SPECTRAL PROPERTIES AND NODAL SOLUTIONS FOR SECOND-ORDER, m-POINT, BOUNDARY VALUE PROBLEMS

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Abstract. We consider the m-point boundary value problem consisting of the equation

$$-u'' = f(u), \quad \text{on } (0,1),$$

where $f : \mathbb{R} \to \mathbb{R}$ is C^1, with $f(0) = 0$, together with the boundary conditions

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i),$$

where $m \geq 3$, $\eta_i \in (0,1)$ and $\alpha_i > 0$ for $i = 1, \dots, m-2$, with

$$\sum_{i=1}^{m-2} \alpha_i < 1.$$

We first show that the spectral properties of the linearisation of this problem are similar to the well-known properties of the standard Sturm-Liouville problem with separated boundary conditions (with a minor modification to deal with the multi-point boundary condition). These spectral properties are then used to prove a Rabinowitz-type global bifurcation theorem for a bifurcation problem related to the above problem. Finally, we use the global bifurcation theorem to obtain nodal solutions (that is, sign-changing solutions with a specified number of zeros) of the above problem, under various conditions on the asymptotic behaviour of f.

1. Introduction

We consider the m-point boundary value problem consisting of the equation

$$-u'' = f(u), \quad \text{on } (0,1), \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is C^1, with $f(0) = 0$, together with the boundary conditions

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \tag{1.2}$$

where $m \geq 3$, $\eta_i \in (0,1)$ and $\alpha_i > 0$ for $i = 1, \dots, m-2$, with

$$A := \sum_{i=1}^{m-2} \alpha_i < 1. \tag{1.3}$$

We study the existence of sign-changing solutions of (1.1)–(1.2), having a given number of zeros (so called 'nodal solutions'). The existence of such solutions has been studied for many types of nonlinear Sturm-Liouville problems with separated boundary conditions — a brief survey of previous references is given in [4]. Also, multi-point boundary conditions of the above form have been considered in many recent papers, again see the references in [4]. However, only [4] seems to have considered nodal solutions of the problem (1.1)–(1.2). In [4], Ma and O'Regan use a standard, global bifurcation method to obtain nodal solutions of (1.1)–(1.2), under various additional, technical hypotheses. Here, we improve the result stated in [4], and dispense with the technical hypotheses imposed there (we also correct some gaps in the proof in [4]). The paper [7] has also obtained sign-changing solutions of (1.1)–(1.2), but no information is obtained regarding the number of zeros of the solution, and the method of proof is entirely different (relying on degree theory in cones).

Previous results on the existence of nodal solutions (for problems with other types of boundary conditions) rely heavily on information about the spectral properties of the linearisation of the problem. When this linearisation is a Sturm-Liouville operator with separated boundary conditions the required spectral properties are well known. However, in the case of multi-point boundary conditions, these spectral properties do not seem to have been obtained previously (some partial results are obtained in [4], and the principal eigenvalue is considered in [6]). Thus, in Section 3 we derive the required results. The results we obtain are similar to the standard spectral theory of the linear, separated Sturm-Liouville problem, with a slight difference in the nodal counting method used, to deal with the multi-point boundary conditions. We also show that the standard counting method is inadequate in this situation, and that the condition (1.3) is optimal for the results in Section 3 to hold. This section is entirely linear.

In Section 4 we consider a bifurcation problem related to (1.1)–(1.2), and prove a Rabinowitz-type global bifurcation theorem for this problem. The proof uses the spectral properties of the linearisation obtained in Section 3. Finally, in Section 5, we use the global bifurcation theorem from Section 4 to obtain nodal solutions of (1.1)–(1.2), under various hypotheses on the asymptotic behaviour of f. Specifically, we consider the cases where f is asymptotically linear, superlinear or jumping (these terms will be explained below).

2. Preliminary definitions and results

For any integer $n \geq 0$, $C^n[0,1]$ will denote the usual Banach space of n-times continuously differentiable functions on $[0,1]$, with the usual sup-type norm, denoted by $|\cdot|_n$. Let

$$X := \{u \in C^2[0,1] : u \text{ satisfies } (1.2)\}, \quad Y := C^0[0,1],$$

with the norms $|\cdot|_2$ and $|\cdot|_0$ respectively. We define a bounded linear operator $L : X \to Y$ by

$$Lu := -u'', \quad u \in X.$$

This operator has a bounded inverse $L^{-1} : Y \to X$, see [6, Section 6]. In addition, for any continuous function $g : \mathbb{R} \to \mathbb{R}$ and any $u \in Y$, we let $g(u) \in Y$ denote the function $g(u(x))$, $x \in [0,1]$. With this notation we can rewrite the problem (1.1), (1.2) as

$$Lu = f(u), \quad u \in X. \tag{2.1}$$

Next, we introduce some notation to describe the nodal properties of solutions of (1.1). Firstly, for any C^1 function g, if $u(x_0) = 0$ then x_0 is a *simple* zero of g if $g'(x_0) \neq 0$. Now, for any integer $k \geq 1$ and any $\nu \in \{\pm\}$, we define sets $S_k^\nu, T_k^\nu \subset C^2[0,1]$ consisting of the set of functions $u \in C^2[0,1]$ satisfying the following conditions:

$\underline{S_k^\nu}$

(i) $u(0) = 0$, $\nu u'(0) > 0$;
(ii) u has only simple zeros in $[0, 1]$ and has exactly $k - 1$ zeros in $(0, 1)$.

$\underline{T_k^\nu}$

(i) $u(0) = 0$, $\nu u'(0) > 0$ and $u'(1) \neq 0$;
(ii) u' has only simple zeros in $(0, 1)$, and has exactly k such zeros;
(iii) u has a zero strictly between each consecutive zero of u'.

Remark 2.1. One could regard the sets S_k^ν as counting zeros of u, while the sets T_k^ν count 'bumps'. The nodal properties of solutions of nonlinear Sturm-Liouville problems with separated boundary conditions are usually described in terms of sets similar to S_k^ν (with an additional condition at $x = 1$ to incorporate the boundary condition there), see, for example, [3, Section 2]. However, it will be seen below that, when considering the multi-point boundary condition (1.2), the sets T_k^ν are in fact more appropriate than the sets S_k^ν.

Remarks 2.2. (i) If $u \in T_k^\nu$ then u has exactly one zero between each consecutive zero of u', and all zeros of u are simple. Thus, u has at least $k - 1$ zeros in $(0, 1)$, and at most k zeros in $(0, 1]$.
(ii) The sets T_k^ν are open in X and disjoint.

3. Eigenvalues of the linear problem

In this section we consider the linear eigenvalue problem

$$Lu = \lambda u, \quad u \in X. \tag{3.1}$$

We call the set of eigenvalues of (3.1) the *spectrum* of L, and denote it by $\sigma(L)$. As usual, the boundary condition (1.2) ensures that if λ is an eigenvalue then the space of eigenfunctions corresponding to λ is one-dimensional. We now prove the following result regarding $\sigma(L)$.

Theorem 3.1. *The spectrum $\sigma(L)$ consists of a strictly increasing sequence of eigenvalues $\lambda_k > 0$, $k = 1, 2, \ldots$, with corresponding eigenfunctions $\phi_k(x) = \sin(\lambda_k^{1/2} x)$. In addition,*
(i) $\lim_{k\to\infty} \lambda_k = \infty$;
(ii) $\phi_k \in T_k^+$, for each $k \geq 1$, and ϕ_1 is strictly positive on $(0, 1)$.

Proof. Suppose that $\lambda = -s^2 < 0$ is an eigenvalue. Then $u(x) = e^{sx} - e^{-sx}$ is a corresponding eigenfunction. Now, u is increasing on $[0, 1]$ so, by (1.2) and (1.3).

$$u(1) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i) \leq u(1) \sum_{i=1}^{m-2} \alpha_i < u(1), \tag{3.2}$$

and this contradiction shows that λ cannot be an eigenvalue. It can be shown similarly that $\lambda = 0$ is not an eigenvalue, so all the eigenvalues of L are strictly positive.

We now define a function $\Gamma : (0, \infty) \to \mathbb{R}$ by

$$\Gamma(s) := \sin s - \sum_{i=1}^{m-2} \alpha_i \sin(\eta_i s), \quad s \in (0, \infty).$$

It is easy to verify that $\Gamma(s) = 0$ if and only if $\lambda = s^2$ is an eigenvalue of (3.1). In addition, the function $u_s(x) := \sin(sx)$ is then an eigenfunction. Hence, it suffices to search for zeros of Γ.

Now, (1.3) implies that $\Gamma(s) > 0$ on the interval $(0, \pi/2]$, that is, Γ has no zero in this interval, and also

$$\Gamma((k \pm \tfrac{1}{2})\pi) \neq 0, \quad k \geq 1. \tag{3.3}$$

For arbitrary $k \geq 1$, let

$$I_k := ((k - \tfrac{1}{2})\pi, (k + \tfrac{1}{2})\pi).$$

It is clear that if $s \in I_k$ then the above function $u_s \in T_k^+$ so, by the above remarks, to prove the theorem (apart from the positivity of ϕ_1) it suffices to show that Γ has exactly one zero in the interval I_k. To prove this we require the following lemma.

Lemma 3.2. *All the zeros of* Γ *are simple.*

Proof. Suppose that s is a double zero of Γ, that is,

$$\sin s = \sum_{i=1}^{m-2} \alpha_i \sin(\eta_i s), \quad \cos s = \sum_{i=1}^{m-2} \alpha_i \eta_i \cos(\eta_i s). \tag{3.4}$$

Then, by (1.3),

$$\begin{aligned} 1 &= \Big(\sum_{i=1}^{m-2} \alpha_i \sin(s\eta_i)\Big)^2 + \Big(\sum_{i=1}^{m-2} \alpha_i \eta_i \cos(s\eta_i)\Big)^2 \\ &\leq \sum_{i,j=1}^{m-2} \alpha_i \alpha_j (|\sin(s\eta_i)\sin(s\eta_j)| + |\cos(s\eta_i)\cos(s\eta_j|) \\ &\leq \Big(\sum_{i=1}^{m-2} \alpha_i\Big)^2 < 1, \end{aligned}$$

which shows that (3.4) cannot hold, and so Γ has only simple zeros. □

We can now prove the required result.

Lemma 3.3. *For each* $k \geq 1$, Γ *has exactly one zero in* I_k.

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ and let D denote the set of $\alpha \in (0,1)^m$ satisfying (1.3). The functions Γ and Γ' depend continuously on α, and Lemma 3.2 and the inequalities (3.3) hold for all $\alpha \in D$ (and also for $\alpha = 0$, where $\Gamma(s)$ is simply $\sin s$). Thus, by the implicit function theorem, the zeros of Γ depend continuously on $\alpha \in D$, and the number of zeros in I_k cannot change as α varies over D (D is connected). In addition, when $\alpha = 0$ it is clear that Γ has exactly one zero in I_k so, by the implicit function theorem, this is true for sufficiently small $\alpha \in D$, and hence for all $\alpha \in D$. □

Finally, to prove that ϕ_1 is strictly positive on $(0,1)$ we note that

$$\Gamma(\pi) = -\sum_{i=1}^{m-2} \alpha_i \sin(\eta_i \pi) < 0,$$

which implies that $s_1 < \pi$. □

The spectral properties of L, as described in Theorem 3.1, are similar to those of the standard, second-order, Sturm-Liouville operator with separated boundary conditions, see [1], except that in the separated case the kth eigenfunction ϕ_k has exactly $k-1$ zeros in $(0,1)$ (that is, $\phi_k \in S_k^+$). The following simple 3-point examples show the necessity of the use of the sets T_k^+ rather than S_k^+ in Theorem 3.1, and also the necessity of hypothesis $A < 1$.

These examples all have $m = 3$ (we do not state this each time), and for simplicity we omit the subscript 1 on α and η.

Example 3.4. Suppose that $\alpha = \frac{1}{2}$. Then simple estimates on the location of the zeros of Γ for this problem show that:

$$\begin{aligned} \eta \in (0, \tfrac{1}{2}) &\implies \phi_2 \in S_3^+, \\ \eta \in (\tfrac{1}{2}, 1) &\implies \phi_2 \in S_2^+, \\ \eta \in (0, \tfrac{1}{3}) \cup (\tfrac{2}{3}, 1) &\implies \phi_3 \in S_3^+, \\ \eta \in (\tfrac{1}{3}, \tfrac{2}{3}) &\implies \phi_3 \in S_4^+. \end{aligned}$$

In particular, if $\eta \in (\frac{1}{3}, \frac{2}{3})$ then both ϕ_2 and ϕ_3 lie in S_3^+, that is, $\phi_2 \notin S_2^+$, and the nodal sets S_k^ν do not distinguish between eigenfunctions corresponding to different eigenvalues. This also shows that varying η continuously causes the number of zeros in $(0,1)$ of the eigenfunctions to change.

The next example shows that Theorem 3.1 need not be true if $A = 1$.

Example 3.5. Let $\alpha = 1$ and $\eta = 1/5$. It can easily be verified that $\Gamma(15\pi/2) = 0$, and the corresponding eigenfunction $u(x) = \sin(\frac{15\pi}{2}x)$ has $u'(1) = 0$, that is u is not in any set T_k^+. In fact, u lies on the boundary of the sets T_7^+ and T_8^+, while the coefficient $\alpha = 1$ lies on the boundary of the set D in the proof of Theorem 3.1.

Of course, it might be possible to change the definition of the sets T_k^ν to deal with the problem in Example 3.5. However, the next example shows that if $A > 1$ then, no matter how close A is to 1, arbitrarily many nodal eigenfunctions may be 'missing'.

Example 3.6. Choose arbitrary $\epsilon > 0$ and let $\alpha = 1 + \epsilon$. Then there exists $\delta \in (0, \pi/2)$ such that

$$\frac{\pi}{2} - \delta < t < \frac{\pi}{2} + \delta \implies \alpha \sin t > 1.$$

Now choose arbitrary $w > 0$, and let $\eta = \delta/w$. Recalling the construction of the eigenvalues in the proof of Theorem 3.1 via the function Γ, we see that

$$0 < \frac{w\pi}{2\delta} - w < s < \frac{w\pi}{2\delta} + w \implies \Gamma(s) \le 1 - \alpha \sin(\delta s/w) < 0,$$

that is, Γ has no zero on an interval W of length $2w$. Thus, if k is such that $I_k \subset W$ then there are no eigenfunctions lying in T_k^+. Since w can be arbitrarily large, this shows that Theorem 3.1 is not true for this problem. We also note that $\alpha\eta = (1+\epsilon)\delta/w$ may be arbitrarily small.

Remark 3.7. It is shown [6, Section 6] that the optimal condition to obtain a positive principal eigenfunction ϕ_1 is the condition $\sum_{m=1}^{m-2} \alpha_i \eta_i < 1$, that is, this condition is sufficient for Theorem 3.1 to be true for $k = 1$. Example 3.6 shows that this condition is not sufficient for Theorem 3.1 to be true for all $k \ge 1$.

4. Global bifurcation

In this section we consider the following bifurcation problem

$$Lu = \mu u + g(u), \quad (\mu, u) \in \mathbb{R} \times X, \tag{4.1}$$

where we suppose that $g : \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies

$$g(0) = g'(0) = 0. \tag{4.2}$$

Clearly, $u \equiv 0$ is a solution of (4.1) for any $\mu \in \mathbb{R}$; such solutions will be called *trivial*. We will prove a Rabinowitz-type global bifurcation theorem for (4.1).

We first prove the following technical result regarding the nodal properties of non-trivial solutions of (4.1). Note that the conditions on g ensure that the solution of the initial value problem for the differential equation in (4.1) is unique — this fact will be used repeatedly in the following proof so, for brevity, it will be abbreviated to 'IVPU'.

Proposition 4.1. *If* $(\lambda, u) \in \mathbb{R} \times X$ *is a non-trivial solution of* (4.1) *then* $u \in T_k^\nu$, *for some* k, ν.

Proof. Since u is non-trivial, IVPU implies that all the zeros of u are simple. In particular, by the boundary condition (1.2), $u(0) = 0$ and so $u'(0) \neq 0$. We now describe the qualitative 'shape' of the solution u.

Without loss of generality we assume that $u'(0) > 0$. Now, u' must have a zero in $[0, 1]$ (otherwise u would be increasing on $[0, 1]$, which leads to a contradiction, as in (3.2)), and we let x_0 be the minimal zero of u'. Then u is strictly increasing on the interval $[0, x_0]$. Now, assuming that $2x_0 \leq 1$, it follows from the fact that g is independent of x and IVPU that

$$u(x_0 + s) = u(x_0 - s), \quad s \in [0, x_0].$$

Hence, on $[0, 2x_0]$ the solution u consists of a single positive 'bump', with zeros at the end points 0 and $2x_0$, and a single zero of u', at x_0. In addition, $u'(2x_0) = -u'(0)$. Continuing this argument, we see that u consists of a sequence of positive and negative bumps, together with a truncated bump at the right end of the interval $[0, 1]$, with the following properties (ignoring the truncated bump):

(a) all the positive (respectively, negative) bumps have the same shape (the shapes of the positive and negative bumps may be different);

(b) each bump contains a single zero of u', and there is exactly one zero of u between consecutive zeros of u';

(c) all the positive (negative) bumps attain the same maximum (minimum) value.

Armed with this information on the shape of u we can continue the proof of the proposition. Suppose that $u'(1) = 0$. Then, $u(1) \neq 0$, and without loss of generality we may suppose that $u(1) > 0$. Now, by property (c) above, $u(x) \leq u(1)$, $x \in [0, 1]$, but then (1.2) and (1.3) lead to a contradiction (as in (3.2)), which proves that in fact $u'(1) \neq 0$. Thus, u satisfies condition (i) in the definition of T_k^ν (for some ν).

Now suppose that there exists $x_0 \in (0, 1)$ such that $u'(x_0) = 0$, $u''(x_0) = 0$. Then by (4.1), $\lambda u(x_0) + g(u(x_0)) = 0$, and so it again follows from IVPU that $u \equiv u(x_0)$ on $[0, 1]$. However, this contradicts the condition $u(0) = 0$ and the non-triviality of u, so we conclude that u satisfies condition (ii) in the definition of T_k^ν (for some k).

Finally, property (b) above shows that u satisfies condition (iii) in the definition of T_k^ν. □

We can now prove the following global bifurcation result for (4.1), analogous to [3, Theorem 2.3] (which considered a general, nonlinear Sturm-Liouville problem, with standard separated, two-point boundary conditions). Note that a *continuum* is a closed, connected set.

Theorem 4.2. *For each $k \geq 1$ and ν there exists a continuum $\mathcal{C}_k^\nu \subset \mathbb{R} \times X$ of solutions of* (4.1) *with the properties:*

(i) $(\lambda_k, 0) \in \mathcal{C}_k^\nu$;
(ii) $\mathcal{C}_k^\nu \setminus \{(\lambda_k, 0)\} \subset \mathbb{R} \times T_k^\nu$;
(iii) $\mathcal{C}_k^\nu$ *is unbounded in* $\mathbb{R} \times X$.

Proof. We define the Banach space

$$E := \{u \in C^1[0,1] : u \text{ satisfies } (1.2)\}.$$

Recalling that L has a bounded inverse $L^{-1} : Y \to X$, it is clear that (4.1) is equivalent to the problem

$$u = \mu L^{-1} u + L^{-1} g(u), \quad (\mu, u) \in \mathbb{R} \times E. \tag{4.3}$$

The restriction of L^{-1} to E can be regarded as a compact operator $L_E^{-1} : E \to E$, and the mapping $u \to L^{-1} g(u) : E \to E$ is continuous and compact, and satisfies $|L^{-1} g(u)|_1 = \mathrm{o}(|u|_1)$ for $|u|_1$ near 0. Thus, the problem (4.3) is of the form considered in [2] and [3] and satisfies the general hypotheses imposed there. To apply the results of [2] and [3] to (4.3) we also require the following lemma (this lemma is proved in [7, Lemma 2.6], but for completeness we give a shorter proof here).

Lemma 4.3. *For each $k \geq 1$, λ_k is a characteristic value of L_E^{-1}, with algebraic multiplicity 1.*

Proof. Clearly, $(I - \lambda_k L_E^{-1})\phi_k = 0$ and $\dim N(I - \lambda_k L_E^{-1}) = 1$. Now suppose that there exists $y \in E$ such that $(I - \lambda_k L_E^{-1}) y = \lambda_k^{-1} \phi_k$. Operating on this equation with L yields $Ly - \lambda_k y = \phi_k$, that is, y satisfies

$$-y'' - s_k^2 y = \sin(s_k x),$$

together with the boundary condition (1.2). The general solution of this differential equation, satisfying (1.2) at $x = 0$, is

$$y = C \sin(s_k x) + \frac{1}{2 s_k} x \cos(s_k x).$$

By the construction of s_k, the term $C \sin s_k x$ satisfies (1.2) at $x = 1$, so the term $x \cos(s_k x)$ must also satisfy (1.2) at $x = 1$. However, this is equivalent to s_k satisfying (3.4), which is impossible, by the proof of Lemma 3.2. □

Now, for any k and ν, Theorem 2 in [2] (together with Theorem 3.1 above) shows that there exists a continuum, $\mathcal{C}_k^\nu \subset \mathbb{R} \times E$, of solutions of (4.3) such that:
(a) $(\lambda_k, 0) \in \mathcal{C}_k^\nu$, and on a sufficiently small ball $B \subset \mathbb{R} \times E$ centred at $(\lambda_k, 0)$, $(\mathcal{C}_k^\nu \setminus \{(\lambda_k, 0)\}) \cap B \subset \mathbb{R} \times T_k^\nu$;
(b) $\mathcal{C}_k^\nu$ is either unbounded or $\mathcal{C}_k^+ \cap \mathcal{C}_k^- \neq \{(\lambda_k, 0)\}$.
In addition, by (4.3) and the continuity of the operator $L^{-1} : Y \to X$, the continuum $\mathcal{C}_k^\nu$ lies in $\mathbb{R} \times X$ and the injection $\mathcal{C}_k^\nu \to \mathbb{R} \times X$ is continuous. Hence, all the above properties (a)–(b) of the sets $\mathcal{C}_k^\nu$ also hold in the space $\mathbb{R} \times X$.

Next, since T_k^ν is open in X it follows from Proposition 4.1 that

$$(\lambda, u) \in \mathcal{C}_k^\nu \cap (\mathbb{R} \times \partial T_k^\nu) \implies u = 0.$$

Using this result it can be shown, as in the proof of [3, Theorem 2.3], that $\mathcal{C}_k^\nu \setminus \{(\lambda_k, 0)\} \subset \mathbb{R} \times T_k^\nu$, and hence that $\mathcal{C}_k^\nu$ is unbounded in $\mathbb{R} \times X$. This completes the proof of Theorem 4.2. □

5. Nodal solutions

We now return to the original problem (2.1), and we obtain nodal solutions of (2.1) under various hypotheses on the behaviour of f at infinity.

5.1. Asymptotically linear f.

The first case we deal with is where f is 'asymptotically linear', in the sense of the following theorem.

Let

$$f_0 := f'(0), \quad f_\infty := \lim_{|s| \to \infty} \frac{f(s)}{s}$$

(we assume that this limits exists).

Theorem 5.1. *Suppose that f_∞ is finite. If, for some $k \ge 1$,*

$$(\lambda_k - f_0)(\lambda_k - f_\infty) < 0, \tag{5.1}$$

then (2.1) has solutions $u_k^\pm \in T_k^\pm$.

Proof. Writing the function f in the form $f(s) = f_0 s + g(s)$, where g satisfies (4.2), we can introduce the bifurcation problem

$$Lu = \mu u + f_0 u + g(u), \quad (\mu, u) \in \mathbb{R} \times X. \tag{5.2}$$

Clearly, any non-trivial solution of (5.2) with $\mu = 0$ yields a non-trivial solution of (2.1). Now, a simple modification of Theorem 4.2 shows that for each ν there exists an unbounded continuum $\mathcal{C}_k^\nu \subset \mathbb{R} \times X$ of solutions of (5.2), bifurcating from $(\mu, u) = (\lambda_k - f_0, 0)$. It can be shown, using the Sturm comparison theorem, that the projection of $\mathcal{C}_k^\nu$ onto $\mathbb{R}$ must be bounded, and hence a standard argument (see, for example, the proof of Theorem 3.1 in [5]) shows that $\mathcal{C}_k^\nu$ contains a sequence (μ_n, u_n), $n = 1, 2, \ldots$, such that $|u_n|_2 \to \infty$ and $\mu_n \to \lambda_k - f_\infty$. Now, since $\mathcal{C}_k^\nu$ is connected, these results, together with (5.1), show that $\mathcal{C}_k^\nu$ intersects the hyperplane $\{0\} \times X$ in $\mathbb{R} \times X$, at a non-trivial solution $(0, u_0)$ say. Clearly, $u_0 \in T_k^\nu$ is a solution of (2.1), which completes the proof of the theorem. □

Remark 5.2. For given f_0 and f_∞, there may of course be several values of k for which (5.1) holds, in which case Theorem 5.1 yields solutions for each such k, that is, Theorem 5.1 also yields multiplicity results.

Remark 5.3. A result similar to Theorem 5.1 is stated in [4, Theorem 5.1], under additional hypotheses on η_i, $i = 1, \ldots, m-2$, f and k. This result is obtained by a similar bifurcation method (indeed, this approach has been used many times to obtain nodal solutions). However, the proof of [4, Theorem 5.1] appears to contain some gaps. Specifically, the sets S_k^ν as defined in [4] are not open in the set E used there, and it is not demonstrated that nodal properties of solutions are preserved along the bifurcating continua (nothing in [4] appears to prevent zeros of solutions entering or leaving the interval $(0, 1)$ through the point $x = 1$). Preservation of nodal properties is required in order to show that the bifurcating continua are unbounded.

5.2. **Superlinear f.** Next, we consider the case where f is 'superlinear', in the sense of the following theorem.

Theorem 5.4. *Suppose that $f_\infty = \infty$. If, for some $k_0 \geq 1$,*

$$f_0 < \lambda_{k_0}, \tag{5.3}$$

then (2.1) has solutions $u_k^\pm \in T_k^\pm$ for all $k \geq k_0$.

Proof. The proof follows the proof of Theorem 5.1. In this case, it can be shown that, for any $k \geq 1$, the bifurcating continuum $\mathcal{C}_k^\nu$ contains a sequence (μ_n, u_n), $n = 1, 2, \ldots$, such that $\mu_n \to -\infty$. Thus, since $\mathcal{C}_k^\nu$ bifurcates from $(\mu, u) = (\lambda_k - f_0, 0)$, it will intersect the hyperplane $\{0\} \times X$ at a non-trivial solution if $\lambda_k - f_0 > 0$, which is exactly the condition in the theorem. □

5.3. **Jumping f.** Finally, we no longer suppose that the limit f_∞ exists, and we allow different asymptotic behaviour of $f(s)$ as $s \to \infty$ and $s \to -\infty$. Such a nonlinearity f is said to be *jumping*. To make this precise, we define

$$\overline{\gamma}^\pm := \limsup_{s \to \pm\infty} \frac{f(s)}{s}, \quad \underline{\gamma}^\pm := \liminf_{s \to \pm\infty} \frac{f(s)}{s},$$

and we assume that these are finite, but they may all be different.

For any $k \geq 1$ we now define the numbers $\overline{\lambda}_k^+$, $\underline{\lambda}_k^+$, to be the unique solutions of the equations

$$\frac{\frac{1}{2}(k+1)\pi}{(\underline{\gamma}^+ + \underline{\lambda}_k^+)^{1/2}} + \frac{\frac{1}{2}k\pi}{(\underline{\gamma}^- + \underline{\lambda}_k^+)^{1/2}} = 1,$$

$$\frac{\frac{1}{2}k\pi}{(\overline{\gamma}^+ + \overline{\lambda}_k^+)^{1/2}} + \frac{\frac{1}{2}(k-1)\pi}{(\overline{\gamma}^- + \overline{\lambda}_k^+)^{1/2}} = 1,$$

and define $\overline{\lambda}_k^-$, $\underline{\lambda}_k^-$, to be the solutions of the equations obtained by interchanging the numerators in the above equations for the corresponding numbers $\overline{\lambda}_k^+$, $\underline{\lambda}_k^+$. Clearly, $\underline{\gamma}^\pm \leq \overline{\gamma}^\pm$, and hence by the above definitions, $\overline{\lambda}_k^\pm \leq \underline{\lambda}_k^\pm$. We can now obtain nodal solutions of (2.1).

Theorem 5.5. *If, for some $k \geq 1$ and ν,*

$$\lambda_k - f_0 < 0 \text{ and } \overline{\lambda}_k^\nu > 0 \quad \text{or} \quad \lambda_k - f_0 > 0 \text{ and } \underline{\lambda}_k^\nu < 0, \tag{5.4}$$

then (2.1) has a solution $u_k^\nu \in T_k^\nu$.

Proof. The proof follows the proof of Theorem 5.1. However, in this case $\mathcal{C}_k^\nu$ contains a sequence (μ_n, u_n), $n = 1, 2, \ldots$, such that $|u_n|_2 \to \infty$ and $\mu_n \to \mu_\infty$, but μ_∞ need not be an eigenvalue. The following lemma estimates μ_∞.

Lemma 5.6. $\overline{\lambda}_k^\nu \leq \mu_\infty \leq \underline{\lambda}_k^\nu$.

Proof. The argument in the proof of Theorem 3.1 in [5] shows that $u_n/|u_n|_2 \to v_\infty$ in X, and v_∞ is non-trivial solution of the eigenvalue type problem

$$Lv_\infty = q^+ v_\infty^+ - q^- v_\infty^- + \mu_\infty v_\infty, \quad v_\infty \in X, \tag{5.5}$$

where $v_\infty^\pm(x) = \max\{\pm v_\infty(x), 0\}$, and the functions $q_\pm \in L^\infty(0,1)$ satisfy

$$\underline{\gamma}^\pm \leq q^\pm(x) \leq \overline{\gamma}^\pm, \quad x \in [0,1]. \tag{5.6}$$

Since v_∞ is a non-trivial solution of (5.5) it has only simple zeros in $[0,1]$, and by its construction as the limit of the sequence $u_n/|u_n|_2$, $n-1,2,\dots$, it has the general shape described in the proof of Proposition 4.1. Specifically, v_∞ consists of a sequence of positive and negative bumps (together with a truncated bump at the right end of the interval $[0,1]$), such that all the positive (respectively, negative) bumps have the same shape (the shapes of the positive and negative bumps may be different), and all these bumps are symmetric about their mid-points. Now, recalling the definition of the set T_k^ν, and 'counting' the truncated bump, we can say, heuristically, that v_∞ has between $k-\frac{1}{2}$ and $k+\frac{1}{2}$ bumps.

Letting d^+ (respectively, d^-) denote the width of a complete (untruncated) positive (respectively, negative) bump of v_∞, it follows from the estimate (5.6) that

$$\frac{\pi}{(\overline{\gamma}^\pm+\mu_\infty)^{1/2}} \le d^\pm \le \frac{\pi}{(\underline{\gamma}^\pm+\mu_\infty)^{1/2}}, \tag{5.7}$$

by (5.5) and the Sturm comparison theorem (see, for example, Theorem 1.2 in Chapter 8 of [1] — this considers continuous coefficients, but can easily be extended to L^∞ coefficients). Now suppose that $k=2l$, for some integer l, and $\nu=+$. Then v_∞ has between l and $l+\frac{1}{2}$ positive bumps, and between $l-\frac{1}{2}$ and l negative bumps, and hence

$$\frac{l\pi}{(\overline{\gamma}^+ +\mu_\infty)^{1/2}} + \frac{(l-\frac{1}{2})\pi}{(\overline{\gamma}^- +\mu_\infty)^{1/2}} \le 1 \le \frac{(l+\frac{1}{2})\pi}{(\underline{\gamma}^+ +\mu_\infty)^{1/2}} + \frac{l\pi}{(\underline{\gamma}^- +\mu_\infty)^{1/2}},$$

from which we deduce that $\overline{\lambda}_{2l}^+ \le \mu_\infty \le \underline{\lambda}_{2l}^+$. The other cases can be dealt with similarly. □

The proof of the theorem is now completed by noting that the condition (5.4) ensures that the bifurcation point $\lambda_k - f_\infty$ and the point μ_∞ have opposite signs, and hence the continuum $\mathcal{C}_k^\nu$ must cross the hyperplane $\{0\}\times X$. □

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