p-LAPLACIAN PROBLEMS WITH JUMPING NONLINEARITIES

BRYAN P. RYNNE

ABSTRACT. We consider the p-Laplacian boundary value problem

$$-(\phi_p(u'(x))' = f(x, u(x), u'(x)), \quad \text{a.e. } x \in (0,1), \tag{1}$$

$$c_{00}u(0) + c_{01}u'(0) = 0, \quad c_{10}u(1) + c_{11}u'(1) = 0, \tag{2}$$

where $p > 1$ is a fixed number, $\phi_p(s) = |s|^{p-2}s$, $s \in \mathbb{R}$, and for each $j = 0, 1$, $|c_{j0}| + |c_{j1}| > 0$. The function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function satisfying, for $(x, s, t) \in [0,1] \times \mathbb{R}^2$,

$$\psi_\pm(x)\phi_p(s) - E(x,s,t) \le f(x,s,t) \le \Psi_\pm(x)\phi_p(s) + E(x,s,t), \quad \pm s \ge 0,$$

where $\psi_\pm$, $\Psi_\pm \in L^1(0,1)$, and E has the form $E(x,s,t) = \zeta(x)e(|s|+|t|)$, with $\zeta \in L^1(0,1)$, $\zeta \ge 0$, $e \ge 0$ and $\lim_{r\to\infty} e(r)r^{1-p} = 0$. This condition allows the nonlinearity in (1) to behave differently as $u \to \pm\infty$. Such a nonlinearity is often termed *jumping*.

Related to (1), (2) is the problem

$$-(\phi_p(u')' = a\phi_p(u^+) - b\phi_p(u^-) + \lambda\phi_p(u), \quad \text{in } (0,1), \tag{3}$$

together with (2), where a, $b \in L^1(0,1)$, $\lambda \in \mathbb{R}$, and $u^\pm(x) = \max\{\pm u(x), 0\}$ for $x \in [0,1]$. This problem is 'positively-homogeneous' and jumping. Values of λ for which (2), (3) has a non-trivial solution u will be called *half-eigenvalues*, while the corresponding solutions u will be called *half-eigenfunctions*.

We show that a sequence of half-eigenvalues exists, the corresponding half-eigenfunctions having certain nodal properties, and we obtain certain spectral and degree theoretic properties of the set of half-eigenvalues. These properties lead to existence and non-existence results for the problem (1), (2). We also consider a related bifurcation problem, and obtain a global bifurcation result similar to the well-known Rabinowitz global bifurcation theorem. This then leads to a multiplicity result for (1), (2).

When the functions a and b are constant the set of half-eigenvalues is closely related to the 'Fučík spectrum' of the problem, and equivalent solvability results are obtained using the two approaches. However, when a and b are not constant the half-eigenvalue approach yields stronger results.

1. INTRODUCTION

We consider the p-Laplacian boundary value problem

$$-(\phi_p(u'(x))' = f(x, u(x), u'(x)), \quad \text{a.e. } x \in (0,1), \tag{1.1}$$

$$c_{00}u(0) = c_{01}u'(0), \quad c_{10}u(1) = c_{11}u'(1), \tag{1.2}$$

where $p > 1$ is a fixed number, $\phi_p(s) = |s|^{p-2}s$, $s \in \mathbb{R}$, and for each $j = 0, 1$, $|c_{j0}| + |c_{j1}| > 0$. The function $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a Carathéodory function, that is, $f(x,s,t)$ is measurable in x for every fixed $(s,t) \in \mathbb{R}^2$, and continuous in (s,t) for a.e. $x \in (0,1)$.

In addition, for $(x, s, t) \in [0, 1] \times \mathbb{R}^2$,

$$\psi_{\pm}(x)\phi_p(s) - E(x, s, t) \leq f(x, s, t) \leq \Psi_{\pm}(x)\phi_p(s) + E(x, s, t), \quad \pm s \geq 0, \tag{1.3}$$

where $\psi_{\pm}$, $\Psi_{\pm} \in L^1(0, 1)$, and E has the form $E(x, s, t) = \zeta(x)e(|s| + |t|)$, with $\zeta \in L^1(0, 1)$, $\zeta \geq 0$ and $e : [0, \infty) \to [0, \infty)$ is increasing with $\lim_{r\to\infty} e(r)r^{1-p} = 0$. Condition (1.3) allows the nonlinearity in (1.1) to behave differently as $u \to \pm\infty$. Such a nonlinearity is often termed *jumping*.

When $p = 2$ the solvability of (1.1), (1.2) (or with periodic boundary conditions) has been studied extensively. Solvability conditions for both the periodic and the separated boundary condition problems have been expressed in terms of either the 'Fučík spectrum' or the 'half-eigenvalues' of the problem. Most of the original results on the Fučík spectrum approach were obtained in [8], and these results are described in detail in the monograph [13]. Both these concepts are discussed in [23] and a relatively detailed comparison of the two approaches is given there. The periodic problem is discussed in [4], and a brief survey of both approaches is given, together with a large, but by no means exhaustive, bibliography of both the periodic and the separated cases. Since our main interest here is the case $p \neq 2$ we will not give further references to the extensive literature on the case $p = 2$.

The case $p \neq 2$ has also received a great deal of attention, although not so much as the case $p = 2$. For instance, the papers [1], [5], [10], [11], [12], [13], [16], [17], [18], [20], [25], [26], all obtain existence results for problems of the above form, although most of these papers impose considerable additional assumptions, of various types, on f. In this paper we extend most of the results obtained in these papers to the more general setting described above (in fact, we extend the results for the case $p = 2$, under fairly general condition on f, to the case $p \neq 2$). We will compare our results with previous results in more detail in Section 7 below. For now, we give a brief description of the contents of the paper.

In Section 3 we define the idea of a 'half-eigenvalue' for a related, 'positively-homogeneous' problem, and we show that half-eigenvalues exist and obtain their basic properties. Additionally, solvability and non-solvability conditions are given for a simplified, positively-homogeneous form of (1.1), (1.2). These conditions are expressed in terms of the half-eigenvalues. In Sections 4 and 5 we extend the solvability and non-solvability results of Section 3 to the problem (1.1), (1.2). Finally, in Section 6, we consider a bifurcation problem related to (1.1), (1.2) (with f small near $u = 0$), and we obtain a global bifurcation result (similar to Rabinowitz' well-known global bifurcation theorem), and then use this to obtain a result on the multiplicity of solutions of (1.1), (1.2).

Finally, in this section, we note that if we define $\widetilde{f}(x, s, t) := f(x, s, t)/\phi_p(s)$, $s \neq 0$, then (1.3) implies that

$$\psi_{\pm}(x) \leq \liminf_{s\to\pm\infty} \widetilde{f}(x, s, t) \leq \limsup_{s\to\pm\infty} \widetilde{f}(x, s, t) \leq \Psi_{\pm}(x), \quad x \in [0, 1]. \tag{1.4}$$

In many previous papers (both with $p = 2$ and with $p \neq 2$) the conditions on f are expressed as inequalities of this form, together with some uniformity condition on the lim sup and lim inf in (1.4). Condition (1.3) can be regarded as encapsulating such a uniformity condition. More restrictively, many papers essentially assume that

$$\psi_{\pm} = \Psi_{\pm}, \tag{1.5}$$

and hence the limits

$$\widetilde{f}^{\infty}_{\pm}(x) := \lim_{s\to\pm\infty} \widetilde{f}(x,s,t) = \psi_{\pm}(x), \quad x \in [0,1],$$

exist. We do not assume this in general, but some of our results are sharper in this case, and we comment on this further below. More restrictively still, most of the cited papers assume that the nonlinearity in (1.1) is independent of u', that is, it has the form $f(x,u)$. Zhang [25] allows f to depend on u'.

2. Preliminaries

For $j \ge 0$, let $C^j[0,1]$ denote the space of j times continuously differentiable functions on $[0,1]$, with the usual sup-norm $|\cdot|_j$, and let $L^1(0,1)$ denote the space of integrable functions on $[0,1]$, with the usual norm $\|\cdot\|_1$ (throughout, all function spaces will be real). Let

$$Y := C^1[0,1], \quad X := \{u \in Y : u \text{ satisfies } (1.2)\}, \quad Z := L^1(0,1)$$

(X will have the norm $|\cdot|_1$). We let ϕ_p, $f : Y \to Z$ denote the Nemitskii operators induced by the functions ϕ_p, f. These operators are continuous, see the proof of Theorem 2.2 in [2].

A *solution* of the problem (1.1), (1.2), and other problems below, is a function $u \in X$ such that u and $\phi_p(u')$ are absolutely continuous on $[0,1]$ and (1.1) holds for a.e. $x \in (0,1)$. In view of this, we define $\Delta_p : D(\Delta_p) \subset X \to Z$ by

$$D(\Delta_p) := \{u \in X : \ \phi_p(u') \text{ is a.c. on } [0,1]\ \},$$
$$\Delta_p(u) := -\phi_p(u')', \quad u \in D(\Delta_p).$$

The problem (1.1), (1.2), can now be rewritten as

$$\Delta_p(u) = f(u), \quad u \in D(\Delta_p). \tag{2.1}$$

For $u \in Y$, let

$$BC_j(u) := c_{j0}u(j) - c_{j1}u'(j), \quad j = 0,\ 1;$$

clearly, $X = \{u \in Y : BC_j(u) = 0,\ j = 0,1\}$. For $(u,h) \in Y \times Z$, $x \in [0,1]$, let

$$I(h)(x) := \int_x^1 h(y)\,dy,$$
$$T_p(u,h)(x) := u(0) + BC_0(u) + \int_0^x \phi_p^{-1}\Big\{\phi_p(u'(1)) + \phi_p(BC_1(u)) + I(h)(y)\Big\}\,dy.$$

It is clear that $I(h) \in C^0[0,1]$, $T_p(u,h) \in Y$ and the mappings $I : Z \to C^0[0,1]$, $T_p : Y \times Z \to Y$ are continuous. Furthermore,

$$u \in Y \text{ and } u - T_p(u,h) = 0 \iff u \in D(\Delta_p) \text{ and } \Delta_p(u) = h. \tag{2.2}$$

This equivalence can readily be obtained by differentiating or integrating the relevant equations, and by noting that

$$u(0) - T_p(u,h)(0) = 0 \implies BC_0(u) = 0,$$
$$u'(1) - T_p(u,h)'(1) = 0 \implies BC_1(u) = 0.$$

Some compactness properties of I and T_p will be required. A set $B \subset Z$ will be said to be *equi-integrable* if there exists $h_B \in Z$ such that for any $h \in B$, $|h(x)| \le h_B(x)$ for a.e. $x \in [0,1]$. Weak convergence in Z will be denoted by $\rightharpoonup$.

We note also that if $(u, h) \in C^0[0,1] \times Z$ and the derivatives $u_n'(0)$, $u_n'(1)$ exist, then $T_p(u,h)$ can be defined as above and $T_p(u,h) \in Y$.

Lemma 2.1. *(i) If $B \subset Z$ is equi-integrable then it is weakly sequentially compact.*
(ii) Suppose that (h_n) is a sequence in Z such that the set $\{h_n\} \subset Z$ is equi-integrable and $h_n \rightharpoonup h_\infty$. Then $I(h_n) \to I(h_\infty)$ in $C^0[0,1]$.
(iii) Suppose that (h_n) satisfies the hypothesis of part (ii), or that $h_n \to h_\infty$ in Z. Also suppose that (u_n) is a sequence in $C^0[0,1]$ such that $u_n \to u_\infty$ in $C^0[0,1]$, and, for $j = 0, 1$, the derivative $u_n'(j)$ exists and $u_n'(j) \to \gamma_j$ in $\mathbb{R}$. Then $T_p(u_n, h_n) \to T_p(u_\infty, h_\infty)$ in Y (where $T_p(u_\infty, h_\infty)$ is defined as above, with γ_j in the place of $u_\infty'(j)$).

Proof. Part (i) follows from Corollary IV.8.11 on p. 294 of [15]. Next, the weak convergence $h_n \rightharpoonup h_\infty$ implies that $I(h_n)(x) \to I(h_\infty)(x)$ for each $x \in [0,1]$, while the equi-integrability of $\{h_n\}$ implies that the set $\{I(h_n)\} \subset C^0[0,1]$ is equi-continuous, so part (ii) follows from the Arzela-Ascoli lemma. Part (iii) now follows readily from part (ii) and the construction of T. □

Finally, to describe the nodal properties of solutions of (2.1) we introduce the following notation. From now on, ν will denote an element of $\{\pm\}$, and $k \geq 0$ will be an integer. For each such ν and k, let S_k^ν denote the set of functions $u \in X$ having only simple zeros in $[0,1]$ (that is, $u' \neq 0$ at each zero of u) and exactly k zeros in $(0,1)$, and with $\nu u > 0$ in a deleted neighbourhood of $x = 0$ (with the obvious interpretation of νu). The set S_k^ν is open in X.

3. Half-eigenvalues and associated spectral theory

For arbitrary a, $b \in Z$, we consider the problem

$$\Delta_p(u) = a\phi_p(u^+) - b\phi_p(u^-) + \lambda\phi_p(u), \quad \lambda \in \mathbb{R},\ u \in D(\Delta_p), \tag{3.1}$$

where $u^\pm(x) = \max\{\pm u(x), 0\}$, $x \in [0,1]$. If u is a solution of (3.1) then tu is also a solution, for any number $t \geq 0$, so (3.1) is *positively-homogeneous*. Furthermore, (3.1) can be rewritten as $\Delta_p(u) = m\phi_p(u)$, where $m := a\chi(u^+) - b\chi(u^-) + \lambda \in Z$, and

$$\chi(s) := \begin{cases} 1, & \text{if } s > 0, \\ 0, & \text{if } s \leq 0, \end{cases}$$

so by Lemma 3.1 in [3], if u is non-trivial then it has only simple zeros, and hence $u \in S_k^\nu$ for some k and ν.

A number λ will be called a *half-eigenvalue* (of (3.1)) if there exists a non-trivial solution u of (3.1), and u will then be called a *half-eigenfunction*. If u is a half-eigenfunction then $u \in S_k^\nu$ for some k and ν, and tu is also a half-eigenfunction, for any $t > 0$. We denote the set of half-eigenvalues of (3.1) by Σ_H.

Theorem 3.1. *For each $k \geq 0$, $\nu \in \{\pm\}$, problem (3.1) has a unique solution $(\lambda, u) = (\lambda_k^\nu, u_k^\nu) \in \mathbb{R} \times S_k^\nu$ with $|u_k^\nu|_1 = 1$, and all the half-eigenfunctions corresponding to λ_k^ν are of the form tu_k^ν, with $t > 0$. The set $\Sigma_H = \{\lambda_k^\nu : k \geq 0,\ \nu \in \{\pm\}\}$. The half-eigenvalues are increasing, in the sense that*

$$k' > k \implies \lambda_{k'}^{\nu'} > \lambda_k^\nu, \quad \textit{for each } \nu,\, \nu' \in \{\pm\}, \tag{3.2}$$

and $\lim_{k\to\infty} \lambda_k^\nu = \infty$.

Proof. A standard method of obtaining spectral properties of the linear Sturm-Liouville problem is by means of the Prüfer transformation — this method is described in detail in Section 2 of Chapter 8 of [6]. To obtain corresponding spectral properties for the p-Laplacian problem, modified Prüfer-type transformations have been constructed in several papers, see for example [3], [20], [26] (there are minor differences between the transformations in these papers). To obtain the desired properties of the half-eigenvalue problem (3.1) we will also use a modified Prüfer-type transformation — specifically, we use the transformation (and the results) of [3], and we refer the reader to [3] for further details of the constructions below.

Let $\pi_p := (2\pi/p)\sin(\pi/p)$ (the notation $\hat{\pi}$ is used in [3]), and define the sine-like function $S_p : \mathbb{R} \to \mathbb{R}$ as in [3]. The function S_p is antisymmetric, $2\pi_p$ periodic, positive on $(0, \pi_p)$ and satisfies $S_p(0) = S_p(\pi_p) = 0$, $S_p'(0) = 1$, and $|S(\theta)|^p + |S'(\theta)|^p = 1$, $\theta \in \mathbb{R}$, see [3]. A Prüfer-type transformation can now be defined via

$$u = \rho S_p(\theta), \quad u' = \rho S_p'(\theta),$$

with $\rho > 0$, which transforms equation (3.1) into the pair of equations

$$\theta_x = \widetilde{p}\big(\lambda|S_p|^p + a|S_p^+|^p + b|S_p^-|^p\big) + |S_p'|^p, \tag{3.3}$$

$$\rho_x = \rho\big((1 - \widetilde{p}\lambda)S_p - \widetilde{p}(aS_p^+ - bS_p^-)\big)|S_p|^{p-2}S_p', \tag{3.4}$$

where $\widetilde{p} := (p-1)^{-1}$ and S_p denotes $S_p(\theta)$. Sections 1 and 2 of [3] describe this transformation, with $a = b = 0$; a similar transformation, in a Fučík setting with $a = \mu s$, $b = \nu s$, $\mu, \nu \in \mathbb{R}$ and $s \in L^\infty(0,1)$, is used in [20], with a slightly different function S. We can now choose $\theta_0 \in [0, \pi_p)$, $\theta_1 \in (0, \pi_p]$, such that the boundary conditions (1.2) correspond to the conditions

$$\theta(0) = \theta_0 + i\pi_p, \quad \theta(1) = \theta_1 + j\pi_p, \tag{3.5}$$

for arbitrary i, $j \in \mathbb{Z}$. Since equation (3.3) does not depend on ρ, to find half-eigenvalues it suffices to find non-trivial solutions of the problem (3.3), (3.5). Furthermore, it suffices to consider only the cases $i = 0$ and $i = 1$ in (3.5) (without the jumping terms we would only need to consider $i = 0$).

Now, for fixed $\lambda \in \mathbb{R}$, let $\theta_\pm(\lambda, \cdot)$ denote the solutions of equation (3.3) with the initial values

$$\theta_+(\lambda, 0) = \theta_0, \quad \theta_-(\lambda, 0) = \theta_0 + \pi_p \tag{3.6}$$

(existence and uniqueness of these solutions, on the interval $[0,1]$, is proved in Lemma 2.1 of [3], which also proves a similar result for equation (3.4)). Clearly, the solutions of (3.1) corresponding to θ_+ and θ_- (for any solution $\rho > 0$ of (3.4)) are positive and negative respectively in a deleted neighbourhood of $x = 0$. Hence, to find all the half-eigenvalues of (3.1) it suffices to consider the functions $\theta_\pm$, and (as in the usual Prüfer transform construction of linear eigenvalues) we obtain half-eigenvalues by finding values of λ for which $\theta_\pm(\lambda, 1)$ satisfy the second boundary condition in (3.5).

It is shown in Lemmas 2.2–2.5 in [3] that $\theta_\pm(\lambda, 1)$ are continuous and strictly increasing functions of λ on $\mathbb{R}$, with

$$\lim_{\lambda \to -\infty} \theta_+(\lambda, 1) = 0, \quad \lim_{\lambda \to -\infty} \theta_-(\lambda, 1) = \pi_p, \quad \lim_{\lambda \to \infty} \theta_\pm(\lambda, 1) = \infty.$$

Hence, for each $k \geq 0$ and $\nu \in \{\pm\}$, we may define half-eigenvalues $\lambda_k^\pm$ to be the unique solutions of the equations

$$\theta_+(\lambda_k^+, 1) = \theta_1 + k\pi_p, \quad \theta_-(\lambda_k^-, 1) = \theta_1 + (k+1)\pi_p. \tag{3.7}$$

It is easy to see that the corresponding half-eigenfunctions belong to $S_k^{\pm}$, and the other properties of the half-eigenvalues stated in the theorem follow immediately from the above properties of $\theta_{\pm}(\lambda, 1)$. This completes the proof of the theorem. □

The half-eigenvalues depend on the variables p, a, b, but normally we will suppress this dependence, except when it is necessary to emphasize it, when we will write $\lambda_k^{\pm}(p, a, b)$. The following theorem shows that the half-eigenvalues depend continuously on $p \in (1, \infty)$ and on a, $b \in L^1(0, 1)$ (with respect to the $L^1(0, 1)$ norm), and are decreasing functions of a, b, for fixed p. To define the idea of 'decreasing' we introduce the following notation. For $(a_i, b_i) \in L^1(0, 1)^2$, $i = 0$, 1, we write $(a_0, b_0) \le (a_1, b_1)$ if

$$a_0(x) \le a_1(x) \quad \text{and} \quad b_0(x) \le b_1(x), \quad \text{a.e. } x \in [0, 1], \tag{3.8}$$

and we write $(a_0, b_0) < (a_1, b_1)$ if $(a_0, b_0) \le (a_1, b_1)$ and *both* the inequalities in (3.8) hold strictly when x lies in some set $J \subset [0, 1]$ having positive measure. The idea of 'decreasing' is now defined in (3.9).

Theorem 3.2. *For each $k \ge 0$:*

(i) *$\lambda_k^{\pm}(p, a, b)$ depends continuously on $(p, a, b) \in (1, \infty) \times L^1(0, 1)^2$.*
(ii) *for fixed $p \in (1, \infty)$,*

$$(a_0, b_0) \le (a_1, b_1) \implies \lambda_k^{\pm}(p, a_0, b_0) \ge \lambda_k^{\pm}(p, a_1, b_1); \tag{3.9}$$

this result remains valid if $\le$ is replaced throughout by $<$.

Proof. Following the proof of Theorem 3.1, but regarding $p \in (1, \infty)$, a, $b \in L^1(0, 1)$, as variables, we construct functions $\theta_{\pm}(\lambda, p, a, b, x)$ by solving the initial value problems (3.3), (3.6). It can be shown that these functions are continuous in all their arguments (continuous dependence on (a, b) is not totally straightforward since we only assume that a, $b \in L^1(0, 1)$, but a similar proof, for the case $p = 2$, is given in the proof of Theorem 2.3 in [4]). In addition, it follows easily from the differential equation (3.3) (using standard results on first-order differential inequalities, see for example Lemma 4 in [20]), that $\theta_{\pm}$ are increasing functions of (a, b), for fixed λ, p, x (here, 'increasing' is defined in the same way that (3.9) defines 'decreasing'). The desired results now follow from these results and the properties of the functions $\theta_{\pm}$ used in the proof of Theorem 3.1. □

Remark 3.3. A similar proof shows that the monotonicity result in Theorem 3.2 also holds for each $k \ge 1$ (so that the half-eigenfunctions change sign) if $(a_0, b_0) \le (a_1, b_1)$ and at least one of the inequalities in (3.8) holds strictly for almost all $x \in [0, 1]$.

For each $k \ge 0$, let $\lambda_k^{\max} := \max\{\lambda_k^+, \lambda_k^-\}$, $\lambda_k^{\min} := \min\{\lambda_k^+, \lambda_k^-\}$, and define the open intervals

$$\Lambda_{-1}^1 = (-\infty, \lambda_0^{\min}), \quad \Lambda_k^1 = (\lambda_k^{\max}, \lambda_{k+1}^{\min}), \quad \Lambda_k^0 = (\lambda_k^{\min}, \lambda_k^{\max}).$$

Intuitively, Theorem 3.1 says that the term $a\phi_p(u^+) - b\phi_p(u^-)$ in equation (3.1) 'splits apart' the usual eigenvalues λ_k into half-eigenvalues λ_k^+, λ_k^-. The interval Λ_k^0 is the gap between the half-eigenvalues $\lambda_k^{\pm}$ produced by this splitting process, and may be empty if the half-eigenvalues coincide. The inequality (3.2) says that in this splitting process, half-eigenvalues with different values of k do not meet each other, so the interval Λ_k^1 between half-eigenvalues corresponding to k and $k + 1$ is non-empty. Also, all these intervals are disjoint and their union comprises $\mathbb{R} \setminus \Sigma_H$.

Where necessary, we will indicate the dependence of these intervals on the coefficient functions a, b by writing $\Lambda_k^0(a,b)$, $\Lambda_k^1(a,b)$.

In addition to eigenvalues, linear spectral theory is also concerned with the solvability of inhomogeneous problems. Accordingly, we will consider the solvability of the equation

$$\Delta_p(u) = a\phi_p(u^+) - b\phi_p(u^-) + \lambda\phi_p(u) + h, \tag{3.10}$$

for general $h \in Z$, when λ is not a half-eigenvalue.

For $\lambda \in \mathbb{R}$, we define the positively homogeneous operator $S_\lambda : D(\Delta_p) \to Z$ by

$$S_\lambda(u) := \Delta_p(u) - a\phi_p(u^+) + b\phi_p(u^-) - \lambda\phi_p(u), \quad u \in D(\Delta_p).$$

Lemma 3.4. *Suppose that* $\lambda \notin \Sigma_H$.
(i) There exists $\delta(\lambda) > 0$ *such that*

$$\|S_\lambda(u)\|_1 \geq \delta(\lambda)|u|_1^{p-1}, \quad u \in D(\Delta_p). \tag{3.11}$$

(ii) The range $R(S_\lambda)$ *is closed.*

Proof. Suppose that there exists a sequence $u_n \in D(\Delta_p)$, $n = 1, 2, \ldots$, such that $|u_n|_1 = 1$ for all n and $\|h_n\|_1 \to 0$, where $h_n := S_\lambda(u_n)$. We may also suppose that $u_n \to u_\infty$ in $C^0[0,1]$, and $u_n'(j) \to \gamma_j$, in $\mathbb{R}$, for $j = 0, 1$. By (2.2), for each n,

$$u_n - T_p(u_n, a\phi_p(u_n^+) - b\phi_p(u_n^-) + \lambda\phi_p(u_n) + h_n) = 0, \tag{3.12}$$

so by Lemma 2.1, $u_n \to u_\infty$ in Y, and hence $u_\infty \neq 0$, and taking the limit in (3.12) yields

$$u_\infty - T_p(u_\infty, a\phi_p(u_\infty^+) - b\phi_p(u_\infty^-) + \lambda\phi_p(u_\infty)) = 0.$$

However, by (2.2), this implies that $\lambda \in \Sigma_H$, which contradicts the hypothesis of the lemma, and so proves part (i). The proof of part (ii) is similar. □

Theorem 3.5. *(i) If* $\lambda \in \Lambda_k^1$, *for some* $k \geq -1$, *then* S_λ *is surjective, that is, for any* $h \in Z$, *equation* (3.10) *has a solution* $u \in D(\Delta_p)$.
(ii) If $\lambda \in \Lambda_k^0$, *for some* $k \geq 0$, *then* S_λ *is not surjective, that is, there exists* $h \in Z$ *such that equation* (3.10) *has no solution* $u \in D(\Delta_p)$.

Proof. Part (i) is a special case of Theorem 4.1 below, while part (ii) is a special case of Theorem 5.1. □

4. Existence of solutions

We now return to the general problem (2.1), and obtain conditions under which this problem has a solution. Recalling assumption (1.3), we will say that an arbitrary pair of functions $(a,b) \in Z^2$ is (ψ,Ψ)*-bounded* if

$$(\psi_+, \psi_-) \leq (a,b) \leq (\Psi_+, \Psi_-).$$

It follows from Theorem 3.2 that if (a,b) is (ψ,Ψ)-bounded then, for any $k \geq -1$ and $j \in \{0, 1\}$ (if $k = -1$ then $j = 0$),

$$\lambda \in \Lambda_k^j(\psi_+,\psi_-) \cap \Lambda_k^j(\Psi_+,\Psi_-) \implies \lambda \in \Lambda_k^j(a,b). \tag{4.1}$$

Theorem 4.1. *Suppose that* (1.3) *holds and, for some* $k \geq -1$,

$$0 \in \Lambda_k^1(\psi_+,\psi_-) \cap \Lambda_k^1(\Psi_+,\Psi_-). \tag{4.2}$$

Then equation (2.1) *has a solution* $u \in D(\Delta_p)$.

Proof. We use the Leray-Schauder continuation theorem to prove the result. The following notation will be used: for any $r > 0$ let $B_r := \{y \in Y : |y|_1 \le r\}$ and, for any completely continuous mapping $T : Y \to Y$, let $\deg(I - T, B_r, 0)$ denote the Leray-Schauder degree of $I - T$ with respect to 0, on the ball B_r, see [9].

Let $\mu_\pm := \frac{1}{2}(\psi_\pm + \Psi_\pm)$, and consider the homotopy $H : [0,1] \times Y \to Y$ defined by

$$H(\tau, u) := T_p(u, (1-\tau)f(u) + \tau(\mu_+\phi_p(u^+) - \mu_-\phi_p(u^-))).$$

If $B \subset Y$ is bounded, then assumption (1.3) implies that $f(B)$ is equi-integrable, so Lemma 2.1, together with the compactness of the embedding $Y \to C^0[0,1]$, shows that the mapping H is completely continuous. To apply the Leray-Schauder theorem we first show that there exists a constant $R > 0$ such that any solution $(\tau, u) \in [0,1] \times Y$ of the equation $H(\tau, u) = u$ satisfies $|u|_1 < R$.

Suppose instead that for each integer $n \ge 1$ there is a solution $(\tau_n, u_n) \in [0,1] \times Y$ with $|u_n|_1 \ge n$, and let $v_n := |u_n|_1^{-1} u_n$. It follows from the compactness of the embedding $Y \to C^0[0,1]$ that, after choosing a subsequence if necessary, there exists $(\tau_\infty, v_\infty) \in [0,1] \times C^0[0,1]$, and $\gamma_0, \gamma_1 \in \mathbb{R}$, such that

$$\tau_n \to \tau_\infty, \quad |v_n - v_\infty|_0 \to 0, \quad v_n'(j) \to \gamma_j, \ j = 0, 1.$$

Furthermore, letting $g_n := |u_n|_1^{1-p} f(u_n) \in Z$, $n \ge 1$, it follows from (1.3) that

$$|g_n(x)| \le |v_n|_0^{p-1} A(x) + e(|u_n|_1)|u_n|_1^{1-p}\eta(x), \quad x \in [0,1], \ n \ge 1, \tag{4.3}$$

for some $A \in Z$. In particular, the set $\{g_n\}$ is equi-integrable and the sequence $\|g_n\|_1$, $n \ge 1$, is bounded. Also, dividing the equation $H(\tau_n, u_n) = u_n$ by $|u_n|_1$ yields

$$v_n = T_p(v_n, (1-\tau_n)g_n + \tau_n(\mu_+\phi_p(v_n^+) - \mu_-\phi_p(v_n^-))), \quad n \ge 1. \tag{4.4}$$

Now suppose that $v_\infty = 0$, that is $|v_n|_0 \to 0$. Then by (4.3), $\|g_n\|_1 \to 0$, and so (4.4), together with Lemma 2.1, implies that $v_n \to v_\infty$ in Y, that is, $|v_n|_1 \to 0$. However, this contradicts the fact that $|v_n|_1 = 1$ for all n, so we must have $v_\infty \ne 0$.

Next, combining (1.3) with a slight extension of the argument on p. 648 of [14], proves the following result.

Lemma 4.2. *After choosing a subsequence, if necessary, there exists a* (ψ, Ψ)-*bounded pair* $(m_+, m_-) \in Z^2$ *such that*

$$g_n \rightharpoonup m_+\phi_p(v_\infty^+) - m_-\phi_p(v_\infty^-). \tag{4.5}$$

Hence, by Lemma 2.1, taking the limit in (4.4) shows that $v_n \to v_\infty$ in Y and $H(\tau_\infty, v_\infty) = v_\infty$. Combining this with (2.2) implies that $v_\infty \in D(\Delta_p)$ and

$$\Delta_p(v_\infty) = \eta_+\phi_p(v_\infty^+) - \eta_-\phi_p(v_\infty^-), \quad v_\infty \ne 0, \tag{4.6}$$

where $\eta_\pm = (1-\tau_\infty)m_\pm + \tau_\infty\mu_\pm$. However, it is clear that (η_+, η_-) is (ψ, Ψ)-bounded, so by (4.1) and (4.2), 0 is not a half-eigenvalue of (4.6), which contradicts (4.6), and so completes the proof that the constant R exists.

To complete the proof of the theorem we must show that $\deg(I - H(1, \cdot), B_R, 0) \ne 0$. To do this we will use p as a homotopy parameter to transform the nonlinear operator $H(1, \cdot)$ to a linear operator (at $p = 2$). This idea is used in the paper [10], although the actual operator used here is slightly more complicated than in [10], which considers a Dirichlet problem (for which Δ_p is invertible).

Suppose that $p < 2$ (the other case is similar). The equation $H(1, u) = u$ is equivalent to the equation

$$\Delta_p(u) = \mu_+\phi_p(u^+) - \mu_-\phi_p(u^-),$$

and $0 \in \Lambda^1_k(\mu_+, \mu_-)$ (since (μ_+, μ_-) is (ψ, Ψ)-bounded). Now, for $\tau \in [p, 2]$ let $\gamma(\tau) = 1 - (\tau - p)/(2 - p)$ (so $\gamma(p) = 1$, $\gamma(2) = 0$). It follows from Theorems 3.1 and 3.2 that we can choose a continuous function $\ell : [p, 2] \to \mathbb{R}$ such that $\ell(p) = 0$ and $\ell(\tau) \in \Lambda^1_k(\gamma(\tau)\mu_+, \gamma(\tau)\mu_-)$ for all $\tau \in [p, 2]$. Hence, for each $\tau \in [p, 2]$ the equation

$$\Delta_\tau(u) = \gamma(\tau)(\mu_+\phi_\tau(u^+) - \mu_-\phi_\tau(u^-)) + \ell(\tau)\phi_\tau(u) \tag{4.7}$$

has no non-trivial solution u. Thus, defining a homotopy $\widetilde{H} : [p, 2] \times Y \to Y$ by,

$$\widetilde{H}(\tau, u) := T_p(u, \gamma(\tau)(\mu_+\phi_\tau(u^+) - \mu_-\phi_\tau(u^-)) + \ell(\tau)\phi_\tau(u)),$$

$\widetilde{H}$ has the following properties:
(i) $\widetilde{H}(\tau, u) = u$ is equivalent to (4.7) at each $\tau \in [p, 2]$;
(ii) $\widetilde{H}(p, \cdot) = H(1, \cdot)$;
(iii) $\widetilde{H}$ is completely continuous;
(iv) $\deg(I - \widetilde{H}(p, \cdot), B_R, 0) = \deg(I - \widetilde{H}(2, \cdot), B_R, 0) \neq 0$ (since $I - \widetilde{H}(2, \cdot)$ is a linear, injective, compact perturbation of the identity, and hence is non-singular).
These results complete the proof of the theorem. □

5. Non-existence of solutions

In this section we will prove a non-existence result for (2.1) with a general function f. We first prove the following result, which shows that we can choose a single $h \in Z$ such that the result in part (ii) of Theorem 3.5 holds for all (ψ, Ψ)-bounded $(a, b) \in Z^2$.

Theorem 5.1. *Suppose that, for some $k \geq 0$,*

$$\lambda \in \Lambda^0_k(\psi_+, \psi_-) \cap \Lambda^0_k(\Psi_+, \Psi_-). \tag{5.1}$$

Then there exists $h_0 \in Z$ such that if $h = h_0$ then for any (ψ, Ψ)-bounded $(a, b) \in Z^2$ equation (3.10) *has no solution $u \in D(\Delta_p)$.*

Proof. A result similar to part (ii) of Theorem 3.5 was proved in part (ii) of Proposition 1 in [8], with $p = 2$ and a, b constants; this was extended to more general a, b in [22] and [23]. However, these proofs relied on a Wronskian type construction, which seems to be difficult to extend to the case $p \neq 2$. Thus, to prove the result here we will use the Prüfer constructions in the proof of Theorem 3.1.

For $\epsilon \in [0, 1)$, let

$$\chi_\epsilon(x) := \begin{cases} 0, & x \in [0, 1 - \epsilon], \\ 1, & x \in (1 - \epsilon, 1], \end{cases}$$

and let $(3.10)_\epsilon$ denote equation (3.10) with $h = \chi_\epsilon$; when $\epsilon = 0$, $(3.10)_0$ reduces to the homogeneous equation (3.1). The hypothesis (5.1) implies that either

$$\lambda^+_k(a, b) < \lambda < \lambda^-_k(a, b), \tag{5.2}$$

for all (ψ, Ψ)-bounded (a, b), or the reverse inequalities hold. We will suppose that k is even and that (5.2) holds, and we show that if $\epsilon > 0$ is sufficiently small (independent of (a, b)) then the result holds with $h_0 = \chi_\epsilon$. The other cases can be tackled similarly, using $h_0 = -\chi_\epsilon$ in some cases.

We first show that we can use the Prüfer transformation to deal with solutions of the inhomogeneous equation $(3.10)_\epsilon$, when ϵ is sufficiently small.

Lemma 5.2. *There exists $\epsilon_0 \in (0,1)$ such that if $\epsilon \in (0,\epsilon_0)$ then any solution u of $(3.10)_\epsilon$ satisfies*

$$|u(x)| + |u'(x)| > 0, \quad x \in [0,1]. \tag{5.3}$$

Proof. Suppose that for each $n = 1,2,\dots,$ there exists $\epsilon_n \in (0,1/n)$ such that $(3.10)_{\epsilon_n}$ has a solution u_n, with $|u_n(x_n)| + |u_n'(x_n)| = 0$, $x_n \in [0,1]$, that is, x_n is a double zero of u_n. By the uniqueness result in Lemma 3.1 in [3], if $x_n \in [0, 1-\epsilon_n]$ then $u_n \equiv 0$ on this interval, so we may suppose that $x_n \in [1-\epsilon_n, 1]$. It can now be verified, using the form of $(3.10)_{\epsilon_n}$, that there exists a continuous, increasing function $M : [0,\infty) \to [0,\infty)$, depending only on the coefficients p, $\psi_\pm$, $\Psi_\pm$, such that $M(0) = 0$ and, if n is sufficiently large then for $x \in [1-\epsilon_n, 1]$,

$$\begin{aligned} |u_n'(x) + |x-x_n|^{\widetilde{p}}| &\le M(|x-x_n|)|x-x_n|^{\widetilde{p}}, \\ |u_n(x) + (1+\widetilde{p})^{-1}|x-x_n|^{1+\widetilde{p}}| &\le M(|x-x_n|)|x-x_n|^{1+\widetilde{p}} \end{aligned} \tag{5.4}$$

(recall that $\widetilde{p} = (p-1)^{-1}$). If $x_n < 1$ for all n, then by (5.4), $u_n(1) \ne 0$, $u_n'(1) \ne 0$, and $u_n(1)/u_n'(1) \to 0$. Substituting this into (1.2) yields

$$0 = c_{10}u_n(1)/u_n'(1) - c_{11} \to -c_{11},$$

which then implies that $u_n(1) = 0$, and so yields a contradiction.

Now suppose that $x_n = 1$ for all n. It then follows from (5.4), with $x = 1-\epsilon_n$, that $u_n \not\equiv 0$ on the interval $[0, 1-\epsilon_n]$. Hence, the Prüfer angle corresponding to u_n is well-defined on this interval, and so we may suppose that it coincides with $\theta_\nu(\lambda,\cdot)$, for some fixed $\nu \in \{\pm\}$ (recall that $\theta_\nu(\lambda,\cdot)$ was defined in the proof of Theorem 3.1). From $\theta_\nu(\lambda,1) = \lim_{n\to\infty} \theta_\nu(\lambda, 1-\epsilon_n)$ and (5.4) we see that $\theta_\nu(\lambda,1) = l\pi_p$, for some integer l, and

$$\theta_\nu(\lambda, 1-\epsilon_n) = l\pi_p + (1+\widetilde{p})^{-1}\epsilon_n(1+\mathrm{O}(1)),$$

as $n \to \infty$. However, putting $\theta_\nu(\lambda,1) = l\pi_p$ into (3.3) yields $\theta_\nu(\lambda, 1-\epsilon_n) = l\pi_p + \epsilon_n(1+\mathrm{O}(1))$. This contradiction completes the proof of the lemma. □

Lemma 5.2 shows that when $\epsilon \in (0,\epsilon_0)$ the Prüfer variables ρ, θ are well-defined on $[0,1]$ for any solution u of $(3.10)_\epsilon$. In addition, it can be verified that the Prüfer equations corresponding to $(3.10)_\epsilon$ are

$$\theta_x = \Theta(\lambda,\theta,x) + \widetilde{p}\rho^{1-p}\chi_\epsilon S_p(\theta), \tag{5.5}$$

$$\rho_x = R(\lambda,\theta,\rho,x) - \widetilde{p}\rho^{2-p}\chi_\epsilon S_p'(\theta), \tag{5.6}$$

where Θ, R, denote the right hand sides of (3.3), (3.4) respectively. To prove the result we will show that if ϵ is sufficiently small then no solution of the pair of equations (5.5)-(5.6) satisfies the boundary conditions (3.5). However, it suffices to show that for any strictly positive $\rho \in C^0[0,1]$, no solution of (5.5) satisfies (3.5), and at $x = 0$ it suffices to consider only the conditions (3.6).

Now, regarding λ as fixed, and a, $b \in Z$, $\epsilon \in [0,\epsilon_0)$ as variables, we construct functions $\theta_\pm(a,b,\epsilon,\cdot) : [0,1] \to \mathbb{R}$ by solving the initial value problems (3.6), (5.5). We note the following:

(a) in the present notation, the functions that were denoted by $\theta_\pm(\lambda,\cdot)$ in the proof of Theorem 3.1 are now $\theta_\pm(a,b,0,\cdot)$;
(b) $\theta_\pm(a,b,\epsilon,\cdot) \equiv \theta_\pm(a,b,0,\cdot)$ on $[0,1-\epsilon]$;

(c) the functions $\theta_\pm(a, b, \epsilon, \cdot)$ depend on the function ρ, but we do not include this dependence explicitly in the notation — it will be seen that the choice of ϵ below is independent of ρ.

It follows from (5.1) and (5.2), together with (3.7) and the monotonicity of $\theta_\pm$ with respect to λ and (a, b) in the proofs of Theorems 3.1 and 3.2, that

$$\theta_1 + k\pi_p < \theta_\pm(\psi_+, \psi_-, 0, 1) \le \theta_\pm(\Psi_+, \Psi_-, 0, 1) < \theta_1 + (k + 1)\pi_p, \tag{5.7}$$

and also, for any (ψ, Ψ)-bounded $(a, b) \in Z^2$,

$$\theta_\pm(\psi_+, \psi_-, 0, x) \le \theta_\pm(a, b, 0, x) \le \theta_\pm(\Psi_+, \Psi_-, 0, x), \quad x \in [0, 1]. \tag{5.8}$$

Lemma 5.3. *There exists sufficiently small $\epsilon \in (0, \epsilon_0)$ such that, for any (ψ, Ψ)-bounded $(a, b) \in Z^2$,*

$$\theta_1 + k\pi_p < \theta_\pm(a, b, \epsilon, 1) < \theta_1 + (k + 1)\pi_p. \tag{5.9}$$

Proof. Let θ^c_ϵ denote the solution of equation (3.3), with the coefficients $a(x)$, $b(x)$ replaced by $\max\{|\psi_+(x)|, |\Psi_+(x)|\}$, $\max\{|\psi_-(x)|, |\Psi_-(x)|\}$, respectively, and satisfying the initial condition $\theta^c_\epsilon(1 - \epsilon) = (k + 1)\pi_p$. In view of (5.7), we can choose ϵ sufficiently small that the following inequalities hold for $x \in [1 - \epsilon, 1]$,

$$\theta_1 + k\pi_p < \theta_\pm(\psi_+, \psi_-, 0, x) \le \theta_\pm(\Psi_+, \Psi_-, 0, x) < \theta_1 + (k + 1)\pi_p, \tag{5.10}$$

$$\theta^c_\epsilon(x) < \theta_1 + (k + 1)\pi_p. \tag{5.11}$$

This choice of ϵ is clearly independent of a, b, and we will show that this ϵ yields the result.

Pick an arbitrary $\nu \in \{\pm\}$, and first suppose that $\theta_\nu(a, b, 0, 1 - \epsilon) \ge (k + 1)\pi_p$. Since $S(l\pi_p) = 0$, $|S'(l\pi_p)| = 1$, for any integer l, it follows from (5.5) that $\theta_\nu(a, b, \epsilon, \cdot)$ cannot decrease below $(k + 1)\pi_p$ on the interval $[1 - \epsilon, 1]$. Hence, since $\chi_\epsilon \ge 0$ and $S_p(\theta) < 0$ for $\theta \in ((k+1)\pi_p, (k+2)\pi_p)$, it follows from (5.5) and (5.10) that

$$(k + 1)\pi_p < \theta_\nu(a, b, \epsilon, x) < \theta_\nu(a, b, 0, x) < \theta_1 + (k + 1)\pi_p, \quad x \in [1 - \epsilon, 1],$$

which proves (5.9) in this case.

Next suppose that $\theta_\nu(a, b, 0, 1 - \epsilon) < (k + 1)\pi_p$. Since $S_p(\theta) > 0$ whenever $\theta \in (k\pi_p, (k + 1)\pi_p)$, it follows from (5.5) that $\theta_\nu(a, b, \epsilon, x) > \theta_\nu(a, b, 0, x)$, so long as $\theta_\nu(a, b, \epsilon, x) < (k + 1)\pi_p$. Thus, if this latter inequality holds on the interval $[1 - \epsilon, 1]$ then we again obtain (5.9).

Finally, suppose that $\theta_\nu(a, b, \epsilon, x_1) = (k+1)\pi_p$, for some $x_1 \in (1-\epsilon, 1]$. We now see that

$$(k + 1)\pi_p \le \theta_\nu(a, b, \epsilon, x) < \theta^c_\epsilon(x) < \theta_1 + (k + 1)\pi_p, \quad x \in [x_1, 1]$$

(using (5.11)), which proves (5.9) in this case, and so completes the proof of the lemma. □

Lemma 5.3 completes the proof of Theorem 5.1. □

To extend the non-existence result in Theorem 5.1, for equation (3.10), to a general non-linear problem we replace f in (2.1) with $f + h$, for arbitrary $h \in Z$, that is, we consider the problem

$$\Delta_p(u) = f(u) + h. \tag{5.12}$$

Theorem 5.4. *Suppose that* (1.3) *and* (5.1) *hold, for some $k \ge 0$. Then there exists $h \in Z$ such that equation* (5.12) *has no solution $u \in D(\Delta_p)$.*

Proof. We will show that if $h = \gamma^{p-1} h_0$, for sufficiently large $\gamma > 0$, then (5.12) has no solution (here, h_0 is as in Theorem 5.1). Suppose instead that for each $n \geq 1$ there exists $\gamma_n \geq n$ and $u_n \in D(\Delta_p)$ satisfying (5.12). Then by (2.2),

$$u_n = T_p(u_n, f(u_n) + \gamma_n^{p-1} h_0), \quad n \geq 1. \tag{5.13}$$

Suppose further that the sequence $|u_n|_1$, $n \geq 1$, is bounded. Dividing (5.13) by γ_n yields

$$\begin{aligned} 0 = \lim_{n\to\infty} \gamma_n^{-1} |u_n|_1 &= \lim_{n\to\infty} |T_p(\gamma_n^{-1} u_n, \gamma_n^{1-p} f(u_n) + h_0)|_1 \\ &= |T_p(0, h_0)|_1 > 0 \end{aligned}$$

(by (1.3) and continuity of T_p). Hence, we may suppose that $|u_n|_1 \to \infty$.

Next, for each $n \geq 1$, dividing (5.13) by $|u_n|_1$ and writing $v_n = |u_n|_1^{-1} u_n$, $g_n = |u_n|_1^{1-p} f(u_n)$ and $\widetilde{\gamma}_n = |u_n|_1^{-1} \gamma_n$, yields the equation

$$v_n = T_p(v_n, g_n + \widetilde{\gamma}_n^{p-1} h_0). \tag{5.14}$$

Since $|v_n|_1 = 1$ for all n, the above argument shows that the sequence $\widetilde{\gamma}_n$, $n \geq 1$, is bounded. Hence, a similar argument to that in the proof of Theorem 4.1 shows that, after taking a subsequence if necessary, there exists $\widetilde{\gamma}_\infty \geq 0$ and $0 \neq v_\infty \in D(\Delta_p)$ such that $\widetilde{\gamma}_n \to \widetilde{\gamma}_\infty$, $|v_n - v_\infty|_1 \to 0$ and

$$\Delta_p(v_\infty) = m_+ \phi_p(v_\infty^+) - m_- \phi_p(v_\infty^-) + \widetilde{\gamma}_\infty^{p-1} h_0, \tag{5.15}$$

where the pair $(m_+, m_-) \in Z^2$ is (ψ, Ψ)-bounded (see Lemma 4.2). There are now two possibilities for the limit $\widetilde{\gamma}_\infty$:

(i) if $\widetilde{\gamma}_\infty = 0$ then (5.15) and the argument after (4.6) yields a contradiction;
(ii) if $\widetilde{\gamma}_\infty > 0$ then (5.15) contradicts Theorem 5.1 (by positive-homogeneity).

These contradictions complete the proof of the theorem. □

6. Global bifurcation and multiplicity results

In this section we suppose that f has the form

$$f(x, s, t) = (w(x) + g(x, s, t))\phi_p(s), \quad (x, s, t) \in [0, 1] \times \mathbb{R}^2, \tag{6.1}$$

where $w \in Z$ and $g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ is Carathéodory and satisfies the following standard 'Carathéodory type' condition: for any bounded set $B \subset \mathbb{R}^2$, there exists $h_B \in Z$ such that

$$|g(x, s, t)| \leq h_B(x), \quad (x, s, t) \in [0, 1] \times B \tag{6.2}$$

(this condition is considerably weaker than condition (1.3); in this section we do not assume (1.3) until Theorem 6.4). We also suppose that

$$g(x, 0, 0) = 0. \tag{6.3}$$

These conditions ensure that the Nemitskii operator $g : Y \to Z$ is continuous and $g(0) = 0$. We consider the bifurcation problem

$$\Delta_p(u) = \lambda \phi_p(u) + f(u), \quad (\lambda, u) \in \mathbb{R} \times D(\Delta_p). \tag{6.4}$$

Clearly, (6.1)–(6.3) implies that $u = 0$ is a solution of (6.4), for all $\lambda \in \mathbb{R}$, and so Theorem 4.1 gives no information about the existence of non-trivial solutions. We will prove a global bifurcation result for (6.4), and then use this to obtain a result on the multiplicity of non-trivial solutions u for fixed λ.

Let $\mu_k(w)$, $k \geq 0$, denote the eigenvalues of the problem $\Delta_p(u) = (\lambda + w)\phi_p(u)$ (the existence and basic properties of these eigenvalues is known, see Theorem 3.1 in [3], or the references therein; the basic results that we require also follow from Theorems 3.1–3.5, by putting $a = b = w$). Let $\mathcal{S} \subset \mathbb{R} \times X$ denote the set of non-trivial solutions of (6.4), and let $\overline{\mathcal{S}}$ denote its closure. For each $k \geq 0$, let $\mathcal{C}_k$ denote the component of $\overline{\mathcal{S}}$, in $\mathbb{R} \times X$, containing the point $(\mu_k(w), 0)$, and let $\mathcal{C}_k^\pm := (\mathcal{C}_k \cap (\mathbb{R} \times S_k^\pm)) \cup \{(\mu_k(w), 0)\}$.

The following theorem is a p-Laplacian version of Theorem 2.3 in [19].

Theorem 6.1. *Suppose that* (6.1)–(6.3) *hold. Then, for each* $k \geq 0$, $\mathcal{C}_k = \mathcal{C}_k^+ \cup \mathcal{C}_k^-$, *and each set* $\mathcal{C}_k^\pm$ *is closed, connected and unbounded in* $\mathbb{R} \times X$.

Proof. Equation (6.4) is equivalent to the problem

$$u = G(\lambda, u) := T_p(u, (\lambda + w + g(u))\phi_p(u)), \quad (\lambda, u) \in \mathbb{R} \times Y.$$

In [19], Rabinowitz deals with a similar problem, where $G(\lambda, u)$ has the form $\lambda L u + H(\lambda, u)$, with L linear and compact, and H completely continuous and $\lim_{\|u\| \to 0} \|H(\lambda, u)\|/\|u\| = 0$, uniformly on compact λ intervals (for suitable norms). With our hypotheses on g it follows from Lemma 2.1 that $G : \mathbb{R} \times Y \to Y$ is completely continuous and $\lim_{|u|_1 \to 0} \|g(u)\|_1 = 0$, but we have homogeneity of the mapping $u \to T(u, (\lambda + w)\phi_p(u))$, rather than linearity of L. However, by some slight amendments of the proofs in [19], these conditions are sufficient to prove the above result. We will sketch some of the details of the amended proof.

Firstly, we observe that an analogue of the basic Lemma 1.24 in [19] holds here, with a similar proof (essentially, this lemma states that if a sequence of non-trivial solutions $(\lambda_n, u_n) \to (\lambda_\infty, 0)$, then λ_∞ must be an eigenvalue $\mu_k(w)$, and u_n must approach zero in the 'direction' of the corresponding eigenfunction). Now suppose that (λ, u) is a non-trivial solution of (6.4). By (6.1), $f(u) = (w + g(u))\phi_p(u)$, and hence the argument at the start of Section 3 shows that $u \in S_k^\nu$, for some k and ν. Thus the argument in the proof of Theorem 2.3 in [19] regarding preservation of the nodal structure of solutions of (6.4) along continua holds here, and shows that for any $k \geq 0$, the continuum $\mathcal{C}_k$ can only meet $\mathbb{R} \times \partial(S_k^+ \cup S_k^-)$ at the point $(\mu_k(w), 0)$, and hence $\mathcal{C}_k \setminus \{(\mu_k(w), 0)\} \subset \mathbb{R} \times (S_k^+ \cup S_k^-)$. All the results of the theorem now follow immediately from this and the definition of the sets $\mathcal{C}_k^\pm$, except the unboundedness of these sets. To prove this we require the following lemma.

Lemma 6.2. *If* $|\lambda - \mu_k(w)| > 0$ *is sufficiently small then* $u = 0$ *is an isolated zero of the operator* $I - G(\lambda, \cdot)$, *and the index of this zero changes as* λ *crosses* $\mu_k(w)$.

Proof. The fact that $u = 0$ is an isolated zero of $I - G(\lambda, \cdot)$ follows from the analogue of Lemma 1.24 in [19] mentioned above. To prove the index jump result we observe that, after using similar homotopies to H and $\widetilde{H}$ in the proof of Theorem 4.1, it suffices to consider the degree $\deg(L(\lambda), B_r(0), 0)$, where $L(\lambda) : Y \to Y$ is the linear operator $L(\lambda)u = L_0 u + \lambda L_1 u := u - T_2(u, \lambda u)$, $u \in Y$, and $r > 0$ is arbitrary. Next, by constructing a similar homotopy of the coefficients c_{ji} in the boundary condition functionals BC_j in the definition of T_2, we may transform the boundary conditions (1.2) into Dirichlet boundary conditions (while keeping the degree constant) — it is at this point that we require the space Y rather than X.

Now suppose that $L(\mu)u = 0$, for some $\mu \in \mathbb{R}$ and $0 \neq u \in Y$ (that is, 0 is an eigenvalue of the operator $L(\mu)$). Suppose further that 0 is not a simple eigenvalue of $L(\mu)$, that is, there exists $v \in Y$ such that $L(\mu)v = u$. As in (2.2), these equations yield

$$u(0) = 0, \;\; u(1) = 0, \;\; -u'' - \mu u = 0, \quad v(0) = 0, \;\; v(1) = -u'(1), \;\; -v'' - \mu v = -u'',$$

and taking the $L^2(0, 1)$ inner product yields

$$0 < \langle -u'', u\rangle = \langle -v'' - \mu v, u\rangle = -[v', u]_0^1 + [v, u']_0^1 = -(u'(1))^2.$$

This contradiction shows that 0 must be a simple eigenvalue of $L(\mu)$.

Similarly, we can show that $R(L(\mu)) \cap \operatorname{span}\{L_1 u\} = \{0\}$. Combining these results shows that a simple eigenvalue of $L(\lambda)$ crosses 0 transversly when λ crosses μ, which implies the required jump in the degree. □

Using the index jump result of Lemma 6.2 we can now follow the proof of Theorem 1.3 in [19] to show that the continuum $\mathcal{C}_k$ must be unbounded in $\mathbb{R} \times X$. It follows immediately from this that at least one of the sets $\mathcal{C}_k^\pm$ must be unbounded, but in general it is rather difficult to show that both the sets $\mathcal{C}_k^\pm$ are unbounded. A proof of this, in a general, abstract setting, in the Rabinowitz type of 'linear' case is given in [7]. This proof could probably be extended to a general 'homogeneous' problem of the above form, but in the current setting a simpler proof is available, due to the preservation of the nodal structure along the continuum $\mathcal{C}_k$, which yields the above simple decomposition of $\mathcal{C}_k$ into the sets $\mathcal{C}_k^\pm$. We will sketch this proof.

Suppose that $\mathcal{C}_k^-$ is bounded (a similar proof holds for $\mathcal{C}_k^+$). We now follow the proof of Theorem 1.27 in [19] to construct a modified function $\widetilde{G} : \mathbb{R} \times Y \to Y$, such that the modified equation $u = \widetilde{G}(\lambda, u)$ has solution continua $\widetilde{\mathcal{C}}_k$ and $\widetilde{\mathcal{C}}_k^\pm$, with the property that $\widetilde{\mathcal{C}}_k^- = \mathcal{C}_k^-$, $\widetilde{\mathcal{C}}_k^+ = -\mathcal{C}_k^-$, and hence $\widetilde{\mathcal{C}}_k = \widetilde{\mathcal{C}}_k^+ \cup \widetilde{\mathcal{C}}_k^-$ is bounded. But the preceding result, applied to the modified equation, implies that $\widetilde{\mathcal{C}}_k$ must be unbounded. This contradiction completes the proof. □

Remark 6.3. We could also allow g to depend on λ in a suitable manner (see Theorem 2.3 in [19], with $p = 2$).

We now use the global bifurcation result to prove the following multiplicity results for (6.4), when f also satisfies (1.3).

Theorem 6.4. *Suppose that* (1.3) *and* (6.1)–(6.3) *hold. If λ satisfies one of the inequalities*

$$\mu_k(w) < \lambda < \lambda_k^\nu(\Psi_+, \Psi_-) \quad \text{or} \quad \lambda_k^\nu(\psi_+, \psi_-) < \lambda < \mu_k(w), \tag{6.5}$$

for some $k \geq 0$ and $\nu \in \{\pm\}$, then equation (6.4) *has at least one solution $u \in S_k^\nu$.*

Proof. Let k and ν be as in the statement of the theorem, and $\mathcal{C}_k^\nu$ be as in Theorem 6.1. Choose a sequence $(\lambda_n, u_n) \in \mathcal{C}_k^\nu$, $n \geq 1$, such that $|\lambda_n| + |u_n|_1 \to \infty$. For each $n \geq 1$, it follows from (6.1) that

$$\Delta_p(u_n) = (\lambda_n + w + g(u_n))\phi_p(u_n),$$

and by (1.3) and (6.2) there exists $A \in Z$ such that $|g(u_n)(x)| \leq A(x)$, $x \in [0, 1]$, so by Theorem 3.2,

$$\mu_k(|w| + A) \leq \lambda_n \leq \mu_k(-|w| - A).$$

This bound on λ_n implies that $|u_n|_1 \to \infty$. Hence, by the argument in the proof of Theorem 4.1, we may suppose that $\lambda_n \to \lambda_\infty$ and there exists a non-zero $v_\infty \in D(\Delta_p)$ such that $|v_n - v_\infty|_1 \to 0$ and

$$\Delta_p(v_\infty) = \lambda_\infty \phi_p(v_\infty) + m_+\phi_p(v_\infty^+) - m_-\phi_p(v_\infty^-),$$

for some (ψ, Ψ)-bounded pair (m_+, m_-). Thus, $v_\infty \in S_k^\nu$, and so $\lambda_\infty = \lambda_k^\nu(m_+, m_-)$ and, by Theorem 3.2,

$$\lambda_k^\nu(\Psi_+, \Psi_-) \le \lambda_\infty \le \lambda_k^\nu(\psi_+, \psi_-). \tag{6.6}$$

It now follows from this, together with $(\mu_k(w), 0) \in \mathcal{C}_k^\nu$, (6.5) and the connectedness of $\mathcal{C}_k^\nu$, that $\mathcal{C}_k^\nu$ must intersect the set $\{\lambda\} \times X$, which proves the result. □

Theorem 6.4 yields the following multiplicity result.

Corollary 6.5. *Suppose that the hypotheses of Theorem 6.4 hold. Then for each pair (k, ν) for which* (6.5) *holds, equation* (6.4) *has at least one solution $u_k^\nu \in S_k^\nu$, and solutions corresponding to different (k, ν) are distinct.*

Remark 6.6. If (1.3) and (6.1)–(6.3) hold then, for each k and ν, (6.6) gives an estimate on where the continuum $\mathcal{C}_k^\nu$ 'meets infinity', In particular, if (1.5) also holds then $\mathcal{C}_k^\nu$ meets infinity precisely at $\lambda = \lambda_k^\nu(\widetilde{f}_+^\infty, \widetilde{f}_-^\infty)$.

7. Comparison with previous results

7.1. **The Fučík spectrum.** An alternative to the half-eigenvalue approach to the problem is to use the so called 'Fučík spectrum'. This is defined to be the set Σ_F consisting of those $(\alpha, \beta) \in \mathbb{R}^2$ for which the equation

$$\Delta_p(u) = \alpha\phi_p(u^+) - \beta\phi_p(u^-)$$

has a non-trivial solution $u \in D(\Delta_p)$. When $p = 2$ this approach is discussed in many papers, see for example [8], [13], [22] or [23] and the references therein. For the case $p \ne 2$, see [5], [12], [13], [20] (a generalization of the p-Laplacian is also considered in [16]). A detailed discussion of the relationship between the Fučík and half-eigenvalue approaches is given in [23] for the case $p = 2$ — this discussion extends to the case $p \ne 2$. We give a brief sketch of the results here.

It is known that for any $p > 1$ the set Σ_F consists of a collection of decreasing curves in $\mathbb{R}^2$, with various geometrical properties, see [20]. The usual hypothesis in the Fučík approach, analogous to the hypotheses (4.2) or (5.1), is of the following form: suppose that (1.4) holds, and there exists points (s_+, s_-), (t_+, t_-), lying on consecutive curves of Σ_F, such that

$$(s_+, s_-) < (\psi_+, \psi_-) \le (\Psi_+, \Psi_-) < (t_+, t_-).$$

Geometrically, this ensures that the rectangle

$$R := \{(r_1, r_2) \in \mathbb{R}^2 : s_+ < r_1 < t_+,\ s_- < r_2 < t_-\},$$

lies between the Fučík curves, with $R \cap \Sigma_F = \emptyset$, and the points $(\psi_+(x), \psi_-(x))$, $(\Psi_+(x), \Psi_-(x))$ lie in $\overline{R}$, for all $x \in [0, 1]$, and in R when x lies in a set of positive measure. In particular, if (1.5) holds then $(\widetilde{f}_+^\infty(x), \widetilde{f}_-^\infty(x)) \in \overline{R}$, $x \in [0, 1]$. Similar hypotheses are used in each of the cited papers to obtain existence results akin to Theorem 4.1.

When a, b are constant functions, equal to α, β say, it is clear that

$$\lambda \in \Sigma_H(\alpha, \beta) \text{ if and only if } (\alpha + \lambda, \beta + \lambda) \in \Sigma_F.$$

Furthermore, if (1.5) holds and $\widetilde{f}_\pm^\infty$ are constant then the two approaches are equivalent, in the sense that existence or non-existence results, stated in terms of either set $\Sigma_H(\alpha, \beta)$, Σ_F (using either hypotheses such as (4.2) or (5.1), or the Fučík type hypothesis mentioned above) can readily be restated in terms of the other set. However, when $\widetilde{f}_\pm^\infty$ are not constant the half-eigenvalue approach can yield considerably better existence and non-existence results. In fact, if $\widetilde{f}_\pm^\infty$ are sufficiently oscillatory that they cannot be bounded by any single pair of points on consecutive Fučík curves then the Fučík approach yields no information — clearly, this can happen for a large class of functions $\widetilde{f}_\pm^\infty$. On the other hand, when (1.5) holds, we can summarise the results of Theorems 4.1, 5.4 as follows

$$0 \in \Lambda_k^1(\widetilde{f}_+^\infty, \widetilde{f}_-^\infty) \implies \text{(5.12) has a solution for all } h,$$
$$0 \in \Lambda_k^0(\widetilde{f}_+^\infty, \widetilde{f}_-^\infty) \implies \text{(5.12) has no solution for some } h,$$

so these theorems distinguish between the two types of behaviour for any $\widetilde{f}_\pm^\infty$ except when $0 \in \Sigma_H(\widetilde{f}_+^\infty, \widetilde{f}_-^\infty)$, which can be regarded as a 'non-generic' situation since the set $\Sigma_H(\widetilde{f}_+^\infty, \widetilde{f}_-^\infty)$ is discrete. See [23] for further details of this when $p = 2$ — the discussion in [23] extends to the case $p \neq 2$.

Finally, we note that the case where $0 \in \Sigma_H(\widetilde{f}_+^\infty, \widetilde{f}_-^\infty)$ is termed 'resonant' and requires additional 'Landesman-Lazer' type conditions to obtain existence results, see for instance [13] (in the Fučík setting).

7.2. **Existence and non-existence results.** All the papers we cite in this section assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, and most assume that f is continuous rather than Carathéodory. In [10] it is assumed that f is continuous, (1.4) holds uniformly (so (1.3) holds) and

$$\psi_+ = \psi_- = \psi, \quad \Psi_+ = \Psi_- = \Psi, \tag{7.1}$$

for some ψ, $\Psi \in Z$. In addition, it is assumed that, for some $k \geq 0$,

$$\mu_k \leq \psi \leq \Psi \leq \mu_{k+1}, \tag{7.2}$$

where $\mu_l := \mu_l(0)$, $l \geq 0$, are the eigenvalues of Δ_p, and the first and last inequalities in (7.2) hold strictly on subsets of $[0,1]$ having positive measure (thus, in fact, (7.2) implies that ψ, $\Psi \in L^\infty(0,1)$). Theorem 5.2 in [10] then obtains a solution of (2.1). To compare this result with Theorem 4.1 above, we observe that, by definition, $\lambda_l^\pm(\mu_l, \mu_l) = 0$, for all $l \geq 0$, and the conditions (7.1), (7.2), imply that

$$(\mu_k, \mu_k) < (\psi, \psi) \leq (\Psi, \Psi) < (\mu_{k+1}, \mu_{k+1}),$$

and hence, by Theorem 3.2, (4.2) holds. Thus, Theorem 4.1 generalises Theorem 5.2 in [10].

Similarly, Theorem 3.3 in [26] obtains a solution of (2.1) under the assumption that f is Carathéodory, that a variant of (1.3) holds with (7.1) and $\psi \geq 0$ (with > 0 on a set of positive measure), and, for some $k \geq 0$,

$$\lambda_k(\psi) < 1, \quad \lambda_{k+1}(\Psi) > 1,$$

where $\lambda_k(w)$, $k \geq 0$, are the eigenvalues of the problem $\Delta_p(u) = \lambda w \phi_p(u)$. Theorem 3.2 again shows that these assumptions imply that condition (4.2) holds, and so Theorem 4.1 also generalises Theorem 3.3 in [26] (except that the condition analogous to (1.3) used in [26] is similar to (1.3) but seems neither stronger nor weaker than (1.3)).

The papers [10] and [26] assume that (1.4) holds with (7.1), that is, they impose the same bounds on $\widetilde{f}(x,s)$ as $\xi \to \pm\infty$, and these bounds are expressed in terms of the eigenvalues (or weighted eigenvalues) of Δ_p. When different bounds are imposed at $\pm\infty$ the Fučík spectrum or half-eigenvalue approach is required. The Fučík type hypotheses bound $\widetilde{f}$ between points on consecutive curves of the Fučík spectrum, and a similar argument to that just given shows that such bounds also imply that our condition (4.2) holds (details of the argument, in the case $p = 2$, are given in [23]), so that Theorem 4.1 also generalises such results. The papers [5], [12], [13], [20] use this approach, and hence Theorem 4.1 generalises the results in these papers. In particular, as mentioned above, the half-eigenvalue approach applies to any problem to which the Fučík approach applies, but also applies to problems for which the Fučík approach simply yields no information.

Non-existence results such as Theorem 5.4 have not previously been proved when $p \neq 2$. When $p = 2$, a similar result was proved in Theorem 4.1 of [23], while such a non-existence result was originally proved in [8] using the Fučík approach.

7.3. **Global bifurcation and multiplicity results.** Theorem 6.1 extends to the p-Laplacian setting Theorem 2.3 in [19], which deals with a standard Sturm-Liouville problem. The papers [11] and [17] also extend Rabinowitz' result to the p-Laplacian setting. In fact, these papers consider a partial differential, p-Laplacian, Dirichlet problem on domains with radial symmetry, and use the radial symmetry to reduce the problem to an ordinary differential equation; [11] also obtains a global continuum of positive solutions of the problem on a general domain — this result extends Theorem 2.12 in [19]. Theorem 14.14 in [13] proves a similar result to Theorem 6.1, for a Dirichlet problem with continuous f.

Theorem 5.3 in [10] imposes hypotheses similar to those of Theorem 6.4 (although using eigenvalues rather than half-eigenvalues, and allowing oscillation of $f(x,s)$ between lim-sup and lim-inf when $s \to 0$, as well as when $s \to \pm\infty$) and obtains a non-trivial solution. However, nodal properties of this solution are not considered, so the result in [10] cannot yield an estimate on the multiplicity of the non-trivial solutions such as given by Corollary 6.5. Theorem 4.2 in [11] obtains a multiplicity result (again using eigenvalues rather than half-eigenvalues) by a similar proof to the above, using global bifurcation and nodal properties. However, the result in [11] does not determine solutions in $S_k^{\pm}$, only in S_k, so in general it only yields half as many solutions as Corollary 6.5.

If the function f in (2.1) or (6.4) depends only on u then solutions can be explicitly constructed by quadratures. This is done in the papers [1] and [18]. A detailed description of the global bifurcation diagram for (6.4) is given in [18], assuming that f is odd and $\widetilde{f}$ is decreasing, with $\widetilde{f}(0) = 0$ and $\lim_{s\to\infty} \widetilde{f}(s) = -\infty$. In [1] it is assumed that (1.5) holds (and hence the limits $\widetilde{f}^{\infty}_{\pm}$ are constant), and half-eigenvalues are then defined and the multiplicity result Theorem 6.4 is obtained. In addition, it is shown that if $\widetilde{f}$ is strictly decreasing then this result yields the exact number of solutions.

For the case $p = 2$, results similar to Theorem 6.4 have been obtained in [21], using a global bifurcation proof similar to the above proof of Theorem 6.4, and in [24], using a Prüfer angle proof.

References

[1] I. Addou Multiplicity of solutions for quasilinear elliptic boundary value problems, *Elec. J. Differential Equns.* **1999** (1999), No. 21, 1–27.

[2] A. Ambrosetti, G. Prodi, *A Primer of Nonlinear Analysis*, CUP, Cambridge (1993).

[3] P. Binding, P. Drábek, Sturm-Liouville theory for the p-Laplacian, *Studia Sci. Math. Hungar.* **40** (2003), 375–396.

[4] P. Binding, B. P. Rynne, Half-eigenvalues of periodic Sturm-Liouville problems, *J. Differential Equns.* **206** (2004), 280–305.

[5] L. Boccardo, P. Drábek, D. Giachetti, M. Kučera, Generalization of Fredholm alternative for nonlinear differential operators, *Nonlinear Anal.* **10** (1986), 1083–1103.

[6] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York (1955).

[7] E. N. Dancer, On the structure of solutions of non-linear eigenvalue problems, *Indiana Univ. Math. J.* **23** (1974), 1069–1076.

[8] E. N. Dancer, On the Dirichlet problem for weakly non-linear elliptic partial differential equations, *Proc. Roy. Soc. Edin.* **76A** (1977), 283–300.

[9] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin (1985).

[10] M. A. del Pino, M. Elgueta, R. A. Manásevich, A homotopic deformation along p of a Leray-Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(0) = 0$, $p > 1$, *J. Differential Equns.* **80** (1989), 1–13.

[11] M. A. del Pino, R. A. Manásevich, Global bifurcation from the eigenvalues of the p-Laplacian, *J. Differential Equns.* **92** (1991), 226-251.

[12] M. A. del Pino, R. A. Manásevich, A. E. Murúa, Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE, *Nonlinear Anal.* **18** (1992), 79–92.

[13] P. Drábek, *Solvability and bifurcations of nonlinear equations*, Pitman Research Notes in Mathematics Series, 264, Longman, Harlow (1992).

[14] P. Drábek, S. Invernizzi, On the periodic BVP for the forced Duffing equation with jumping nonlinearity, *Nonlinear Analysis* **10** (1986), 643–650.

[15] N Dunford, J. T. Schwartz, *Linear Operators, Part I*, Wiley (1988).

[16] M. Garcia-Huidobro, R. A. Manásevich, F. Zanolin, On a pseudo Fučík spectrum for strongly nonlinear second-order ODEs and an existence result, *J. Comput. Appl. Math.* **52** (1994), 219–239.

[17] J. Garcia-Melian, J. Sabina de Lis, A local bifurcation theorem for degenerate elliptic equations with radial symmetry, *J. Differential Equns.* **179** (2002), 27–43.

[18] M. Guedda, L. Veron, Bifurcation phenomena associated to the p-Laplace operator, *Trans. Amer. Math. Soc.* **310** (1988), 419–431.

[19] P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, *J. Funct. Analysis* **7** (1971), 487-513.

[20] W. Reichel, W. Walter, Sturm-Liouville type problems for the p-Laplacian under asymptotic non-resonance conditions, *J. Differential Equns.* **156** (1999), 50–70.

[21] B. Ruf, Multiplicity and eigenvalue intersecting nonlinearities, *Z. Angew. Math. Phys.* **40** (1989), 774–779.

[22] B. P. Rynne, The Fučík spectrum of general Sturm-Liouville problems, *J. Differential Equations* **161** (2000), 87–109.

[23] B. P. Rynne, Non-resonance conditions for semilinear Sturm-Liouville problems with jumping non-linearities, *J. Differential Equns.* **170** (2001), 215–227.

[24] F. Sadyrbaev, Multiplicity of solutions for two-point boundary value problems with asymptotically asymmetric nonlinearities, *Nonlinear Analysis* **27**, (1996), 999-1012.

[25] M. Zhang Nonuniform nonresonance at the first eigenvalue of the p-Laplacian, *Nonlinear Analysis* **29** (1997), 41–51.

[26] M. Zhang, Nonuniform nonresonance of semilinear differential equations, *J. Differential Equns.* **166** (2000), 33–50.

Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, Scotland.

E-mail address: bryan@ma.hw.ac.uk