be set aside for their loans. Meanwhile AIG, being subject to different regulations and despite being exposed to the losses on the loans, did not have to hold the full value of the European regulatory capital.

Type of CDS	Notional amount at	Collateral posting at			
	Dec 31 2007	Dec 31 2007	Mar 31 2008	Jun 30 2008	Sep 30 2008
	US$ million	US$ million			
Regulatory capital	379,000	0	212	319	443
Corporate loans	70,000	161	368	259	902
Multi-sector CDO	78,000	2,718	7,590	13,241	31,469
Total	527,000	2,879	8,170	13,819	32,814

Table 1: Collateral postings by AIG to its counterparties in respect of the three types of CDSs it wrote.

The remaining notional amount of CDSs sold by AIGFP was split almost evenly between those written on portfolios of corporate debt and collaterized loan obligations (US$70 billion), which we call "corporate loan CDSs", and those written on portfolios of multi-sector CDOs (US$78 billion), which we call "multi-sector CDO CDSs". A multi-sector CDO is a CDO with an underlying portfolio consisting of loans, asset-backed securities and mortgage-backed securities. This means that a multi-sector CDO is exposed to portfolios of assets from multiple sectors, such as residential mortgage loans, commercial mortgages, loans, auto loans and credit card receivables. AIG wrote protection on mostly the super-senior tranches of these multi-sector CDOs. Unfortunately for AIG, many of the multi-sector CDOs on which it sold CDSs were based on residential mortgage-backed securities, whose assets included subprime mortgage loans. Typically about 50% of the multi-sector CDOs on which AIG wrote CDSs was exposed to subprime mortgages; see AIG (2007b, page 28). By 2005, according to another presentation by AIG, they made the decision to stop committing to any new multi-sector CDOs which had subprime mortgages in their underlying portfolios; see AIG (2007a, Slide 16). They also saw evidence that underwriting standards in subprime mortgages were beginning to decline in a material way.

For several of its counterparties, AIGFP had collateral arrangements nearly all of which were written under a Credit Support Annex to an ISDA Master Agreement. The intent of these arrangements was to hedge against counterparty credit risk exposures. The amount of collateral was primarily based either on the replacement value of the derivative or the market value of the reference asset. It was also affected by AIG's credit rating and that of the reference assets.

In mid-2007, the defaults by borrowers of subprime mortgages started to ripple down the chain of financial contracts based on them. This led to massive write-downs in AIGFP's portfolio, totalling US$11.2 billion in 2007 and US$19.9 billion for the first nine months of 2008. More importantly, the effects of the defaults were collateral posting requirements. As the values of the CDOs on which AIGPF had sold CDSs declined, AIGFP was required to post more and more collateral. Between July 1 2008 and August 31 2008, AIGFP either posted or agreed to post US$6 billion in collateral. This represented 34% of the US$17.6 billion that AIG had in cash and cash equivalents available on July 1 2008. In Table 1, we show the collateral postings on AIGFP's super-senior tranche CDS porfolio; see AIG (2009, page 144) and AIG (2008, page 122). The first column of figures in Table 1 relates to the nominal amounts of the CDS at December 31 2007, which we discussed above. These are shown to provide the reader with a sense of the magnitude of the multi-sector CDO CDS portfolio relative to the other two CDS portfolios. Moving from left to right across the table, we see that the amount of collateral postings on the multi-sector CDO CDSs increases dramatically and comprises 96% of the total amount of collateral postings of US$32.8 billion as at September 30 2008.

Adding to AIG's cash woes was its Securities Lending Program, which was a centrally managed program facilitated by AIG Investments. Through this program, certain of AIG's insurance companies lent securities to other financial institutions, primarily banks and brokerage firms. In exchange, AIG received partially cash collateral equal to 102% of the fair value of the loaned securities from the borrowers. In 2008, as the borrowers learned of AIG's cashflow problems, they asked for their collateral back in exchange for the return of the securities. From September 12 2008 to September 30 2008, borrowers demanded the return of around US$24 billion in cash; see Dinallo (2009).

A typical securities lending program reinvests the collateral in short duration instruments such as treasuries and commercial paper. AIG's Securities Lending Program did not do this. Instead, they invested most of the collateral in longer duration, AAA-rated residential mortgage-backed securities; see AIG (2008, page 108) and Dinallo (2009). As the effects of the defaults of subprime mortgage holders continued to ripple through the financial markets, these mortgage-backed securities declined substantially in value and became illiquid. As a result, the Securities Lending Program had insufficient funds to pay back the collateral it had taken in exchange for lending AIG's securities. AIG was forced to transfer billions in cash to the Securities Lending Program to pay back the collateral.

By early September 2008, AIG's cash situation was dire. It was unable to raise additional capital due to the seizing up of liquidity in the markets. As a result of all these events, AIG was downgraded. The downgrade triggered additional collateral postings in excess of US$20 billion on the CDSs sold by AIGFP.

On September 16 2008, the Federal Reserve Board, with the support of the U.S. Department of the Treasury, announced that it had authorized the Federal Reserve Bank of New York to lend up to US$85 billion to AIG. This was to allow AIG to sell certain of its businesses in an orderly manner, with the least possible disruption to the overall economy. According to a Federal Reserve Board press release, it had determined that, in the current circumstances, a disorderly failure of AIG could add to already significant levels of financial market fragility and lead to substantially higher borrowing costs, reduced household wealth, and materially weaker economic performance; see Federal Reserve Board (2008a). By October 1 2008, AIG had drawn down approximately US$61 billion from the credit facility; see Kohn (2009).

In November 2008, it appeared that another downgrade of AIG's credit rating was looming. This would have triggered additional collateral calls, and would probably have led to the collapse of AIG. Wishing to avoid this, the Federal Reserve Board and the U.S. Department of the Treasury announced a series of mitigating actions on November 10 2008; see Kohn (2009). In the press release, the Federal Reserve Board stated that these new measures were to establish a more durable capital structure, resolve liquidity issues, facilitate AIG's execution of its plan to sell certain of its businesses in an orderly manner, promote market stability, and protect the interests of the US government and taxpayers; see Federal Reserve Board (2008b).

AIG's net losses for 2008 were about US$99 billion, of which approximately US$62 billion was in the last quarter of 2008. By August 2009, according to the magazine The Economist, the total value of US government help that was distributed to AIG was around US$145 billion, which is equivalent to about 1% of the GDP of the US.

8.2 Risk management issues

The risk management failings at AIG were seen at other global firms. For many firms, the shortcomings of the risk management practices were translated into huge financial losses. In AIG's case, they almost bankrupted the firm.

The biggest failure of risk management at AIG was in not appreciating the risk inherent in the super-senior tranches of the multi-sector CDOs. Indeed, Rutledge (2008) points to exposure to the US subprime market as driving the losses at some major global financial organizations, with the key driver being exposure to the super-senior tranches of CDOs. Firms thought that the super-senior tranches were practically risk-free. According to Rutledge (2008), prior to the third quarter of 2007, few firms used valuation models to model their exposure to super-senior tranches related to subprime mortgages. It seems that AIG was also guilty of this to some degree. Until 2007, it did not model the liquidity risk that it was exposed to from writing CDSs with collateral posting provisions; see Mollenkamp et al. (2008) and St Denis (2008).

Inadequate mathematical modeling meant that AIG were not able to quantify properly the risks in their CDS portfolio. Moreover, by ignoring the liquidity risk they were exposed to until after the Crisis had begun, AIG could not take early action to reduce their exposure to potential collateral postings. By the end of 2007, when substantial declines in the subprime mortgage market were already occurring, it would have been very expensive to do this.

Some firms survived the Crisis better than others. Based on a sample of eleven global banking organizations and securities firms, Rutledge (2008) identified the key risk management practices which differentiated the performance of firms during the Crisis. From them, we take some relevant lessons.

One is the need for firms to embrace quantitative risk management. This provides them with a means of quantifying and aggregating risks on a firm-wide basis. Relying on human judgment, while still essential, is not enough, especially in huge firms like AIG.

The mathematics to measure many of the risks that firms face are well-developed. For example, while the Crisis is considered an extreme event, there does exist a mathematical framework to assess the risk of such events. As mentioned before, Extreme Value Theory has been an active area of mathematical research since the 1950s, with early publications going back to the 1920s, and has been applied to the field of finance for over 50 years. There has been much research on the quantification of the interdependence and concentration of risks, and the aggregation of risks. A starting place for learning about the concepts and techniques of quantitative risk management is McNeil et al. (2005).

9 Summary

The prime aim of this paper is not to give a detailed overview of all that went wrong leading up to and during the Crisis, but instead a rather personal account of the important issues of which an actuarial audience should be aware. The pessimist may say "The only thing we learn from history is that we learn nothing from history", a quote attributed to Friedrich Hegel, among others. However, we hope that future generations of actuarial students will read this paper even after the Crisis has become part of economic history and avoid the errors of the current generation. As a consequence, we have left out numerous important aspects of the Crisis, but hopefully we have compensated that oversight with some general references which we found useful. By now, new publications appear every day which makes choosing "what to read" very difficult indeed.

We have concentrated on two aspects of the Crisis which are relevant from an actuarial viewpoint: the use of an actuarial formula (that is, the Gaussian copula model) far beyond the level it was originally created for and the near-bankruptcy of an insurance giant, AIG, in part because of the failure of internal risk management practices.

If we consider the concepts behind the numerous acroynms like RM, IRM, ERM, QRM,..., the message is screaming out to us: we need to learn from these recent events. As stated on several occasions in our paper, it is totally preposterous to blame one man or one model for all or part of the Crisis. In reaction to the Financial Times article Jones (2009) we wrote the following letter:

> Dear Sir
>
> The article "Of couples and copulas", published on 24 April 2009, suggests that David Li's formula is to blame for the current financial crisis. For me, this is akin to blaming Einstein's $E = mc^2$ formula for the destruction wreaked by the atomic bomb. Feeling like a risk manager whose protestations of imminent danger were ignored, I wish to make clear that many well-respected academics have pointed out the limitations of the mathematical tools used in the finance industry, including Li's formula. However, these warnings were either ignored or dismissed with a desultory response: "It's academic".
>
> We hope that we are listened to in the future, rather than being made a convenient scapegoat.
>
> Yours, etc

It was unfortunately not published!

As actuaries, both in practice and in academia, we must think more carefully on how to communicate the use and potential misuse of the concepts, techniques and tools in use in the risk management world. Two things are clear from the Crisis: "we were not listened to", but also "we did not know". First on the latter: as academics, we have to become much more involved with macroeconomic reality. The prime example is the astronomical nominal value invested in credit derivatives and why our risk management technology did not have all the red warning lights flashing much earlier. Second, as a final remark, we want to say something on the former "we were not listened to". This leads us to the problem of communication, a stage on which actuaries are not considered the best actors. Too many papers are currently written, post-event, on "why we got into this mess". We need to learn why certain warnings were not heeded. Below we give some personal recollections on this matter.

In 2001, the second author contributed to the 17 page document Daníelsson et al. (2001) which was mailed in the same year as an official reply to the Basel Committee in the wake of the new Basel II guidelines. The academic authors were a mixture of microeconomists, macroeconomists, econometricians and actuaries. Due to its relevance to the Crisis, we quote from the Executive Summary:

> It is our view that the Basel Committee for Banking Supervision, in its Basel II proposals, has failed to address many of the key deficiencies of the global financial regulatory system and even created the potential for new sources of instability. . . .
>
> - The proposed regulations fail to consider the fact that risk is endogenous. Value-at-Risk can destabilise an economy and induce crashes when they would not otherwise occur.
> - Statistical models used for forecasting risk have been proven to give inconsistent and biased forecasts, notably under-estimating the joint downside risk of different assets. . . .
> - Heavy reliance on credit rating agencies for the standard approach to credit risk is misguided as they have been shown to provide conflicting and inconsistent forecasts of individual clients' creditworthiness. . . .
> - Financial regulation is inherently procyclical. Our view is that this set of proposals will, overall, exacerbate this tendency significantly. In so far as the purpose of financial regulation is to reduce the likelihood of systemic crisis, these proposals will actually tend to negate, not promote this useful purpose.

The introduction of Daníelsson et al. (2001) concludes with

> Perhaps our most serious concern is that these proposals, taken altogether, will enhance both the procyclicality of regulation and the susceptibility of the financial system to systemic crises, thus negating the central purpose of the whole exercise. *Reconsider before it is too late.*

The authors of Daníelsson et al. (2001) could not have been more forceful and explicit, and yet the reaction from the Basel Committee was basically nil. Along these lines, every beginning actuary should read Markopolos (2005). In this 2005 document addressed to the SEC, one of several by Markopolos over the period 2000-2008, Markopolos proves that Madoff Investment Securities is a Ponzi scheme, and yet the SEC did nothing. We do not enter into the reasons why this happened, but simply note that every quantitatively trained finance expert would have immediately reacted upon reading the very detailed and point-by-point accusation made in Markopolos (2005).

Though the above example is somewhat discouraging, we have to keep vigilant and communicate in a forceful way those actuarial, technical findings which are of societal importance. We can do this through our publications, societies and conferences. A key research theme that we have to address more explicitly going forward is that of model uncertainty. From a technical perspective, this means explaining the precise conditions under which a particular model can be used. But at the same time, we have to be aware, or become aware, where and how these models *are* used in a non-trivial way.

It always pays to be be humble in the face of real application. Shakespeare's Hamlet formulated this as follows: "There are more things in heaven and earth, Horatio, than are dreamt of in your philosophy." New generations of actuarial students will have to use the tools and techniques of quantitative risk management wisely in a world where the rules of the game will constantly change. A message we would like to give them on this path is to be always scientifically critical, socially honest and to adhere to the highest ethical principles, especially in the face of temptation...which will come!

Acknowledgements

The authors would like to extend their gratitude to the organisers of the 39th International ASTIN Colloquium in Helsinki, June 1-4 2009, for having invited Paul Embrechts to present an early version of this paper. Catherine Donnelly thanks RiskLab at ETH Zürich for financial support. Research was partially completed while Paul Embrechts was visiting the Institute of Mathematical Sciences, National University of Singapore in November-December 2009.

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