THE SPECTRUM OF THE PERIODIC p-LAPLACIAN

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ABSTRACT. We consider one dimensional p-Laplacian eigenvalue problems of the form

$$-\Delta_p u = (\lambda - q)|u|^{p-1}\mathrm{sgn}\, u, \quad \text{on } (0, b),$$

together with periodic or separated boundary conditions, where $p > 1$, Δ_p is the p-Laplacian, $q \in C^1[0, b]$, and $b > 0$, $\lambda \in \mathbb{R}$.

It will be shown that when $p \neq 2$, the structure of the spectrum in the general periodic case (that is, with $q \neq 0$ and periodic boundary conditions), can be completely different from those of the following known cases: (i) the general periodic case with $p = 2$, (ii) the periodic case with $p \neq 2$ and $q = 0$, and (iii) the general separated case with any $p > 1$.

1. INTRODUCTION

Eigenvalue problems involving the one dimensional p-Laplacian Δ_p have been investigated for many years. When $p = 2$ the classical Sturm-Liouville operator is involved, but for $p \neq 2$ one can already find modified variational and Prüfer methods in [9] (with references to earlier work) and [8], respectively. Despite a considerable amount of activity since, significant questions still remain concerning the nature of the spectrum (which can be defined in different ways) for various boundary conditions.

We consider the equation

$$-\Delta_p u = (\lambda - q)|u|^{p-1}\mathrm{sgn}\, u, \quad \text{on } (0, \pi_p), \tag{1.1}$$

where $q \in C^1[0, \pi_p]$, $\lambda \in \mathbb{R}$ and π_p will be defined in Section 2.1, together with either nontrivial, separated boundary conditions of the form

$$c_{00}u(0) + c_{01}u'(0) = 0, \quad c_{10}u(\pi_p) + c_{11}u'(\pi_p) = 0, \tag{1.2}$$

or periodic boundary conditions

$$u(0) = u(\pi_p), \quad u'(0) = u'(\pi_p). \tag{1.3}$$

In what follows, in either the separated or the periodic case, an eigenvalue is a value λ for which (1.1) has a nontrivial solution u, interpreted in the classical sense, and satisfying the boundary conditions. We call u an eigenfunction, and an eigenvalue λ will be called simple if all its eigenfunctions are proportional to each other. We use σ to denote the spectrum, i.e., the set of eigenvalues λ, of the problem.

Most of the early work focussed on Dirichlet and Neumann boundary conditions, and it was only recently that a p-Laplacian version of the classical Sturm oscillation theorem was established for general separated boundary conditions. To be precise, we have

Theorem 1.1. *For any $p > 1$, the spectrum σ of* (1.1), (1.2)*, consists of a sequence of simple eigenvalues $\lambda_0 < \lambda_1 < \dots$, accumulating at $+\infty$. For each $k \geq 0$, $\sigma_k = \{\lambda_k\}$.*

Here, σ_k denotes the set of eigenvalues λ whose corresponding eigenfunctions u have k zeros in the interval $(0, \pi_p)$ (by Theorem 5 in [13], if u is a non-trivial solution of (1.1) then it has only simple zeros).

When $p = 2$, Theorem 1.1 is a classical result of Sturm-Liouville theory and can be found in various books, e.g., [4, Theorem 2.1]. For $p \neq 2$, see [13, Theorem 1] (we remark that these references apply to equations with more general coefficients, which can even belong to $L^1(0, \pi_p)$ as in [1], but the conditions of (1.1) will suffice for the purposes below). We also note that the structure of the separated spectrum described in Theorem 1.1 is the same for both the linear problem when $p = 2$ and the nonlinear one when $p \neq 2$.

Non-separated boundary conditions have been studied rather less, but the periodic p-Laplacian has been investigated by many authors — see, for example, [3, 6, 7, 11, 12, 14]. The constant coefficient case (say $q = 0$ after a translation) can be solved explicitly — see [11] or Section 2 below. In this case, if we replace (1.2) by (1.3), and let σ_k correspond to eigenvalues λ whose eigenfunctions have $2k$ zeros in the interval $[0, \pi_p)$, then Theorem 1.1 continues to hold, except that the eigenvalues λ_k, $k \geq 1$ are no longer simple. More specifically, for each $k \geq 1$, the set E_k of periodic eigenfunctions corresponding to λ_k is a two-dimensional (punctured) linear space if $p = 2$, while if $p \neq 2$ then E_k is a two-dimensional manifold (without boundary). For further details, see Section 2.

For general q the periodic spectrum is less well understood, although we do have the following result.

Theorem 1.2. *For any $p > 1$, the spectrum σ of* (1.1), (1.3) *contains sequences of eigenvalues $\underline{\lambda}_k$, $\overline{\lambda}_k$, $k \geq 1$, satisfying*

$$\underline{\lambda}_1 \leq \overline{\lambda}_1 < \underline{\lambda}_2 \leq \overline{\lambda}_2 < \dots, \tag{1.4}$$

and accumulating at $+\infty$. For each $k \geq 1$, $\underline{\lambda}_k, \overline{\lambda}_k \in \sigma_k$.

(There is also a simple periodic principal eigenvalue $\lambda_0 \in \sigma_0$, but this will not be relevant for our later discussion).

For $p = 2$, this is again a classical result — see [4, Theorem 3.1] for a stronger statement. For $p \neq 2$, Theorem 1.2 was established recently by Zhang [14, Theorem 3.3]. (Actually the cited result does not mention oscillation, but this can be obtained from the rotation number construction used by Zhang, and the accumulation statement follows from his asymptotic estimate on [14, p. 136]). We note that $\underline{\lambda}_k = \overline{\lambda}_k$ in the constant coefficient case discussed above.

For $p = 2$, it is well known (cf. [4], loc. cit.) that

$$\sigma_k = \{\underline{\lambda}_k, \overline{\lambda}_k\}, \tag{1.5}$$

in the periodic case, for any q, and Zhang has conjectured [14, p. 142] that this remains true for $p \neq 2$. It will be seen below that this conjecture is incorrect. Indeed, we will show, for $p \neq 2$ and $q \neq 0$, that the periodic spectrum σ can be much richer than the sets of eigenvalues of Theorems 1.1 and 1.2. In fact our main result is as follows.

Theorem 1.3. *For any* $p > 1$, $p \neq 2$, *and integers* k, $n \geq 1$, *there exists a function* $q = q_{k,n} \in C^1[0, \pi_p]$ *such that* σ_k *for* (1.1), (1.3) *contains at least* n *distinct eigenvalues.*

The proof of this result is quite involved, and requires considerable preparation. The basic construction of the eigenvalues is via a perturbation away from the constant coefficient case, for which the eigenvalues and eigenfunctions are explicitly known. We describe the required details of this case in Section 2. The perturbation technique also requires various differentiability properties of the inverse p-Laplacian, and related operators. When $p = 2$, these properties are well understood, but when $p \neq 2$, they are non-trival and will be established in Section 3. These results are also of independent interest, and are related to those obtained in [10] (for the simpler mixed Neumann-Dirichlet operator), and used there for a bifurcation analysis. We shall discuss the relation between our work and the possible validity of [10] for different ranges of p in Section 3. Armed with these preparations, we use a succession of projections, applications of the implicit function theorem and degree theory to prove Theorem 1.3 in Section 4.

To conclude this introduction we shall comment heuristically on why a given constant coefficient periodic eigenvalue λ_k, $k \geq 1$, can split into an arbitrarily large number of eigenvalues under a small perturbation when $p \neq 2$, but not when $p = 2$. In the latter (linear) case the set E_k of eigenfunctions has two dimensional span, and a linear perturbation produces at most two eigenvalues. When $p \neq 2$, however, the manifold E_k turns out to have infinite dimensional (linear) span, allowing greater freedom for perturbation – see the remarks at the end of Section 2 for further details.

2. Preliminaries

2.1. **General concepts and notation.**

Differentiability will be a key issue in our analysis and we start with our notations for derivatives. If f is a function between Banach spaces then $Df(u)$ denotes the Fréchet derivative of f at u. Partial derivatives will be indicated by subscripts, e.g., $D_u g(u, v)$, $D_v g(u, v)$ are the partial derivatives of a two argument function g. The special cases D_x and D_t will be denoted by the customary prime and dot.

The underlying Banach spaces that we shall need are as follows. For $j = 0, 1$, we let $C^j[0, \pi_p]$ denote the space of j times continuously differentiable functions on $[0, \pi_p]$, with the usual sup-norm $|\cdot|_j$ (throughout, all function spaces will be real). $L^1(0, \pi_p)$, with norm denoted by $\|\cdot\|_1$, will be the usual space of integrable functions on $[0, \pi_p]$, and $W^{1,1}(0, \pi_p)$, with norm denoted by $\|\cdot\|_{1,1}$, will be the usual Sobolev space of absolutely continuous (AC) functions u on $[0, \pi_p]$, with derivative $u' \in L^1(0, \pi_p)$. It turns out that the ranges $p < 2$ and $p > 2$ will require different analysis in later sections, but a degree of unification will be achieved by writing

$$B_p := \begin{cases} C^1[0, \pi_p], & 1 < p \leq 2, \\ W^{1,1}(0, \pi_p) & p > 2. \end{cases} \tag{2.1}$$

We turn now to notation for (1.1). We start with the signed power function in the form $[x]^\alpha := |x|^\alpha \operatorname{sgn} x$, for α, $x \in \mathbb{R}$. We first note that this function satisfies the simple identities $[x]^\alpha = x|x|^{\alpha-1}$ and $[[x]^\alpha]^\beta = [x]^{\alpha\beta}$, for $\alpha, \beta > 0$, $x \in \mathbb{R}$, and, for a differentiable function f, $([f]^\alpha)'(x) = \alpha|f(x)|^{\alpha-1} f'(x)$, when $f(x) \neq 0$. Now

(1.1) can be written in the form

$$-\big([u']^{p-1}\big)' = (\lambda - q)[u]^{p-1}, \quad \text{on } (0, \pi_p). \tag{2.2}$$

The above notation will clarify the various detailed power estimates that will occur in our analysis.

We shall also need to view the signed powers above as operators, and for these we use the notation (which is more common but masks the powers)

$$\phi_p : x \mapsto [x]^{p-1}, \quad \Delta_p : u \mapsto (\phi_p(u'))'.$$

In this notation, (1.1) takes the form

$$-\Delta_p u = (\lambda - q)\phi_p(u),$$

and classical solutions u must therefore satisfy $u, \phi_p(u') \in C^1$. In particular, the boundary conditions (1.2) and (1.3) make sense.

In general, we shall simplify our notation by keeping the same symbols for operators and their restrictions. For example, the operator of differentiation (denoted by D as above) can map AC to L^1, C^1 to C^0, etc. Similarly for the operator $\mathcal{I}$ of integration in Section 3, Δ_p and its inverse, and so on.

We shall also need the following lemma at various points.

Lemma 2.1. *Suppose that $\alpha > -1$ and $v_0 \in C^1[0, \pi_p]$ has only simple zeros. Then there exists a neighbourhood V_0 of v_0 in $C^1[0, \pi_p]$ such that if $v \in V_0$ then $|v|^\alpha \in L^1(0, \pi_p)$ and the mapping $v \to |v|^\alpha$ is continuous from $V_0 \to L^1(0, \pi_p)$.*

Proof. If $\alpha \geq 0$ then the result is clear, so we assume $-1 < \alpha < 0$. We denote the zero set of v_0 by Z, consisting of N points, say. We also write $Z(\eta)$ for an η-neighbourhood of Z – this is a union of N open intervals of length 2η. Let $B(\delta)$ be the ball of radius δ and centre v_0 in $C^1[0, \pi_p]$. Given $\epsilon > 0$, we shall choose η and then δ so that

$$\| \, |v|^\alpha - |v_0|^\alpha \, \|_1 < 2\epsilon$$

for all $v \in B(\delta)$.

It is clear that for small enough δ, η, there is $\zeta > 0$, so that

$$|v'(x)| > \zeta \text{ for all } v \in B(\delta),\ x \in Z(\eta).$$

It then follows that we can choose $\eta > 0$ so that

$$\int_{[0,\pi_p] \cap Z(\eta)} | \, |v|^\alpha - |v_0|^\alpha \, | < \int_{[0,\pi_p] \cap Z(\eta)} (|v|^\alpha + |v_0|^\alpha) < \epsilon \tag{2.3}$$

for all $v \in B(\delta)$.

Outside $Z(\eta)$, for small enough δ, we see that $v\,v_0 > 0$ and $|v|$ is positively bounded below, for all $v \in B(\delta)$. Thus outside $Z(\eta)$, $|v|^\alpha$ obeys a Lipschitz condition and so for small enough δ we obtain

$$\int_{[0,\pi_p] \setminus Z(\eta)} | \, |v|^\alpha - |v_0|^\alpha \, | \leq \epsilon \tag{2.4}$$

for all $v \in B(\delta)$. The result follows from (2.3) and (2.4). □

2.2. **The constant coefficient case.** In this case, by translating the eigenparameter, we may assume that $q = 0$. Then (2.2) takes the form

$$-([u']^{p-1})' = \lambda [u]^{p-1}. \tag{2.5}$$

We denote the (unique) maximal solution of the initial value problem for (2.5) with $\lambda = 1$, $u(0) = 0$, $u'(0) = 1$, by $\sin_p$. A construction of this function is described in [13] and shows that $\sin_p$ is a C^1 function on $\mathbb{R}$, and is $2\pi_p$-periodic, where $\pi_p := 2(p-1)^{1/p}(\pi/p)/\sin(\pi/p)$. Moreover $\sin_p(x+\pi_p) = -\sin_p(x)$ and $\sin_p(m\pi_p) = 0$, $\sin_p'((m+\frac{1}{2})\pi_p) = 0$, $m \in \mathbb{Z}$. Thus the graph of $\sin_p$ resembles a sine wave, and indeed, $\sin_2$ reduces to the usual sin function, and $\pi_2 = \pi$.

Remark 2.2. The notation $\sin_p$ (and π_p) has also been used for different functions (and their zeros) in several works. See [3] for further details.

Later we shall need the following smoothness and nonsmoothness properties of these functions.

Lemma 2.3. *The function* $\sin_p$ *is real analytic except at integer multiples of* $\pi_p/2$. *If* $p < 2$ (*respectively,* $p > 2$) *then* $\sin_p$ *is not* C^3 *at even* (*respectively, not* C^2 *at odd*) *multiples of* $\pi_p/2$.

Proof. The first assertion follows from the analyticity of (2.5) except where $u = 0$ or $u' = 0$ (see [4, Theorem 8.1, Ch. 1]). The final assertions can be proved by using the known relation (e.g., [13, Lemma 1])

$$(p-1)^{-1}|\sin_p|^p + |\sin_p'|^p = 1$$

to calculate $\sin_p''$ and $\sin_p'''$, together with $\sin_p 0 = 0 = \sin_p' \frac{\pi_p}{2}$ and the periodicity properties of $\sin_p$. (For further details, see [3, Lemma 5.1]). ☐

To determine the eigenvalues and eigenfunctions of (1.1), (1.3), we introduce the functions $e_k(t) \in B_p$, for integer $k \geq 0$ and $t \in \mathbb{R}$, defined by

$$e_0(t)(x) = 1, \quad e_k(t)(x) = \sin_p(2k(x+t)), \quad x \in [0, \pi_p]. \tag{2.6}$$

It is clear that the mappings $t \to e_k(t) : \mathbb{R} \to B_p$ are π_p-periodic.

Lemma 2.4. *For* $q = 0$ *and* $k \geq 0$, *the* k*th periodic eigenvalue* λ_k *equals* $(2k)^p$, *with corresponding eigenfunctions* $e_k(t)$, $t \in \mathbb{R}$. *There are no other periodic eigenvalues, and* (*up to scaling*) *no other eigenfunctions.*

This is a straightforward calculation (cf. [11, pp. 442-3], where other boundary conditions are also considered). We remark that the eigenvalues in Lemma 2.4 are to be understood in our standing sense of classical solutions, and are numbered without attempting to count any "multiplicity".

Lemma 2.4 also shows that for any $k \geq 1$, the eigenvalue λ_k is not simple. Let us consider the mapping $e_k : t \to e_k(t) : \mathbb{R} \to B_p$ in more detail. It will be shown in Lemma 3.6 that this mapping is C^1, and by periodicity, $e_k(t)$ parametrises a non-trivial closed loop of eigenfunctions in B_p. Also, denoting the set of all eigenfunctions corresponding to λ_k by E_k, we see from the homogeneity of the problem that E_k is parametrised by the mapping $(s,t) \to se_k(t) : \mathbb{R}\backslash\{0\} \times \mathbb{R} \to B_p$. Thus E_k is a two-dimensional, C^1 manifold in B_p, and the tangent space of E_k at the point $e_k(t)$ has a basis given by $e_k(t)$ and the t derivative $\dot{e}_k(t)$. This tangent space will play an important rôle for us as the nullspace of an appropriate linearisation of (1.1), (1.3).

As mentioned earlier, E_k is a 2-dimensional (punctured) plane in B_p, in the linear case $p = 2$. By contrast, when $p \neq 2$, it follows from the proof of Lemma 4.10 below that the linear span of E_k is infinite dimensional. Lemma 2.3 above provides a crucial step in the argument for Lemma 4.10, which in turn is one of the keys to proving Theorem 1.3.

3. THE INVERSE p-LAPLACIAN

¿From now on, Δ_p will denote the periodic p-Laplacian, with (maximal) domain consisting of u such that

$$u,\ \phi_p(u') \text{ are } AC \text{ and satisfy (1.3).} \tag{3.1}$$

As indicated earlier, we shall also use Δ_p to denote restrictions as needed. We consider the problem

$$\Delta_p u = h, \quad h \in L^1(0, \pi_p). \tag{3.2}$$

Since we allow $h \in L^1(0, \pi_p)$ in (3.2), this equation is taken to hold a.e. on $(0, \pi_p)$, in the Carathéodory sense.

3.1. Existence of Δ_p^{-1}.

We first define

$$Mu(x) := \frac{1}{\pi_p} \int_0^{\pi_p} u, \quad u \in L^1(0, \pi_p),\ x \in [0, \pi_p],$$

so M maps $L^1(0, \pi_p)$ to constant functions. By integrating (3.2) over $[0, \pi_p]$ and using (1.3) we obtain $Mh = 0$, so

$$M\Delta_p u = 0, \tag{3.3}$$

for all u in the domain of Δ_p. In view of this we define

$$E := \{v \in L^1(0, \pi_p) : Mv = 0\}, \quad E^j := E \cap C^j[0, \pi_p],\ j = 0, 1, \tag{3.4}$$

and so $R(\Delta_p) \subset E$.

To construct the operator Δ_p^{-1} we define operators $\mathcal{I} : L^1(0, \pi_p) \to W^{1,1}(0, \pi_p)$ and $T_p : C^0[0, \pi_p] \to C^1[0, \pi_p]$ by

$$\begin{aligned} \mathcal{I}(h)(x) &:= \int_0^x h(s)\, ds, \quad && h \in L^1(0, \pi_p), \\ T_p(g) &:= \mathcal{I}([g]^{p^*}), && g \in C^0[0, \pi_p], \end{aligned}$$

where from now on we write

$$p^* = (p - 1)^{-1}. \tag{3.5}$$

Clearly, these operators are continuous, and $\mathcal{I}$ is also linear.

Theorem 3.1. *If $h \in E$ then $\Delta_p u = h$ has a unique solution $u = \Delta_p^{-1}(h) \in E^1$ given by*

$$\Delta_p^{-1}(h) := T_p(\gamma_1(h) + \mathcal{I}(h)) + \gamma_2(h), \quad h \in E, \tag{3.6}$$

where the constant functions $\gamma_1(h)$, $\gamma_2(h)$ satisfy the equations

$$T_p(\gamma_1(h) + \mathcal{I}(h))(1) = 0, \tag{3.7}$$

$$M\big(T_p(\gamma_1(h) + \mathcal{I}(h)) + \gamma_2(h)\big) = 0. \tag{3.8}$$

Hence, $R(\Delta_p) = E$, and the operator $\Delta_p^{-1} : E \to E^1$ is continuous.

Proof. Since the basic ideas are already in [11, Theorem 20], we will simply sketch the proof.

First, direct verification shows that solutions of $\Delta_p u = h$ must be of the form given in (3.6), with (3.7), (3.8) corresponding to (1.3). We note that for any given $h \in E$, $T_p(y + \mathcal{I}(h))(1)$ is a continuous, strictly increasing function of y, and tends to $\pm\infty$ as $y \to \pm\infty$. Hence, a unique solution $\gamma_1(h)$ of (3.7) exists, and then equation (3.8) is equivalent to $\gamma_2(h) = -M\big(T_p(\gamma_1(h) + \mathcal{I}(h))\big)$. Thus $\Delta_p^{-1}(h)$ is well-defined by (3.6), and since both $\gamma_1(h), \gamma_2(h)$ depend continuously on h, the operator $\Delta_p^{-1} : E \to C^1[0, \pi_p]$ is continuous. $\square$

3.2. **Differentiability of Δ_p^{-1}.** We now discuss the differentiability of Δ_p^{-1}. Examining (3.6), we find that differentiability of T_p is the most complicated part. In particular, the results will depend on the value of p, and we use p-dependent choices of domain and range for T_p, recalling the notation B_p from (2.1).

Theorem 3.2. (A) *Suppose that* $1 < p < 2$. *Then* $T_p : C^0[0, \pi_p] \to B_p$ *is* C^1, *and for any* g, $\bar g \in C^0[0, \pi_p]$,

$$DT_p(g)\bar g = p^* \mathcal{I}(|g|^{p^*-1}\bar g). \tag{3.9}$$

(B) *Suppose that* $p > 2$ *and* $g \in C^1[0, \pi_p]$ *has only simple zeros. Then* $T_p : C^1[0, \pi_p] \to B_p$ *is* C^1 *on a neighbourhood of* g *in* $C^1[0, 1]$, *with derivative given by* (3.9) *for* $\bar g \in C^1[0, \pi_p]$).

Proof. For any g, $\bar g \in C^0[0, \pi_p]$ and $x \in [0, \pi_p]$, let

$$\begin{aligned}\Xi(x) &:= \Big|\Big(T_p(g + \bar g)(x) - T_p(g)(x) - DT_p(g)\bar g(x)\Big)'\Big| \\ &= \Big|[g(x) + \bar g(x)]^{p^*} - [g(x)]^{p^*} - p^*|g(x)|^{p^*-1}\bar g(x)\Big|,\end{aligned} \tag{3.10}$$

where $DT_p(g)\bar g$ is taken to be the right hand side of (3.9).

Lemma 3.3. *For any* a, $\delta \in \mathbb{R}$,

$$\big|[a + \delta]^\alpha - [a]^\alpha - \alpha|a|^{\alpha-1}\delta\big| \le \begin{cases} c_\alpha|\delta|^2(|a|^{\alpha-2} + |\delta|^{\alpha-2}), & \alpha \ge 2, \\ c_\alpha|\delta|^\alpha, & 1 \le \alpha < 2, \\ c_\alpha|\delta|^{1+\alpha/2}|a|^{-1+\alpha/2}, & 0 < \alpha < 1, \quad a \ne 0, \end{cases}$$

where c_α *depends only on* α.

Proof. We may assume that $a \ge 0$. Suppose first that $|\delta| > a/2$. The result is clear if $\alpha \ge 1$, so assume that $0 < \alpha < 1$ and $a > 0$. By considering a/δ in the ranges $(-2, -1)$, $[-1, 0)$ and $(0, 2)$, we see that

$$|\delta|^{-\alpha}\big|[a + \delta]^\alpha - [a]^\alpha\big| \le 1,$$

so

$$\big|[a+\delta]^\alpha - [a]^\alpha - \alpha|a|^{\alpha-1}\delta\big| \le |\delta|^{1+\alpha/2}(|\delta|^{-1+\alpha/2} + \alpha a^{-1+\alpha}|\delta|^{-\alpha/2}) \le 4|\delta|^{1+\alpha/2}a^{-1+\alpha/2}.$$

Now suppose that $0 < |\delta| \le a/2$. Then, applying the the mean value theorem twice, there exists $\eta \in (0, 1)$ such that

$$\big|[a + \delta]^\alpha - [a]^\alpha - \alpha|a|^{\alpha-1}\delta\big| = \alpha(\alpha - 1)|\delta|^2(a + \eta\delta)^{\alpha-2} \le c_\alpha|\delta|^2a^{\alpha-2}.$$

This proves the result when $\alpha \geq 2$. If $\alpha < 2$ then

$$|\delta|^2 a^{\alpha-2} \leq \begin{cases} |\delta|^\alpha (|\delta|/a)^{2-\alpha} \leq |\delta|^\alpha, & 1 \leq \alpha < 2, \\ |\delta|^{1+\alpha/2}(a/2)^{1-\alpha/2} a^{\alpha-2} < |\delta|^{1+\alpha/2} a^{-1+\alpha/2}, & 0 < \alpha < 1, \end{cases}$$

which completes the proof of Lemma 3.3. □

We now return to the proof of the theorem, setting $\alpha = p^*$ in Lemma 3.3. In case (A), $\alpha > 1$, so it follows from Lemma 3.3 that $|\Xi|_0/|\bar{g}|_0 \to 0$ as $|\bar{g}|_0 \to 0$, from which (3.9) can readily be proved. Continuity of $DT_p(g)$ with respect to g is clear from (3.9).

In case (B), $0 < \alpha < 1$, so if $g(x) \neq 0$ then it follows from Lemma 3.3 that $|\Xi(t)|/|\bar{g}|_1 \to 0$ as $|\bar{g}|_1 \to 0$. By the dominated convergence theorem, $\|\Xi\|_1/|\bar{g}|_1 \to 0$ as $|\bar{g}|_1 \to 0$, so (3.9) follows in this case. Continuity of $DT_p(g)$ now follows from Lemma 2.1 together with (3.9). This completes the proof of Theorem 3.2. □

Now we are ready for the differentiability of Δ_p^{-1}.

Theorem 3.4. *For $h \in E$, let $u = u(h) := \Delta_p^{-1}(h)$.*
(A) Suppose that $1 < p < 2$. Then $\Delta_p^{-1} : E \to B_p$ is C^1, and for $\bar{h} \in E$,

$$D\Delta_p^{-1}(h)\bar{h} = p^* \mathcal{I}\left(|u'|^{2-p}(D\gamma_1(h)\bar{h} + \mathcal{I}(\bar{h}))\right) + D\gamma_2(h)\bar{h}, \tag{3.11}$$

$$v = D\Delta_p^{-1}(h)\bar{h} \implies -(|u'|^{p-2}v')' = p^*\bar{h}. \tag{3.12}$$

(B) Suppose that $p > 2$ and $h_0 \in E^0$ is such that $u_0'(x) = 0 \implies h_0(x) \neq 0$, for $x \in [0, \pi_p]$ (with $u_0 = u(h_0)$). Then there exists a neighbourhood V_0 of h_0 in E^0 such that $h \to |u(h)'|^{2-p} : V_0 \to L^1(0, \pi_p)$ is continuous, $\Delta_p^{-1} : V_0 \to B_p$ is C^1, and its derivative satisfies (3.11) and (3.12) (for $\bar{h} \in E^0$).

Proof. (A) In view of Theorem 3.2 (A) and the form of Δ_p^{-1} in (3.6), to prove differentiability of Δ_p^{-1} we only need to show that the functions γ_1, γ_2 are differentiable. This follows readily from Theorem 3.2 (A) and equations (3.7) and (3.8), by the implicit function theorem. Hence, (3.11) follows from (3.6), and we obtain (3.12) simply by differentiating (3.11) with respect to x (clearly, $(D\gamma_j(h)\bar{h})' = 0$ for $j = 0, 1$).

(B) For $h \in E^0$, let $g(h) := \gamma_1(h) + \mathcal{I}(h) \in C^1[0, \pi_p]$. The mapping $g : E^0 \to C^1[0, \pi_p]$ is continuous. Differentiating $u = \Delta_p^{-1}(h)$ with respect to x and using (3.6), we obtain

$$\phi_p(u') = \gamma_1(h) + \mathcal{I}(h) = g(h), \quad h = \phi_p(u')' = g(h)'. \tag{3.13}$$

Hence $u' \in C^0[0, \pi_p]$, and also, by the hypothesis in the theorem,

$$g(h_0)(x) = 0 \implies \phi_p(u_0')(x) = 0 \implies u_0'(x) = 0 \implies g(h_0)'(x) = h_0(x) \neq 0,$$

so $g(h_0)$ has only simple zeros. Continuity of $h \to |u(h)'|^{2-p} = |g(h)|^{p^*-1} : V^0 \to L^1(0, \pi_p)$ now follows from continuity of g and Lemma 2.1. Differentiability of Δ_p^{-1} then follows from Theorem 3.2 (B) and the preceding argument for case (A). □

Remark 3.5. Corresponding operators Δ_p^{-1} can be constructed for a broad class of separated boundary conditions — see [11] for details in the cases of Dirichlet, Neumann and mixed conditions. The main difference from the above construction is that in the separated case only one constant $\gamma(h)$ occurs and, except for the Neumann case, the inverse will generally be defined on the whole of $L^1(0, \pi_p)$, not just on E. Then an almost identical argument to the above yields differentiability of the cited inverse operators.

3.3. **Connections with e_k, ϕ_p and M.** We start with some additional properties of the functions e_k, $k \geq 1$, defined in (2.6).

Lemma 3.6. *For any $p > 1$ $(p \neq 2)$ and $k \geq 1$, the mapping $e_k : \mathbb{R} \to B_p$ is C^1. For any $t \in \mathbb{R}$,*

$$e_k(t) = -\Delta_p^{-1}(\lambda[e_k(t)]^{p-1}) \tag{3.14}$$

and

$$M(e_k(t)) = M([e_k(t)]^{p-1}) = M(\dot{e}_k(t)) = M(|e(t)|^{p-2}\dot{e}_k(t)) = 0. \tag{3.15}$$

Proof. Equations (3.14), (3.15) are immediate from the constructions of $e_k(t)$, Δ_p^{-1} and M. Next, $\sin_p$ is C^1 on $\mathbb{R}$, so the mapping $e_k : \mathbb{R} \to C^1[0, \pi_p]$ is continuous. It remains to show that $\dot{e}_k : \mathbb{R} \to B_p$ is continuous.

For each $t \in \mathbb{R}$, $\dot{e}_k(t) = e_k(t)'$, and $e_k(t)'$ is an absolutely continuous function, with

$$\dot{e}_k(t)' = e_k(t)'' = -p^*|e_k(t)'|^{2-p}\lambda[e_k(t)]^{p-1}. \tag{3.16}$$

If $1 < p < 2$ then the mappings $[e_k(\cdot)]^{p-1}$, $|e_k(\cdot)'|^{2-p} : \mathbb{R} \to C^0[0, \pi_p]$ are continuous, so the proof is clear in this case. On the other hand, if $p > 2$ then $[e_k(\cdot)]^{p-1}$ is continuous from $\mathbb{R}$ to E^0 – see (3.15). Thus by (3.14), (3.16) and part (B) of Theorem 3.4, the mapping $|e_k(\cdot)'|^{2-p} : \mathbb{R} \to L^1(0, \pi_p)$ is continuous, and the result again follows. ☐

Next, M and $I - M$ are projections on $L^1(0, \pi_p)$ and are $\langle \cdot\,, \cdot \rangle$-symmetric, in the sense that

$$\langle Mu_1, u_2 \rangle = (\pi_p)^{-1} \int_0^{\pi_p} u_1 \int_0^{\pi_p} u_2 = \langle u_1, Mu_2 \rangle, \quad u_1, u_2 \in L^1(0, \pi_p). \tag{3.17}$$

Moreover Δ_p commutes with M and with $I - M$ — these are separate statements since Δ_p is nonlinear. More precisely, we have the following

Lemma 3.7. *M is C^1 from $L^1(0, \pi_p)$ to $C^1[0, \pi_p]$, and for any u in the domain of Δ_p (given by (3.1)),*

$$M\Delta_p u = \Delta_p M u = 0, \quad (I - M)\Delta_p u = \Delta_p(I - M)u. \tag{3.18}$$

In particular, Δ_p^{-1} commutes with M and with $I - M$ on $R(\Delta_p) = E = R(I - M)$.

Proof. It is clear that M is linear and continuous from $L^1(0, \pi_p)$ into $C^1[0, \pi_p]$. Next,

$$M\Delta_p u = 0 \tag{3.19}$$

follows from (1.3) and (3.6). Since Mu is constant, $((I - M)u)' = u'$ whence $\Delta_p((I - M)u) = \Delta_p u$, so (1.3) follows from (3.3) and (3.19). The commutativity statement then follows from a standard argument. ☐

We also note the following properties of $\phi_p : f \to [f]^{p-1}$.

Lemma 3.8. *(A) Suppose that $1 < p < 2$ and $g \in C^1[0, \pi_p]$ has only simple zeros. Then $\phi_p : C^1[0, \pi_p] \to L^1(0, \pi_p)$ is C^1 on a neighbourhood of g in $C^1[0, \pi_p]$, and for any $\bar{g} \in C^1[0, \pi_p]$,*

$$D\phi_p(g)\bar{g} = (p - 1)|g|^{p-2}\bar{g}. \tag{3.20}$$

(B) Suppose that $p > 2$. Then $\phi_p : C^0[0, \pi_p] \to C^0[0, \pi_p]$ is C^1, with derivative as in (3.20) for any g, $\bar{g} \in C^0[0, \pi_p]$.

Proof. Referring to (3.10), we see that the proof of Theorem 3.2 actually establishes this result (we replace p^* with $p-1$; note that cases (A) and (B) are interchanged by this replacement). $\square$

Combining these results we have the following conclusion, which will be needed in the next section. We write $\Phi_p := \Delta_p^{-1} \circ (I-M) \circ \phi_p$.

Theorem 3.9. *(A) Suppose that* $1<p<2$. *Then the operator* $\Phi_p : C^1[0,\pi_p] \to B_p$ *is* C^1 *on a neighbourhood of* $e_k(t)$, $t \in \mathbb{R}$.
(B) Suppose that $p>2$. *Then the operator* $\Phi_p : C^0[0,\pi_p] \to B_p$ *is* C^1 *on a neighbourhood of* $e_k(t)$, $t \in \mathbb{R}$.

In each case, the derivative $D\Phi_p(u)$ *is compact on the specified spaces.*

Proof. (A) Differentiability of Φ_p follows from Lemmas 3.7 and 3.8, together with part (A) of Theorem 3.4 and the chain rule. Compactness of the derivative follows from (3.11), (3.20), compactness of the embedding $C^1[0,\pi_p] \to C^0[0,\pi_p]$ and continuity of $\Delta_p^{-1} : E \to C^0[0,\pi_p]$.

(B) In this case, $u'(x) = 0 \implies \phi_p(u(x)) \neq 0$ so the hypothesis in part (B) of Theorem 3.4 holds with $h = \phi_p(u)$. Then differentiability of Φ_p near u follows similarly to (A) above. Compactness of the derivative follows from compactness of the operator $\Delta_p^{-1} : E^0 \to W^{1,1}(0,\pi_p)$, which follows easily from the form of Δ_p^{-1} in (3.11). $\square$

Remark 3.10. Differentiability of Δ_p^{-1}, in the case of mixed Dirichlet and Neumann boundary conditions (when the γ_j are unnecessary) is discussed in [10, Section 2]. In particular, Corollary 6 of [10] corresponds to the hypotheses and conclusions of part (B) of Theorem 3.4, but for all $p>1$. When $1<p<2$, however, this result is weaker than part (A) of Theorem 3.4. In particular, when $1<p<2$ the results in [10] are not strong enough to yield the differentiability of Φ_p as in Theorem 3.9 above. (We note that M is not needed in [10]).

This lack of differentiability seems to be the reason why the bifurcation results in [10, Section 4] are only proved for $p>2$ (see the remarks on p. 37 of [10]). We also remark that the arguments of [10, Section 2] seem to be incomplete, and so our results not only provide a valid basis for those of [10] when $p>2$, but also open the possibility of their validity for $1<p<2$, although we have not checked this.

4. Proof of Theorem 1.3

The remainder of our analysis is devoted to proving Theorem 1.3. To construct a suitable $q_{k,n}$ we consider the equation

$$-\Delta_p(u) + \epsilon q \phi_p(u) = (\lambda_k + \epsilon\mu)\phi_p(u), \tag{4.1}$$

where $q \in C^1[0,\pi_p]$, $\epsilon \in \mathbb{R}$ and $\lambda_k = (2k)^p$ is the constant coefficient eigenvalue constructed in Lemma 2.4. By Lemma 3.6, when $\epsilon = 0$, the mapping $t \to e_k(t)$ gives a closed, C^1 curve of solutions of (4.1) in B_p. We will find $q \in C^1[0,\pi_p]$ such that solutions “bifurcate” from this curve when $\epsilon \neq 0$. This has some resemblance to [2, Theorem 4.2], but the perturbation in (4.1) is different from that in [2], and the problem considered in [2] is semilinear (the analogue of Δ_p in [2] is linear and non-singular, neither of which are true here). These considerations lead to major differences in the analysis, so we give the entire proof.

¿From now on we simplify our notation by suppressing the subscripts from λ_k and e_k. We also suppose throughout this section that $p > 1$, $p \neq 2$.

We first reformulate (4.1) as a functional equation. Defining

$$Q(\mu, u, \epsilon) := (\epsilon(q - \mu) - \lambda)\phi_p(u),$$

for $(\mu, u, \epsilon) \in \mathbb{R} \times B_p \times \mathbb{R}$, we can rewrite (4.1) as

$$\Delta_p u = Q(\mu, u, \epsilon). \tag{4.2}$$

Now define $F : \mathbb{R} \times B_p \times \mathbb{R} \to B_p$ by

$$F(\mu, u, \epsilon) := u - \Delta_p^{-1}(I - M)Q(\mu, u, \epsilon) - M(u + Q(\mu, u, \epsilon)). \tag{4.3}$$

Lemma 4.1. *Equation* (4.1) *is equivalent to the equation*

$$F(\mu, u, \epsilon) = 0. \tag{4.4}$$

Moreover

$$F(\mu, e(t), 0) = 0, \quad (\mu, t) \in \mathbb{R}^2. \tag{4.5}$$

Proof. Suppose that (4.1), and hence (4.2), is satisfied. Operating by M and using Lemma 3.7, we have $MQu = 0$ — here and below we write Qu for $Q(\mu, u, \epsilon)$ when there is no confusion. Operating by $(I - M)\Delta_p^{-1}$ on (4.2), and using Lemma 3.7, we obtain $(I - M)u - \Delta_p^{-1}(I - M)Qu = 0$. Addition then yields (4.4).

Conversely, suppose that (4.4) is satisfied. Applying M and $I - M$ and using Lemma 3.7, we obtain $MQu = 0$ and $(I - M)u = (I - M)\Delta_p^{-1}Qu$. With Lemma 3.7 again, these yield $M(\Delta_p u - Qu) = 0 = (I - M)(\Delta_p u - Qu)$, and (4.2) follows.

Finally, (4.5) follows from Lemma 3.6 (in fact, $F(\mu, u, 0)$ is independent of μ). □

4.1. **Linearisation and projections.** By Lemmas 3.7 and 3.8, and Theorem 3.9 (modified to deal with the term $\lambda + \epsilon(\mu - q)$, but this is trivial since q is C^1), $F(\mu, u, \epsilon)$ is C^1 in a neighbourhood of the point $(\mu, e(t), 0)$, for any $t \in \mathbb{R}$. Thus we may define the operator

$$L(t) := D_u F(\mu, e(t), 0) : B_p \to B_p,$$

and the mapping $t \to L(t)$ is C^0 on $\mathbb{R}$. Writing

$$D\Delta_p^{-1}(t) := D\Delta_p^{-1}(\lambda\phi_p(e(t))),$$

we see from (3.20) that, for any $v \in B_p$,

$$\begin{aligned} L(t)v &= v + D\Delta_p^{-1}(t)[(I - M)\lambda D\phi_p(e(t))v] + M[-v + \lambda D\phi_p(e(t))v] \\ &= (I - M)v + D\Delta_p^{-1}(t)[(I - M)\lambda(p - 1)|e(t)|^{p-2}v] + M[\lambda(p - 1)|e(t)|^{p-2}v]. \end{aligned} \tag{4.6}$$

There is an alternative characterisation of the operator $L(t)$, more in keeping with the original operator Δ_p.

Lemma 4.2. *For any $t \in \mathbb{R}$ and $v \in B_p$, if $w = L(t)v$ then*

$$-(|e(t)'|^{p-2}(v - w)')' = \lambda(I - M)(|e(t)|^{p-2}v). \tag{4.7}$$

Proof. We can rewrite (4.6) as

$$v - w - M[v + \lambda|e(t)|^{p-2}v] = D\Delta_p^{-1}(\lambda\phi_p(e(t)))(I - M)\big(\lambda(p - 1)|e(t)|^{p-2}v\big).$$

Now, using (3.12) (with $h = \lambda\phi_p(e(t))$, $\bar{h} = (I - M)\lambda(p - 1)|e(t)|^{p-2}v$), (3.14) and $(Mg)' = 0$ for any $g \in L^1$, we obtain (4.7). □

The operator $L(t)$ is not one-to-one. In fact we have the following result.

Lemma 4.3. *For each* $t \in \mathbb{R}$,

$$N(L(t)) = \operatorname{span}\{e(t), \dot{e}(t)\}, \tag{4.8}$$

and $R(L(t))$ *is closed, with* $\operatorname{codim} R(L(t)) = 2$.

Proof. Differentiating (4.5) with respect to t, and the identity $F(\mu, se(t), 0) \equiv 0$ with respect to $s \in \mathbb{R}$, at $s = 1$, we obtain

$$L(t)e(t) \equiv 0, \quad L(t)\dot{e}(t) \equiv 0, \tag{4.9}$$

and so $\dim N(L(t)) \geq 2$.

Now suppose that $v \in N(L(t))$. Then, from Lemma 3.7 and (4.6),

$$0 = M(L(t)v) = \lambda(p-1)M(|e(t)|^{p-2}v), \tag{4.10}$$

so Lemma 4.2 (with $w = L(t)v = 0$) yields

$$-(|e(t)'|^{p-2}v')' = \lambda|e(t)|^{p-2}v.$$

By [10, Theorem 7] the set of solutions of this differential equation is 2-dimensional (this is not trivial, since the equation has either degeneracies or singularities at the zeros of $e(t)'$, depending on whether $p < 2$ or $p > 2$ — [10] deals with both these cases). Hence, $\dim N(L(t)) \leq 2$, which with (4.9) proves (4.8). Furthermore, by Theorem 3.9 and the definition of the space B_p, the operator $L(t) : B_p \to B_p$ has the form identity+compact. Thus the results regarding $R(L(t))$ follow immediately from the properties of the null-space $N(L(t))$. □

The operator $L(t)$ is not $\langle\cdot\,,\cdot\rangle$-symmetric, but by introducing some new inner products we can obtain a result close to this, and also define a type of orthogonal projection on to $N(L)$. For each $t \in \mathbb{R}$ let

$$\langle v_1, v_2\rangle_t := \langle v_1, v_2|e(t)|^{p-2}\rangle, \quad v_1, v_2 \in B_p.$$

Lemma 4.4. *The mapping* $(v_1, v_2, t) \to \langle v_1, v_2\rangle_t : B_p^2 \times \mathbb{R} \to \mathbb{R}$ *is well-defined and continuous.*

Proof. If $p > 2$ this follows immediately from Lemma 3.6, while if $1 < p < 2$ then it follows from Lemmas 2.1 and 3.6. □

Lemma 4.5. *For any* $v_1, v_2 \in B_p$ *and* $t \in \mathbb{R}$,

$$\langle (L(t) + M)\, v_1, v_2\rangle_t = \langle v_1, (L(t) + M)\, v_2\rangle_t.$$

Proof. Let $L(t)v_i = w_i$, $i = 1, 2$. As for (4.10),

$$Mw_i = ML(t)v_i = \lambda(p-1)M(|e(t)|^{p-2}v_i),$$

and hence, by (3.17),

$$\langle w_1, M(|e(t)|^{p-2}v_2)\rangle = \lambda(p-1)\langle M(|e(t)|^{p-2}v_1), |e(t)|^{p-2}v_2\rangle. \tag{4.11}$$

This is symmetric in v_1 and v_2 (by (3.17) again), so

$$\langle w_1, M(|e(t)|^{p-2}v_2)\rangle = \langle M(|e(t)|^{p-2}v_1), w_2\rangle. \tag{4.12}$$

Now, by repeated usage of (3.17), Lemma 4.2, (4.12) and integration by parts, we obtain

$$\begin{aligned}
&\lambda\langle L(t)v_1, v_2\rangle_t = \langle w_1, \lambda|e(t)|^{p-2}v_2\rangle\\
&= \langle w_1, -\big(|e(t)'|^{p-2}(v_2-w_2)'\big)'\rangle + \lambda\langle w_1, M(|e(t)|^{p-2}v_2)\rangle\\
&= \langle -\big(|e(t)'|^{p-2}w_1'\big)', v_2 - w_2\rangle + A\big]_0^{\pi_p} + \lambda\langle w_1, M(|e(t)|^{p-2}v_2)\rangle\\
&= \langle -\big(|e(t)'|^{p-2}v_1'\big)' - \lambda|e(t)|^{p-2}v_1, v_2 - w_2\rangle + A\big]_0^{\pi_p} + \lambda\langle M(|e(t)|^{p-2}v_1), v_2 - w_2\rangle\\
&\qquad + \lambda\langle w_1, M(|e(t)|^{p-2}v_2)\rangle.\\
&= \langle v_1, -\big(|e(t)'|^{p-2}(v_2-w_2)'\big)' + (A+B)\big]_0^{\pi_p} - \lambda|e(t)|^{p-2}v_2\rangle + \lambda\langle v_1, |e(t)|^{p-2}w_2\rangle\\
&\qquad + \lambda\langle M(|e(t)|^{p-2}v_1), v_2\rangle\\
&= (A+B)\big]_0^{\pi_p} - \lambda\langle v_1, M(|e(t)|^{p-2}v_2)\rangle + \lambda\langle v_1, L(t)v_2\rangle_t + \lambda\langle v_1, Mv_2\rangle_t,
\end{aligned}$$

where the boundary terms $(A+B)\big]_0^{\pi_p}$ arising from the integrations by parts satisfy

$$A + B = |e(t)'|^{p-2}((v_2-w_2)'(v_1-w_1) - (v_1'-w_1')(v_2-w_2)).$$

Applying M to (4.7) we find that $|e(t)'|^{p-2}((v_2-w_2)'\big]_0^{\pi_p} = 0$. Moreover, if $u \in R(\Delta_p^{-1})$ then $u(0) = u(\pi_p)$ so from (4.6) and the fact that M takes constant values we see that $(v_1 - w_1)\big]_0^{\pi_p} = 0$. Thus the first product in $(A+B)\big]_0^{\pi_p}$ vanishes, and the second vanishes similarly. The required result now follows readily. ☐

Now, for any $t \in \mathbb{R}$ we define $P(t) : B_p \to N(L(t))$ by

$$P(t)v := \frac{\langle v, e(t)\rangle_t}{\langle e(t), e(t)\rangle_t}e(t) + \frac{\langle v, \dot e(t)\rangle_t}{\langle \dot e(t), \dot e(t)\rangle_t}\dot e(t), \quad v \in B_p, \tag{4.13}$$

and we let $Q(t) := I - P(t)$. By the above results, the operator functions P, Q are C^0 on $\mathbb{R}$.

Lemma 4.6. *For each $t \in \mathbb{R}$,*

$$\langle e(t), \dot e(t)\rangle_t = 0, \tag{4.14}$$

and hence $P(t)$, $Q(t)$ are $\langle\cdot\,,\cdot\rangle_t$-symmetric projections from B_p to $N(L(t))$ and $R(L(t))$, respectively. Moreover

$$Q(t)e(t) = 0, \quad Q(t)\dot e(t) = 0, \quad P(t)L(t) = 0. \tag{4.15}$$

Proof. We start by defining

$$\gamma := \int_0^{\pi_p} |e(t)|^p \, dx. \tag{4.16}$$

By $2\pi_p$-periodicity of $\sin_p$, we see that γ is independent of t, so differentiation of (4.16) with respect to t yields (4.14). It follows from (4.13) and (4.14) that $P(t)$ and $Q(t)$ are $\langle\cdot\,,\cdot\rangle_t$-symmetric projections.

Suppose that $w = L(t)v \in R(L(t))$. Then by (3.15), (4.9) and Lemma 4.5,

$$\langle w, e(t)\rangle_t = \langle v, L(t)e(t)\rangle_t + \langle v, Me(t)\rangle_t - \langle v, M(\phi_p(e(t)))\rangle = 0,$$

and similarly $\langle w, \dot e(t)\rangle_t = 0$. Hence, $R(L(t)) \subset R(Q(t))$, and so the result follows from Lemma 4.3. ☐

4.2. **A bifurcation equation.** We now use the projections P, Q to reformulate (4.4) as a bifurcation-type equation on the null-spaces $N(L(t))$, $t \in \mathbb{R}$.

Let t_0 and μ_0 be arbitrary fixed numbers, and write $P_0 := P(t_0)$, $Q_0 := Q(t_0)$, $W_0 := R(Q_0)$. We look for solutions (μ, u, ϵ) of (4.4) near to $(\mu_0, e(t_0), 0)$, with u having the form $u = e(t) + w$, where $w \in W_0$ is small. Equation (4.4) is equivalent to the pair of equations

$$Q(t)F(\mu, e(t) + w, \epsilon) = 0, \tag{4.17}$$

$$P(t)F(\mu, e(t) + w, \epsilon) = 0, \tag{4.18}$$

and it is clear by (4.5) that $(w, \epsilon) = (0, 0)$ satisfies (4.17)-(4.18) for all $(\mu, t) \in \mathbb{R}^2$. The function F is C^1 (when w, ϵ are small), but P, Q are only C^0, so the functions on the left hand sides of (4.17) and (4.18) are C^1 with respect to (μ, w, ϵ) and C^0 with respect to t. Also, denoting the left hand side of (4.17) by $F_Q(\mu, t, w, \epsilon)$, we see from (4.5) that

$$F_Q(\mu, t, 0, 0) \equiv 0, \quad D_w F_Q(\mu_0, t_0, 0, 0)\bar{w} = L(t_0)\bar{w}, \quad \bar{w} \in W_0.$$

By construction and Lemma 4.6, the mapping $L(t_0) : W_0 \to W_0$ is linear and bijective, so is non-singular. Thus, by the implicit function theorem given in [5, Theorem 15.1], equation (4.17) has a solution $w(\mu, t, \epsilon)$, which is defined and continuous on a neighbourhood of $(\mu_0, t_0, 0)$, and

$$w(\mu, t, 0) \equiv 0. \tag{4.19}$$

Also, by the smoothness properties of F_Q mentioned above and a slight extension of the above theorem in [5], the derivative $D_{(\mu,\epsilon)} w(\mu, t, \epsilon)$ exists and is continuous on this neighbourhood. Substituting the solution w into (4.18), we see that (4.1) is locally equivalent to the equation

$$F_P(\mu, t, \epsilon) := P(t)F(\mu, e(t) + w(\mu, t, \epsilon), \epsilon) = 0.$$

Since $D_\epsilon w$ is continuous it follows from (4.19) that we may write $\epsilon\widetilde{w} := w$, with $\widetilde{w}$ continuous. Also, by (4.5),

$$F_P(\mu, t, 0) = P(t)F(\mu, e(t), 0) \equiv 0, \tag{4.20}$$

for (μ, t) near to (μ_0, t_0), so that

$$F_P(\mu, t, \epsilon) = \epsilon P(t)[D_u F(\mu, e(t), 0)\widetilde{w} + D_\epsilon F(\mu, e(t), 0) + \mathrm{o}(1)]$$

as $\epsilon \to 0$.

Now, by the definition of F and (4.6) with $v = \lambda^{-1} p^* (\mu - q) e(t)$,

$$\begin{aligned} D_\epsilon F(\mu, e(t), 0) &= D\Delta_p^{-1}(t)[(I - M)(\mu - q)\phi_p(e(t))] + M[(\mu - q)\phi_p(e(t))] \\ &= \lambda^{-1} p^* (L(t) - I + M)[(\mu - q)e(t)], \end{aligned}$$

so

$$\begin{aligned} F_P(\mu, t, \epsilon) &= \epsilon P(t)[L(t)\widetilde{w} + \lambda^{-1} p^* (L(t) - I + M)\big((\mu - q)e(t)\big) + \mathrm{o}(1)], \\ &= \epsilon \lambda^{-1} p^* P(t)[(I - M)(q - \mu)e(t) + \mathrm{o}(1)] \end{aligned}$$

by (4.15). Thus we may define a continuous function $G : V_0 \to N(L)$, by

$$G(\mu, t, \epsilon) := \begin{cases} \epsilon^{-1}\lambda(p - 1)F_P(\mu, t, \epsilon), & \epsilon \neq 0, \\ P(t)\big((I - M)(q - \mu)e(t)\big), & \epsilon = 0, \end{cases}$$

where $V_0 \subset \mathbb{R}^3$ is a suitable neighbourhood of $(\mu, t, \epsilon) = (\mu_0, t_0, 0)$. Now, by virtue of Lemma 4.3 we obtain our desired bifurcation-type equation, which we state in the following lemma.

Lemma 4.7. *For $\epsilon \neq 0$, equation* (4.1) *is locally equivalent to the equation*

$$H(\mu, t, \epsilon) := \begin{pmatrix} \langle G(\mu, t, \epsilon), e(t)\rangle_t \\ \langle G(\mu, t, \epsilon), \dot{e}(t)\rangle_t \end{pmatrix} = 0. \tag{4.21}$$

4.3. **Solutions of the bifurcation equation.** We will use the implicit function theorem to construct solutions of (4.21). The required non-singularity condition on DH can be expressed in terms of the function J given by

$$J(t, q) := \int_0^{\pi_p} q|e(t)|^p \, dx, \quad t \in \mathbb{R}. \tag{4.22}$$

Although later the q dependence of $J(t, q)$ will be important, for now we regard $q \in C^1[0, \pi_p]$ as fixed and we simply write $J(t)$.

Lemma 4.8. *The functional $J : \mathbb{R} \to \mathbb{R}$ is C^2, and for any $t \in \mathbb{R}$,*

$$\dot{J}(t) = p\langle q\phi_p(e(t)), \dot{e}(t)\rangle. \tag{4.23}$$

Proof. Differentiation of (4.22) with respect to t yields (4.23). Now, Lemma 3.6 shows that the mappings $t \to e(t)$, $\dot{e}(t) : \mathbb{R} \to C^0[0, \pi_p]$ are continuous, so it follows from (4.23) that $\dot{J}(t)$ depends continuously on t.

Next, differentiation of (4.23) yields

$$\ddot{J}(t) = p(p-1)\langle q\dot{e}(t), \dot{e}(t)\rangle_t + p\langle q\phi_p(e(t)), \dot{e}(t)'\rangle$$

(recall that $\ddot{e}(t) = \dot{e}(t)'$). By Lemmas 3.6 and 4.4, the above inner products depend continuously on t, so we conclude that $\ddot{J}(t)$ is C^2 on $\mathbb{R}$. □

If $\dot{J}(t) = 0$ then t is a *critical point* of J, with *critical value* $J(t)$; a critical point t is *non-degenerate* if $\ddot{J}(t) \neq 0$. We are now ready to establish existence of solutions to (4.1).

Theorem 4.9. *Suppose that t_0 is a non-degenerate critical point of J. Then there is an $\epsilon_0 > 0$ such that if $|\epsilon| < \epsilon_0$ then* (4.1) *has an eigenvalue $\lambda(\epsilon) \in \sigma_k(\epsilon q)$ of the form $\lambda(\epsilon) = \lambda + \epsilon\mu(\epsilon)$, where $\mu(\epsilon) \to J(t_0)/\gamma$ as $\epsilon \to 0$, and γ is defined in* (4.16).

Proof. The function $H : V_0 \to \mathbb{R}^2$ of (4.21) is C^0. Also, when $\epsilon = 0$ it follows from Lemmas 4.3 and 4.6 and (3.15) that

$$H(\mu, t, 0) = \begin{pmatrix} \langle (I - M)(q - \mu)e(t), e(t)\rangle_t \\ \langle (I - M)(q - \mu)e(t), \dot{e}(t)\rangle_t \end{pmatrix} = \begin{pmatrix} \langle (q - \mu)e(t), e(t)\rangle_t \\ \langle (q - \mu)e(t), \dot{e}(t)\rangle_t \end{pmatrix}.$$

By virtue of (4.14), (4.16), (4.22) and (4.23), then, (4.21) becomes

$$H(\mu, t, 0) = \begin{pmatrix} J(t) - \mu\gamma \\ \dot{J}(t)/p \end{pmatrix} = 0 \tag{4.24}$$

for $\epsilon = 0$. Thus, by Lemma 4.8, $H(\mu, t, 0)$ is differentiable with respect to (μ, t), and

$$D_{(\mu,t)}H(\mu, t, 0) = \begin{pmatrix} \dot{J}(t) & -\gamma \\ \ddot{J}(t)/p & 0 \end{pmatrix}. \tag{4.25}$$

We conclude from (4.24) and (4.25) that $(\mu_0, t_0, 0)$ is a solution of (4.21) if and only if t_0 is a critical point of J with critical value $\mu_0\gamma$, and if this critical point is non-degenerate then $D_{(\mu,t)}H(\mu_0,t_0,0)$ is non-singular. Thus, by a simple degree argument, for sufficiently small $\epsilon \neq 0$ a non-degenerate critical point t_0 of J gives rise to a solution $(t(\epsilon),\mu(\epsilon),\epsilon)$ of equation (4.21) close to $(t_0,J(t_0)/\gamma,0)$. This corresponds, via Lemma 4.7, to a solution $(\lambda(\epsilon),u(\epsilon))$ of (4.1) of the form

$$\lambda(\epsilon)=\lambda+\epsilon\mu(\epsilon), \quad u(\epsilon)=e(t(\epsilon))+w(t(\epsilon),\mu(\epsilon),\epsilon).$$

Now let $\epsilon\to 0$. Then the above results show that $u(\epsilon)\to e(t_0)$ in $C^0[0,\pi_p]$, so if ϵ is sufficiently small the solution $(\lambda(\epsilon),u(\epsilon))$ is non-trivial, and hence $\lambda(\epsilon)$ is an eigenvalue. Also, by Lemma 4.1,

$$-u(\epsilon)=\Delta_p^{-1}(I-M)[(\lambda+\epsilon(\mu(\epsilon)-q))\phi_p(u(\epsilon))]+M[-u(\epsilon)+(\lambda+\epsilon(\mu(\epsilon)-q))\phi_p(u(\epsilon))],$$

so by Theorem 3.1 and Lemma 3.7, $u(\epsilon)\to e(t_0)$ in $C^1[0,\pi_p]$. Thus, if ϵ is sufficiently small then $u(\epsilon)$ has exactly $2k$ simple zeros in $[0,\pi_p)$, and so $\lambda(\epsilon)\in\sigma_k(\epsilon q)$. This completes the proof of Theorem 4.9. ☐

Our final lemma shows that we can choose a function q for which the corresponding functional $J(\cdot,q)$ has sufficiently many non-degenerate critical points.

Lemma 4.10. *For each k, $n\geq 1$, there exists a function $q_{k,n}\in C^1[0,\pi_p]$, such that the functional $J(\cdot,q_{k,n})$ has at least n non-degenerate critical points in $(0,\pi_p)$, with distinct critical values, and no degenerate critical points.*

Proof. Suppose that n is even (if not, replace n by $n+1$), and choose points $0<t_1<\cdots<t_{n+2}<\pi_p/(8k)$. For each $i=1,\ldots,n+2$, Lemma 2.3 shows that the function $e(t_i)$ is analytic on $\mathbb{R}$, except at the points $x=t_i+m\pi_p/(4k)$, $m\in\mathbb{Z}$. Since these points are distinct for all i and m, the set of functions $\{|e(t_i)|^p\}$ is linearly independent over the interval $[0,\pi_p]$, and so there exists a function $\widetilde{q}_n\in C^1[0,\pi_p]$ such that, for each $i=1,\ldots,1+n/2$,

$$J(t_{2i-1},\widetilde{q}_n)=\int_0^{\pi_p}\widetilde{q}_n|e(t_{2i-1})|^p\,dx<-1, \quad J(t_{2i},\widetilde{q}_n)=\int_0^{\pi_p}\widetilde{q}_n|e(t_{2i})|^p\,dx>1.$$

Thus the functional $J(\cdot,\widetilde{q}_n)$ has at least n critical points in $(0,\pi_p)$. A genericity argument now shows that there exists a nearby function $q_{k,n}\in C^1[0,\pi_p]$ such that $J(\cdot,q_{k,n})$ has at least n non-degenerate critical points in $(0,\pi_p)$, with distinct critical values, and no degenerate critical points. ☐

We can now substitute $q=q_{k,n}$ from Lemma 4.10 into Theorem 4.9 to complete the proof of Theorem 1.3.

Remark 4.11. Since the construction of the eigenvalues and eigenfunctions in the proof of Theorem 1.3 used the implicit function theorem and degree theory, we can also conclude that they persist (at least locally) under small perturbations of the coefficient function q in $C^1[0,\pi_p]$. Thus, these large collections of eigenvalues in σ_k can occur for a relatively "large" set of q.

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