NONRESONANCE CONDITIONS FOR GENERALISED ϕ-LAPLACIAN PROBLEMS WITH JUMPING NONLINEARITIES

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Abstract. We consider the boundary value problem

$$-\psi(x, u(x), u'(x))' = f(x, u(x), u'(x)), \quad \text{a.e. } x \in (0, 1), \tag{1}$$

$$c_{00}u(0) = c_{01}u'(0), \quad c_{10}u(1) = c_{11}u'(1), \tag{2}$$

where $|c_{j0}| + |c_{j1}| > 0$, for each $j = 0, 1$, and ψ, $f : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions, with suitable additional properties. The differential operator generated by the left-hand side of (1), together with the boundary conditions (2), is a generalisation of the usual p-Laplacian, and also of the so called ϕ-Laplacian (which corresponds to $\psi(x, s, t) = \phi(t)$, with ϕ an odd, increasing homeomorphism). For the p-Laplacian problem (and more particularly, the semilinear case $p = 2$), 'nonresonance conditions' which ensure the solvability of the problem (1), (2), have been obtained in terms of either eigenvalues (for non-jumping f) or the Fučík spectrum or half-eigenvalues (for jumping f) of the p-Laplacian. In this paper, under suitable growth conditions on ψ and f, we extend these conditions to the general problem (1), (2).

1. Introduction

We consider the boundary value problem

$$-\psi(x, u(x), u'(x))' = f(x, u(x), u'(x)), \quad \text{a.e. } x \in (0, 1), \tag{1.1}$$

$$c_{00}u(0) = c_{01}u'(0), \quad c_{10}u(1) = c_{11}u'(1), \tag{1.2}$$

where $|c_{j0}|+|c_{j1}| > 0$, for each $j = 0, 1$, and ψ, $f : [0, 1]\times\mathbb{R}^2 \to \mathbb{R}$ are Carathéodory functions (see Section 2 for a definition), with the following properties.

(A) For a.e. $x \in [0, 1]$ and all $s \in \mathbb{R}$, the function $\psi(x, s, \cdot)$ is strictly increasing.

(B) There exists an odd, increasing homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that, for a.e. $x \in [0, 1]$, and all $(s, t) \in \mathbb{R}^2$,

$$\big|\psi(x, s, t) - \phi(t)\big| \leqslant e_\phi(|t|), \tag{1.3}$$

$$\xi_\pm(x)\phi(s) - \zeta(x)e_\phi(|s|) \leqslant f(x, s, t) \leqslant \Xi_\pm(x)\phi(s) + \zeta(x)e_\phi(|s|), \quad \pm s \geqslant 0, \tag{1.4}$$

where:

(i) $\xi_\pm$, $\Xi_\pm$, $\zeta \in L^1(0, 1)$, with $\zeta \geqslant 0$;

(ii) $e_\phi : [0, \infty) \to [0, \infty)$ is decreasing, with $\lim_{r\to\infty} e_\phi(r)/\phi(r) = 0$.

When $\psi(x, s, t) = \phi_p(t) := |t|^{p-1}\operatorname{sgn} t$, $t \in \mathbb{R}$, for some $p > 1$, the differential operator generated by the left-hand side of (1.1), together with the boundary conditions (1.2), is the usual p-Laplacian. More generally, when $\psi(x, s, t) = \phi(t)$ (with ϕ an odd, increasing homeomorphism), the operator is called the ϕ-Laplacian. The above operator, with general ψ, is usually called a Leray-Lions operator, but here we will call it the *ψ-Laplacian*. If (1.3), (1.4), hold with $\phi = \phi_p$, we will say that

ψ (and the ψ-Laplacian) is *asymptotically homogeneous*, while f is also asymptotically homogeneous if $\xi_+ = \xi_- = \Xi_+ = \Xi_-$ (the asymptotic homogeneity is with respect to different variables for ψ and f). In general, a nonlinearity satisfying (1.4) with functions $\xi_\pm$, $\Xi_\pm$, not all equal will be termed *jumping* (a jumping f satisfies different asymptotic bounds as $s \to \pm\infty$).

In this paper we will consider so-called 'nonresonance' conditions which ensure the solvability of (1.1), (1.2). In the case of the p-Laplacian (and more particularly, the semilinear case $p = 2$), such conditions have been discussed extensively in the literature, with nonresonance conditions described in terms of either eigenvalues of the p-Laplacian (for non-jumping f) or the Fučík spectrum or half-eigenvalues of the p-Laplacian (for jumping f). A survey of such results is given in Section 7 of [16] (and in Section 1 of [3] for the case $p = 2$), so we omit a detailed discussion of this case here. However, to introduce some terminology and to motivate our discussion of the general ψ–Laplacian problem, we will give a brief description of the p-Laplacian results in the following paragraph. Nonresonance conditions for the ϕ-Laplacian have also been considered in many recent papers, see for example, [1], [2], [8], [12], [13], [14], and the survey [15], and the references therein. Nonresonance conditions of the type we consider here do not appear to have been discussed for the general ψ-Laplacian, except for a partial result, using eigenvalues, in the paper [1].

Consider the boundary value problem consisting of (1.2) together with the equation

$$-(\phi_p(u'))' = m_+\phi_p(u^+) - m_-\phi_p(u^-) + \lambda\phi_p(u), \quad \text{in } (0,1), \tag{1.5}$$

for suitable coefficient functions $m_\pm \in L^1(0,1)$ (related to the functions $\xi_\pm$, $\Xi_\pm$). Here, $\lambda \in \mathbb{R}$, and $u^\pm(x) = \max\{\pm u(x), 0\}$, $x \in [0,1]$. If u is a solution of (1.2), (1.5), then tu is also a solution, for any number $t \geqslant 0$, so this problem is said to be *positively-homogeneous*. Values of λ for which (1.2), (1.5), have a non-trivial solution u will be called *half-eigenvalues* of the p-Laplacian, and the solutions u are *half-eigenfunctions* (if $m_+ = m_-$ these are eigenvalues and eigenfunctions respectively). Nonresonance conditions for jumping f, with general L^1 coefficients $\xi_\pm$, $\Xi_\pm$, are obtained in [16] in terms of half-eigenvalues of the p-Laplacian; other types of nonresonance conditions are described in [16, Section 7], and in the references therein. Nonresonance conditions for non-jumping f, in terms of eigenvalues, are described in, for example, [1], [10]. When f is jumping, but the functions $\xi_\pm$, $\Xi_\pm$, are constants, nonresonance conditions have been given in terms of the so-called 'Fučík spectrum', see the references in [16]. However, it is shown in [16] that in this case the half-eigenvalue and Fučík spectrum conditions are equivalent.

More generally, if ψ is not asymptotically homogeneous one would not expect to be able to give nonresonance conditions for (1.1), (1.2) in terms of the eigenvalues or half-eigenvalues of the p-Laplacian. Given the conditions (1.3), (1.4), one might hope to obtain such nonresonance conditions from the properties of an associated 'simpler' ϕ-Laplacian problem. However, for general ϕ, the ϕ-Laplacian operator is not positively homogeneous, so it is not clear how to define eigenvalues, or half-eigenvalues, of this operator to replace those of the p-Laplacian (the 'eigenvalues' would depend on the norm of the 'eigenfunctions'). The paper [13] considers the problem (1.1), (1.2) with $\psi = \phi$, non-jumping f, and $\xi_\pm$, $\Xi_\pm$ constant, and by imposing certain growth conditions on ϕ obtains nonresonance conditions in terms of certain 'asymptotic eigenvalues'. A similar problem is considered in [14], with jumping f, and similar growth conditions are used to obtain nonresonance conditions in terms of an 'asymptotic Fučík spectrum'. Unfortunately, the constructions of the asymptotic eigenvalues or Fučík spectrum in [13] and [14] are long and complicated, and it is not clear how the resulting nonresonance conditions could be verified. Also, these constructions depend on a time-mapping approach, so would not extend to the case where ψ depends on x or u. Moreover, in Example 1 in Section 5 of [13] the asymptotic eigenvalues actually reduce to the eigenvalues of the p-Laplacian operator, so the nonresonance conditions in this case reduce to the standard p-Laplacian conditions.

In this paper, rather than search for nonresonance conditions in terms of a general ϕ-Laplacian, and motivated by the above mentioned example in [13], we will utilise a slightly stronger growth condition on ϕ which will enable us to obtain nonresonance conditions in terms of the eigenvalues or half-eigenvalues of the p-Laplacian, even when ϕ is not asymptotically homogeneous. In effect, we will regard the p-Laplacian as a basic, 'simple' operator, and search for solvability conditions for the ψ-Laplacian problem in terms of the p-Laplacian.

From now on we suppose that ϕ satisfies the following growth condition: there exists $p > 1$ such that

$$\lim_{t\to\infty} \frac{\phi(\sigma t)}{\phi(t)} = \sigma^{p-1}, \quad \sigma \in (0,1). \tag{1.6}$$

A simple example of a function ϕ which is not asymptotically homogeneous, but which satisfies (1.6) is given by $\phi(t) = (\log(2+|t|))^\alpha \phi_p(t)$, $t \in \mathbb{R}$, for arbitrary $p > 1$ and $\alpha \in \mathbb{R}$ (the function in the example [13] is essentially of this form, with $p = 3$, $\alpha = 1$).

The growth condition used in [13], [14], is

$$0 < \liminf_{t\to\infty} \frac{\phi(\sigma t)}{\phi(t)} \leqslant \limsup_{t\to\infty} \frac{\phi(\sigma t)}{\phi(t)} < 1, \quad \sigma \in (0,1).$$

The condition (1.6) is more restrictive than this, but we obtain more specific solvability criteria for (1.1), (1.2), in terms of the basic p-Laplacian operator. In addition, our form of the ψ-Laplacian allows considerably more general dependence on x, u and u' than is allowed in [13], [14], where in fact $\psi(x,s,t) = \phi(t)$, and f has the form $f(x,s)$.

Finally, we note that periodic problems can be considered in a similar manner. We give some further brief comments on this case in Remark 4.7.

2. Preliminaries

We will use the standard spaces $C^j[0,1]$, $j \geqslant 0$, $L^r(0,1)$ and $W^{1,r}(0,1)$, $r \geqslant 1$, with norms denoted by $|\cdot|_j$, $\|\cdot\|_r$ and $\|\cdot\|_{1,r}$ respectively (throughout, all function spaces will be real). The generalised derivative of a function $v \in W^{1,r}(0,1)$ will be denoted by $v' \in L^r(0,1)$. If $v \in W^{1,1}(0,1)$ then v is absolutely continuous on $[0,1]$, with generalised derivative $v' \in L^1(0,1)$.

2.1. Carathéodory functions, growth conditions and Nemitskii operators.

We begin with some immediate consequences of (1.6). In all the estimates below, C will denote a positive constant which depends only on the above coefficients and functions, and which may differ at each occurrence. We also note that the function ϕ_p is invertible, with inverse $\phi_{p'}$, where $p' = p/(p-1)$.

Lemma 2.1. (*a*) *If* $0 < \delta < p-1$*, then there exists* $t_\delta \geqslant 1$ *such that*

$$|t|^{p-1-\delta} \leqslant |\phi(t)| \leqslant |t|^{p-1+\delta}, \quad |t| \geqslant t_\delta. \tag{2.1}$$

(*b*) *The function* ϕ^{-1} *satisfies* (1.6) *and* (2.1)*, with* p *replaced by* p'*.*

Proof. By (1.6), for given $\delta > 0$ there exists $t_0 \geqslant 1$ such that

$$\phi(2t) \leqslant \phi(t) 2^{p-1+\delta/2}, \quad t \geqslant t_1.$$

Now, for arbitrary $n \geqslant 1$, if $2^n t_1 < t \leqslant 2^{n+1} t_1$, then

$$\phi(t) \leqslant \phi(2^{n+1}t_1) \leqslant (2^{n+1})^{p-1+\delta/2}\phi(t_1) \leqslant (2/t_1)^{p-1+\delta/2}\phi(t_1) t^{p-1+\delta/2}.$$

This proves one of the inequalities in (2.1), and the other inequality can be proved similarly.

Next, for any fixed $\sigma \in (0,1)$ we can write (1.6) as

$$\frac{\phi(\sigma t)}{\phi(t)} = \sigma^{p-1}(1 + \mathrm{o}(1)), \quad t \to \infty,$$

where the term $\mathrm{o}(1) \to 0$ as $t \to \infty$. Also, it follows from (1.6) and the monotonicity of ϕ that

$$\phi(t(1 + \mathrm{o}(1))) = \phi(t)(1 + \mathrm{o}(1)), \quad t \to \infty. \tag{2.2}$$

Now, setting $\beta = \sigma^{p'-1}$, $s = \phi^{-1}(t)$, and using (2.2), we see that

$$\frac{\phi^{-1}(\sigma t)}{\phi^{-1}(t)} = \frac{\phi^{-1}(\beta^{p-1}\phi(s))}{s} = \frac{\phi^{-1}(\phi(\beta s(1 + \mathrm{o}(1))))}{s} = \beta(1 + \mathrm{o}(1)),$$

which proves that ϕ^{-1} satisfies (1.6), with p replaced by p'. The analogue of (2.1) for ϕ^{-1} now follows from part (a). □

A function $g : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ is *Carathéodory* if, for every fixed $\mathbf{u} \in \mathbb{R}^n$, the function $g(\cdot, \mathbf{u})$ is measurable, while for almost all $x \in [0,1]$, the function $g(x, \cdot)$ is continuous. A Carathéodory function induces a Nemitskii operator on a suitable domain, depending on the growth rate of the function, see [17, Section 26.3]. We will use the Nemitskii operators induced by the functions ψ and f below. However, our constructions will not require the most general domains for these operators, so we simply use the domains, and estimates, which yield the results we require.

By [17, Proposition 26.6] and the growth conditions (1.3), (1.4) and (2.1), for any $q > p-1$ the functions ψ, f induce continuous Nemitskii operators $\psi, f : W^{1,q}(0,1) \to L^1(0,1)$ (we use the same notation for a function and the corresponding induced Nemitskii operator), and we have the estimate

$$\|f(u)\|_1 \leqslant C(1 + \phi(|u|_0)), \quad u \in W^{1,q}(0,1). \tag{2.3}$$

From now on, in fact, we will let q denote an arbitrary fixed number satisfying $q > \max\{p, p'\}$.

Next, for any $u \in C^0[0,1]$ we define $\psi_u : [0,1] \times \mathbb{R} \to \mathbb{R}$ by

$$\psi_u(x,t) := \psi(x, u(x), t), \quad (x,t) \in [0,1] \times \mathbb{R}.$$

The function ψ_u is Carathéodory (measurability of the function $x \to \psi(x, u(x), t)$, for each $t \in \mathbb{R}$, follows from [17, Appendix (12)]), and satisfies a similar bound to (1.3). In addition, for almost all $x \in [0,1]$, the function $\psi_u(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, whose inverse we will denote by $\psi_u^{-1}(x, \cdot)$. The function

$\psi_u^{-1} : [0,1] \times \mathbb{R} \to \mathbb{R}$ is Carathéodory (measurability of the function $x \to \psi_u^{-1}(x,t)$, for each $t \in \mathbb{R}$, follows from the equality

$$\{x : \psi_u^{-1}(x,t) \leqslant c\} = \{x : t \leqslant \psi_u(x,c)\}, \quad t,\, c \in \mathbb{R},$$

and the fact that ψ_u is Carathéodory). We now show that ψ_u^{-1} satisfies a similar growth condition to (1.3).

Lemma 2.2. *There exists a decreasing function $\tilde{e} : [0,\infty) \to [0,\infty)$ such that $\lim_{r\to\infty} \tilde{e}(r)/\phi^{-1}(r) = 0$, and, for any $u \in C^0[0,1]$,*

$$\left|\psi_u^{-1}(x,t) - \phi^{-1}(t)\right| \leqslant \tilde{e}(|t|), \tag{2.4}$$

for a.e. $x \in (0,1)$, and all $t \in \mathbb{R}$.

Proof. By (1.3) and (2.2),

$$\psi_u(x,\tilde{t}) = \phi(\tilde{t})(1 + \mathrm{o}(1)) = \phi(\tilde{t}(1 + \mathrm{o}(1))), \quad |\tilde{t}| \to \infty,$$

and putting $t = \psi_u(x,\tilde{t})$ yields

$$\psi_u^{-1}(x,t) = (1 + \mathrm{o}(1))\phi^{-1}(t), \quad |t| \to \infty,$$

from which we obtain (2.4). □

Lemma 2.2 now yields the following result.

Lemma 2.3. *For any $u \in C^0[0,1]$, the function ψ_u^{-1} induces a continuous, injective Nemitskii operator $\psi_u^{-1} : W^{1,1}(0,1) \to L^q(0,1)$, such that*

$$\|\psi_u^{-1}(w)\|_q \leqslant C(1 + \phi^{-1}(|w|_0)), \quad w \in W^{1,1}(0,1), \tag{2.5}$$

and for any $\epsilon > 0$ there exists C_ϵ such that,

$$\|\psi_u^{-1}(w) - \phi^{-1}(w)\|_q \leqslant C_\epsilon + \epsilon\phi^{-1}(|w|_0), \quad w \in W^{1,1}(0,1), \tag{2.6}$$

where C, C_ϵ are independent of u.

2.2. **The ψ-Laplacian operator.** In this section we define what we mean by a solution of the boundary value problem (1.1), (1.2), and we define a 'ψ-Laplacian' operator Δ_ψ to represent the differential operator on the left-hand side of (1.1), together with the boundary conditions (1.2).

To construct a suitable space of functions on which to define Δ_ψ we let

$$D_\psi^1 := \{u \in W^{1,q}(0,1) : \psi(u) = w \ \text{ a.e. in } [0,1], \text{ for some } w \in W^{1,1}(0,1)\},$$

and, for $u \in D_\psi^1$, let

$$E_u := \{x \in [0,1] : u \text{ is differentiable at } x \text{ and } \psi(u) = w \text{ holds at } x\}$$

(by definition, the set E_u has full measure in $[0,1]$). We observe that, for a general Carathéodory function ψ, although the function $\psi(u)$ agrees almost everywhere with a continuous function $w \in W^{1,1}(0,1)$, neither u' nor $\psi(u)$ need be defined everywhere in $[0,1]$, and u' need not coincide with a continuous function, even on the set E_u. However, if ψ is continuous then we have the following result.

Lemma 2.4. *If ψ is continuous on $[0,1] \times \mathbb{R}^2$, then $D_\psi^1 \subset C^1[0,1]$.*

Proof. Suppose that $u \in D_\psi^1$. Then, by definition, $u' = \psi_u^{-1}(w)$ a.e. on $[0,1]$, and it follows from the continuity of ψ that $\psi_u^{-1}(w) \in C^0[0,1]$. Thus, u' coincides with a continuous function a.e. on $[0,1]$, so by integrating u' we see that $u \in C^1[0,1]$. □

We now define a *solution* of equation (1.1) to be a function $u \in D^1_\psi$ (with corresponding function w) for which $-w' = f(u)$ (in $L^1(0,1)$ — by definition, each side of this equation is a well-defined element of $L^1(0,1)$).

Next, we must consider the meaning of the boundary conditions (1.2) for functions $u \in D^1_\psi$. If $c_{j1} = 0$, for either $j = 0$ or $j = 1$, then the corresponding boundary condition (1.2) at $x = j$ is simply the Dirichlet condition $u(j) = 0$, which has a natural meaning for $u \in D^1_\psi$, since $W^{1,q}(0,1) \subset C^0[0,1]$. However, if $c_{j1} \neq 0$, for some $j = 0$, 1, then it is not immediately clear how to assign a meaning to the condition (1.2) at $x = j$ since, of course, the value of $u'(j)$ is not, in general, well-defined for $u \in W^{1,q}(0,1)$. Naturally, one could consider using a weak formulation of the problem, together with a suitable 'boundary form' at $x = j$, based on the value of $u(j)$. However, a straightforward approach to such a weak formulation leads naturally to the function values $\psi(j, u(j), \gamma_j u(j))$, where $\gamma_j = c_{j0}/c_{j1}$. Unfortunately, under the above Carathéodory hypotheses on ψ, these values need not be well-defined, even when $u \in C^1[0,1]$. To deal with this problem, from now on we impose the following additional assumption on ψ.

Assumption $(H\psi)$. If $c_{j1} \neq 0$, for some $j = 0$, 1, then ψ is continuous at (j,s,t), for all $(j,s,t) \in \mathbb{R}^2$, and for each $s \in \mathbb{R}$ the function $\psi(j,s,\cdot)$ is strictly increasing.

If $c_{j1} \neq 0$ then it follows from Assumption $(H\psi)$ that the estimate (1.3) holds at $x = j$, for all $(s,t) \in \mathbb{R}^2$, so the function $\psi(j,s,\cdot) : \mathbb{R} \to \mathbb{R}$ is a homeomorphism for all $s \in \mathbb{R}$. We now have the following result regarding the derivative u' of a function $u \in D^1_\psi$, at the point $x = j$.

Lemma 2.5. *If $c_{j1} \neq 0$, for some $j = 0$, 1, and $u \in D^1_\psi$, then u is differentiable at $x = j$, and the function $u' : E_u \to \mathbb{R}$ is continuous there.*

Proof. We suppose that $j = 0$. By definition, $\psi(u) = w$ a.e. in $[0,1]$, for some $w \in W^{1,1}(0,1)$. Let t_0 be the unique solution of the equation

$$\psi(0, u(0), t_0) = w(0)$$

(all the quantities in the equation are well-defined, by continuity). Now, for arbitrary $\epsilon > 0$, it follows from the continuity of u and w, and the properties of ψ, that there exists $\delta > 0$ such that

$$x \in E_u \text{ and } x < \delta \implies |u'(x) - t_0| < \epsilon, \tag{2.7}$$

and hence it follows from

$$u(x) - u(0) = \int_0^x u', \quad x \in [0,1],$$

that

$$0 < x < \delta \implies \left| \frac{u(x) - u(0)}{x} - t_0 \right| < \epsilon.$$

This shows that u is differentiable at $x = 0$, with $u'(0) = t_0$, and, by (2.7), u' is continuous at $x = 0$, on the set E_u. □

We can now define the desired *ψ-Laplacian* operator. In view of the above discussion we can define the domain

$$D(\Delta_\psi) := \{u \in D^1_\psi : (1.2) \text{ holds}\},$$

and we define the operator $\Delta_\psi : D(\Delta_\psi) \to L^1(0,1)$ by

$$\Delta_\psi(u) := \psi(u)', \quad u \in D(\Delta_\psi).$$

Remark 2.6. If $\psi(x,s,t) = \phi_p(t)$ then Δ_ψ is in fact the usual p-Laplacian and we will denote it by Δ_p; if $\psi(x,s,t) = \phi(t)$ then Δ_ψ is usually called the ϕ-Laplacian and we will denote it by Δ_ϕ.

2.3. **An integral equation formulation.** With the preceding notation the problem (1.1), (1.2), can be rewritten as

$$-\Delta_\psi(u) = f(u), \quad u \in D(\Delta_\psi). \tag{2.8}$$

We now reformulate this equation as an equivalent integral equation. The cases $c_{j1} = 0$ and $c_{j1} \neq 0$, $j = 0, 1$, require slightly different treatment. For each j for which $c_{j1} \neq 0$ we define a functional $b_j(\cdot\,;\psi) : W^{1,q}(0,1) \to \mathbb{R}$ by

$$b_j(u;\psi) := \psi(j, u(j), \gamma_j u(j)), \quad u \in W^{1,q}(0,1) \tag{2.9}$$

(by Assumption $(H\psi)$, the right hand side of (2.9) is well-defined, by continuity). The estimate (1.3) holds at $x = j$, so the functional $b_j(\cdot\,;\psi)$ is continuous, and satisfies

$$|b_j(u;\psi)| \leqslant C(1 + \phi(|u|_0)), \quad u \in W^{1,q}(0,1). \tag{2.10}$$

From the properties of ψ we also have the following result.

Lemma 2.7. *If $c_{j1} \neq 0$ then a function $u \in D^1_\psi$ satisfies the boundary condition* (1.2) *at $x = j$ iff*

$$\psi(u)(j) = b_j(u;\psi) \tag{2.11}$$

(*here, $\psi(u)(j)$ denotes the value of the* (*continuous*) *function $\psi(u)$ at $x = j$*).

Next, for $v \in L^1(0,1)$, let

$$\mathcal{I}(v)(x) := \int_0^x v(y)\,dy, \quad x \in [0,1].$$

Clearly, $\mathcal{I} : L^r(0,1) \to W^{1,r}(0,1)$ is a bounded linear operator, for any $r \geqslant 1$, and $\mathcal{I}(v)' = v$, for any $v \in L^1(0,1)$. Now, for any $u \in W^{1,q}(0,1)$, let

$$d(u;\phi,\psi,f) := \phi^{-1}\big[b_1(u;\psi) - b_0(u;\psi) + \mathcal{I}(f(u))(1)\big],$$
$$G(u;\psi,f) := \mathcal{I}\big\{\psi_u^{-1}\big[b_0(u;\psi) - \mathcal{I}(f(u))\big]\big\}.$$

By the mapping properties described above, $G(u;\psi,f) \in W^{1,q}(0,1)$, the operators $d(\cdot\,;\phi,\psi,f) : W^{1,q}(0,1) \to \mathbb{R}$, $G(\cdot\,;\psi,f) : W^{1,q}(0,1) \to W^{1,q}(0,1)$ are continuous, and by (1.6), (2.3) and (2.5),

$$|d(u;\phi,\psi,f)| + \|G(u;\psi,f)\|_{1,q} \leqslant C(1 + |u|_0), \quad u \in W^{1,q}(0,1). \tag{2.12}$$

Lemma 2.8. *Suppose that $c_{j1} \neq 0$ for each $j = 0, 1$. Then the following statements are equivalent*:

(*i*) *$u \in D(\Delta_\psi)$ satisfies* (2.8);

(*ii*) *$u \in W^{1,q}(0,1)$ satisfies the equation*

$$u = u(0) + d(u;\phi,\psi,f) + G(u;\psi,f). \tag{2.13}$$

Proof. Suppose that $u \in W^{1,q}(0,1)$ satisfies (2.13), that is, (ii) holds. We first note that evaluating (2.13) at the point $x = 0$ yields $d(u; \phi, \psi, f) = 0$, that is,

$$b_1(u;\psi) = b_0(u;\psi) - \mathcal{I}(f(u))(1). \tag{2.14}$$

Next, differentiating (2.13) yields

$$u' = \psi_u^{-1}\big[b_0(u;\psi) - \mathcal{I}(f(u))\big]$$

(this equation holds in $L^q(0,1)$), and hence, by the definition of ψ_u,

$$\psi(u) = b_0(u;\psi) - \mathcal{I}(f(u)), \tag{2.15}$$

in $L^1(0,1)$. However, since $f(u) \in L^1(0,1)$, we have $\mathcal{I}(f(u)) \in W^{1,1}(0,1)$, so in fact $\psi(u) \in W^{1,1}(0,1)$, and differentiating (2.15) now yields $-\psi(u)' = f(u)$. Furthermore, evaluating (2.15) at the point $x = 0$ yields $\psi(u)(0) = b_0(u;\psi)$, that is, (2.11) holds when $j = 0$. Finally, evaluating (2.15) at $x = 1$ and using (2.14) yields

$$\psi(u)(1) = b_0(u;\psi) - \mathcal{I}(f(u))(1) = b_1(u;\psi),$$

that is, (2.11) also holds when $j = 1$. This completes the proof that $u \in D(\Delta_\psi)$ and that u satisfies (2.8), that is, (ii) implies (i) in the lemma. The reverse implication can be proved similarly. □

If $c_{j1} = 0$, for either j, then the integral equation (2.13) in Lemma 2.8 must be modified slightly — we now briefly describe the required modifications. If $c_{01} \neq 0$, $c_{11} = 0$ then we replace (2.13) with the equation

$$u = -\mathcal{I}^1\big\{\psi_u^{-1}\big[b_0(u;\psi) - \mathcal{I}(f(u))\big]\big\}, \quad u \in W^{1,q}(0,1),$$

where $\mathcal{I}^1(v)(x) := \int_x^1 v(y)\,dy$, for $v \in L^1(0,1)$, $x \in [0,1]$; by following the previous proof it can be verified that the analogue of Lemma 2.8 again holds in this case. If $c_{01} = 0$, $c_{11} \neq 0$ then a similar modification of (2.13) again yields a suitable integral equation. Finally, suppose that $c_{01} = c_{11} = 0$. It is clear from the properties of ψ that the equation

$$0 = \mathcal{I}\big\{\psi_u^{-1}\big[\kappa - \mathcal{I}(f(u))\big]\big\}(1), \quad u \in W^{1,q}(0,1),$$

has a unique solution $\kappa = \kappa(u;\psi,f)$, and the function $\kappa(\cdot\,;\psi,f) : W^{1,q}(0,1) \to \mathbb{R}$ is continuous. We now replace (2.13) with the equation

$$u = \mathcal{I}\big\{\psi_u^{-1}\big[\kappa(u;\psi,f) - \mathcal{I}(f(u))\big]\big\}, \quad u \in W^{1,q}(0,1).$$

Remark 2.9. The proof of Theorem 4.1 below will rely on the above integral equations, depending on the specific boundary conditions (although the result does not depend on the specific boundary conditions). For brevity, in the proofs below we will only consider the integral equation (2.13) (that is, the case $c_{j1} \neq 0$, $j = 0, 1$) considered in Lemma 2.8. The proofs involving the other integral equations are similar.

Remark 2.10. If ψ is continuous then $D(\Delta_\psi) \subset C^1[0,1]$, and in this case the similar (but somewhat simpler) integral equation formulation used in [16] can also be used here.

2.4. **Compactness properties.** Clearly, the functionals $b_j(\cdot\,;\psi)$, $d(\cdot\,;\phi,\psi,f)$, and $\kappa(\cdot\,;\psi,f)$ are continuous and bounded (that is, they map bounded sets to bounded sets), and so are completely continuous. We now show that G is completely continuous.

A set $B \subset L^1(0,1)$ is said to be *equi-integrable* if there exists $\zeta_B \in L^1(0,1)$ such that, for any $v \in B$, $|v(x)| \leqslant \zeta_B(x)$ for almost all $x \in [0,1]$. Weak convergence will be denoted by $\rightharpoonup$ (the space in which the weak convergence occurs will be specified explicitly when necessary).

Lemma 2.11 ([16, Lemma 2.1])**.** *(i) If $B \subset L^1(0,1)$ is equi-integrable then it is weakly sequentially compact.*
(ii) Suppose that (u_n) is a sequence in $L^1(0,1)$ such that the set $\{u_n\} \subset L^1(0,1)$ is equi-integrable and $u_n \rightharpoonup u_\infty$ in $L^1(0,1)$. Then $\mathcal{I}(u_n) \to \mathcal{I}(u_\infty)$ in $C^0[0,1]$.

We now prove the basic compactness that we require.

Lemma 2.12. *The mapping $G(\cdot\,;\psi,f) : W^{1,q}(0,1) \to W^{1,q}(0,1)$ is completely continuous.*

Proof. We have seen that the mapping $G(\cdot\,;\psi,f)$ is continuous. Suppose that (u_n) is a bounded sequence in $W^{1,q}(0,1)$. Then, after taking a suitable subsequence, we may suppose that (u_n) converges in $C^0[0,1]$. Also, by (2.10), the sequence $(b_0(u_n;\psi))$ is bounded so, after taking a subsequence, the sequence $(b_0(u_n;\psi))$ is convergent. Now, by (1.4), the set $\{f(u_n)\}$ is equi-integrable so, by Lemma 2.11, after taking a subsequence, the sequence $(\mathcal{I}(f(u_n))$ converges in $C^0[0,1]$. Hence, by [17, Proposition 26.6], the sequence $(\psi_{u_n}^{-1}[b_0(u_n;\psi) - \mathcal{I}(f(u_n))])$ converges in $L^q(0,1)$ and so, finally, the sequence $(G(u_n;\psi,f))$ converges in $W^{1,q}(0,1)$, which completes the proof. □

3. Half-eigenvalues of Δ_p

For an arbitrary pair of coefficient functions $m_\pm \in L^1(0,1)$, we recall the half-eigenvalue problem (1.2), (1.5), which we can now write in the form

$$-\Delta_p(u) = h_{(m_+,m_-)}(u) := m_+\phi_p(u^+) - m_-\phi_p(u^-) + \lambda\phi_p(u), \quad u \in D(\Delta_p). \tag{3.1}$$

We denote the set of half-eigenvalues of (3.1) by $\Sigma_H(m_+,m_-)$. We briefly recall some results from [16]; further details are given in [16].

Theorem 3.1. *The set Σ_H consists of a sequence*

$$\lambda_0^{\min}(m_+,m_-) \leqslant \lambda_0^{\max}(m_+,m_-) < \lambda_1^{\min}(m_+,m_-) \leqslant \lambda_1^{\max}(m_+,m_-) < \dots,$$

with $\lim_{k\to\infty} \lambda_k^{\min\,/\,\max}(m_+,m_-) = \infty$.

When $m_+ = m_- = 0$, the problem (3.1) is simply the usual eigenvalue problem for the p-Laplacian, and in this case it is clear that the half-eigenvalues coincide with the eigenvalues of the p-Laplacian, which we denote by λ_k, $k \geqslant 0$. Intuitively, Theorem 3.1 says that the jumping term $m_+\phi_p(u^+) - m_-\phi_p(u^-)$ in equation (3.1) 'splits apart' each eigenvalue λ_k into a pair of half-eigenvalues $\lambda_k^{\min\,/\,\max}(m_+,m_-)$. We denote the intervals between the half-eigenvalues by

$$\Lambda_{-1}^1 = (-\infty,\lambda_0^{\min}), \quad \Lambda_k^1 = (\lambda_k^{\max},\lambda_{k+1}^{\min}), \quad \Lambda_k^0 = (\lambda_k^{\min},\lambda_k^{\max}), \quad k \geqslant 0$$

(omitting the arguments (m_+,m_-) for clarity). For each $k \geqslant 0$, the interval $\Lambda_k^0(m_+,m_-)$ may be empty if the corresponding half-eigenvalues coincide, while the interval $\Lambda_k^1(m_+, m_-)$ must be non-empty. The following monotonicity result is a consequence of [16, Theorem 3.2].

Theorem 3.2. *Suppose that*

$$\xi_\pm \leqslant m_\pm \leqslant \Xi_\pm, \quad a.e. \text{ on } [0,1]. \tag{3.2}$$

Then, for any $k \geqslant -1$,

$$\lambda \in \Lambda_k^1(\xi_+, \xi_-) \cap \Lambda_k^1(\Xi_+, \Xi_-) \implies \lambda \in \Lambda_k^1(m_+, m_-). \tag{3.3}$$

4. Existence of solutions

We now return to the general problem (2.8) and describe a nonresonance conditions which ensures that this problem has a solution.

Theorem 4.1. *Suppose that* (1.3) *and* (1.4) *hold and, for some* $k \geqslant -1$,

$$0 \in \Lambda_k^1(\xi_+, \xi_-) \cap \Lambda_k^1(\Xi_+, \Xi_-). \tag{4.1}$$

Then equation (2.8) *has a solution* $u \in D(\Delta_\psi)$.

Proof. We use the Leray-Schauder continuation theorem to prove the result. The following notation will be used. For any $r > 0$, let

$$B_r := \{y \in W^{1,q}(0,1) : \|y\|_{1,q} \leqslant r\}$$

and, for any completely continuous mapping $T : W^{1,q}(0,1) \to W^{1,q}(0,1)$, let $\deg(I - T, B_r, 0)$ denote the Leray-Schauder degree of $I - T$ with respect to 0, on the ball B_r, see [9] (where I denotes the identity operator).

As mentioned in Section 2.3, we will only prove the result in the case $c_{j1} \neq 0$, $j = 0$, 1, considered in Lemma 2.8, using the integral equation formulation (2.13); the other cases can be proved in an identical manner, using the appropriate integral equation described in Section 2.3.

We start by constructing a homotopy of equation (2.13) to a simpler equation. Let $\mu_\pm := \frac{1}{2}(\xi_\pm + \Xi_\pm)$, and for any $\tau \in [0,1]$, $(x,s,t) \in [0,1] \times \mathbb{R}^2$, let

$$\psi_\tau(x,s,t) := (1-\tau)\psi(x,s,t) + \tau\phi(t),$$
$$f_\tau(x,s,t) := (1-\tau)f(x,s,t) + \tau(\mu_+(x)\phi(s^+) - \mu_-(x)\phi(s^-)).$$

We also define an operator $\psi^\#_{\tau,u} : L^1(0,1) \to L^q(0,1)$ by

$$\psi^\#_{\tau,u}(v) := (1-\tau)\psi_u^{-1}(v) + \tau\phi^{-1}(v), \quad v \in L^1(0,1)$$

(it would be natural to use the operator $\psi_{\tau,u}^{-1}$, but the operator $\psi^\#_{\tau,u}$ is slightly simpler to use). Next, we define a homotopy $H : [0,1] \times W^{1,q}(0,1) \to W^{1,q}(0,1)$ by

$$H(\tau,u) := u(0) + d(u;\phi,\psi_\tau,f_\tau) + \mathcal{I}\big\{\psi^\#_{\tau,u}\big[b_0(u;\psi_\tau) - \mathcal{I}(f_\tau(u))\big]\big\},$$

for $(\tau,u) \in [0,1] \times W^{1,q}(0,1)$, and we consider the corresponding fixed point equation

$$u = H(\tau,u). \tag{4.2}$$

Clearly, when $\tau = 0$ equation (4.2) reduces to equation (2.13). Also, by Lemma 2.12, the homotopy H is completely continuous.

To apply the Leray-Schauder theorem we first show that there exists a constant $R > 0$ such that any solution (τ,u) of equation (4.2) satisfies $\|u\|_{1,q} < R$.

Suppose, on the contrary, that for each integer $n \geqslant 1$ there exists a solution (τ_n, u_n) with $\|u_n\|_{1,q} \geqslant n$. Let $v_n := u_n/\|u_n\|_{1,q}$, and define

$$\widetilde{b}_{j,n} := \frac{b_j(u_n;\psi_{\tau_n})}{\phi(\|u_n\|_{1,q})}, \quad \widetilde{d}_n := \frac{d(u_n;\phi,\psi_{\tau_n},f_{\tau_n})}{\|u_n\|_{1,q}}, \quad \widetilde{f}_n := \frac{f_{\tau_n}(u_n)}{\phi(\|u_n\|_{1,q})}, \tag{4.3}$$

$$g_n := b_0(u_n;\psi_{\tau_n}) - \mathcal{I}(f_{\tau_n}(u_n)), \quad \widetilde{g}_n := \frac{g_n}{\phi(\|u_n\|_{1,q})} \tag{4.4}$$

(cf. (2.3), (2.10) and (2.12)). Dividing the equation $u_n = H(\tau_n, u_n)$ by $\|u_n\|_{1,q}$ yields

$$v_n = v_n(0) + \widetilde{d}_n + \frac{1}{\|u_n\|_{1,q}}\mathcal{I}\{\psi^{\#}_{\tau_n,u_n}(g_n)\}. \tag{4.5}$$

Naturally, we now wish to take a limit as $n \to \infty$ in (4.5). We consider the individual terms in (4.5) in the following lemmas.

Lemma 4.2. *After choosing a suitable subsequence, there exists $\tau_\infty \in [0,1]$ and $v_\infty \in W^{1,q}(0,1)$, with $v_\infty \neq 0$, such that*

$$\tau_n \to \tau_\infty, \quad |v_n - v_\infty|_0 \to 0, \quad v_n' \rightharpoonup v_\infty'$$

(here $\rightharpoonup$ denotes weak convergence in $L^q(0,1)$).

Proof. The sequence (v_n) is bounded in $W^{1,q}(0,1)$, so the convergence results are clear. Now suppose that $v_\infty = 0$, that is $|v_n|_0 \to 0$, as $n \to \infty$. Then, by (2.3) and (2.10), for $j = 0$, 1, $\epsilon > 0$ and n sufficiently large,

$$|\widetilde{b}_{j,n}| + |\widetilde{d}_n| + \|\widetilde{f}_n\|_1 \leqslant C\frac{\phi(|v_n|_0\|u_n\|_{1,q})}{\phi(\|u_n\|_{1,q})} \leqslant C\frac{\phi(\epsilon\|u_n\|_{1,q})}{\phi(\|u_n\|_{1,q})} \to C\epsilon^{p-1},$$

by (1.6). Hence,

$$|\widetilde{b}_{j,n}| + |\widetilde{d}_n| + \|\widetilde{f}_n\|_1 \to 0,$$

and so, by (2.5),

$$\frac{\|\psi^{\#}_{\tau_n,u_n}(g_n)\|_{1,q}}{\|u_n\|_{1,q}} \leqslant C\frac{|u_n|_0}{\|u_n\|_{1,q}} = |v_n|_0 \to 0. \tag{4.6}$$

Now, differentiating (4.5) and combining this with (4.6) shows that $\|v_n'\|_q \to 0$. However, this contradicts the fact that $\|v_n\|_{1,q} = 1$ for all n, so we must have $v_\infty \neq 0$. □

Lemma 4.3.

$$\frac{\phi(u_n)}{\phi(\|u_n\|_{1,q})} \to \phi_p(v_\infty),$$

in $L^q(0,1)$. Also,

$$\widetilde{b}_{j,n} \to b_j(v_\infty;\phi_p), \quad j = 0,\ 1 \tag{4.7}$$

(here, the function ϕ_p in the term $b_j(v_\infty;\phi_p)$ denotes the function $(x,s,t) \to \phi_p(t)$).

Proof. For arbitrary $\epsilon > 0$ and $x \in [0,1]$, if n is sufficiently large then by (1.6),

$$\frac{\phi(u_n(x))}{\phi(\|u_n\|_{1,q})} = \frac{\phi(v_n(x)\phi(\|u_n\|_{1,q}))}{\phi(\|u_n\|_{1,q})} \leqslant \frac{\phi((v_\infty(x)+\epsilon)\phi(\|u_n\|_{1,q}))}{\phi(\|u_n\|_{1,q})} \to \phi_p(v_\infty(x)+\epsilon),$$

and combining this with a similar lower bound shows that

$$\frac{\phi(u_n(x))}{\phi(\|u_n\|_{1,q})} \to \phi_p(v_\infty(x)), \quad x \in [0,1]. \tag{4.8}$$

In addition, the sequence is uniformly bounded on $[0,1]$, so the result follows from the dominated convergence theorem. A similar argument shows that

$$\widetilde{b}_{j,n} = \frac{\psi(j, u_n(j), \gamma_j u_n(j))}{\phi(\|u_n\|_{1,q})} \to \phi_p(\gamma_j v_\infty(j)) = b_j(v_\infty;\phi_p), \quad j = 0,\, 1$$

(using (1.3), and the definitions (2.9) and (4.3)). □

Lemma 4.4. *After choosing a suitable subsequence, there exists a pair of functions* $m_\pm \in L^1(0,1)$ *such that* (3.2) *holds and*

$$\widetilde{f}_n \rightharpoonup h_{(m_+,m_-)}(v_\infty) = m_+\phi_p(v_\infty^+) - m_-\phi_p(v_\infty^-), \tag{4.9}$$

in $L^1(0,1)$. *Furthermore,*

$$\mathcal{I}(\widetilde{f}_n) \to \mathcal{I}(h_{(m_+,m_-)}(v_\infty)), \tag{4.10}$$

in $C^0[0,1]$.

Proof. It follows from (1.4) that the set $\{\widetilde{f}_n\}$ is equi-integrable so, by Lemma 2.11, after taking a subsequence there exists $\widetilde{f}_\infty \in L^1(\Omega)$ such that $\widetilde{f}_n \rightharpoonup \widetilde{f}_\infty$ in $L^1(\Omega)$. Now define the sets

$$V_0 := \{x \in [0,1] : v_\infty(x) = 0\}, \qquad V_\pm := \{x \in [0,1] : \pm v_\infty(x) > 0\},$$

and the functions

$$m_\pm(x) := \begin{cases} \frac{1}{2}(\xi_\pm(x) + \Xi_\pm(x)), & x \in V_0 \cup V_\mp, \\ \widetilde{f}_\infty(x)/\phi_p(v_\infty(x)), & x \in V_\pm. \end{cases}$$

By (1.4),

$$\widetilde{f}_n(x) \leqslant \Xi_+(x)\frac{\phi(u_n(x))}{\phi(\|u_n\|_{1,q})} + \zeta(x)\frac{e_\phi(|u_n(x)|)}{\phi(\|u_n\|_{1,q})}, \quad x \in V_+.$$

Hence, for any measurable set $W \subset V_+$, with characteristic function χ_W, it follows from the weak convergence, (4.8) and the dominated convergence theorem that

$$\begin{aligned} \int_W \widetilde{f}_\infty &= \lim_{n\to\infty}\int_0^1 \widetilde{f}_n\chi_W \leqslant \lim_{n\to\infty}\int_W \left[\Xi_+\frac{\phi(|u_n|)}{\phi(\|u_n\|_{1,q})} + \zeta\frac{e_\phi(|u_n|)}{\phi(\|u_n\|_{1,q})}\right] \\ &= \int_W \Xi_+\phi_p(v_\infty). \end{aligned}$$

Combining this with a similar lower bound, and similar estimates on arbitrary subsets of V_- and V_0, now shows that

$$\xi_\pm\phi_p(v_\infty) \leqslant \widetilde{f}_\infty \leqslant \Xi_\pm\phi_p(v_\infty), \quad \text{a.e. on } V_\pm,$$

and $\widetilde{f}_\infty = 0$ on V_0. Hence, $\widetilde{f}_\infty$ has the representation (4.9), with (m_+, m_-) satisfying (3.2). Finally, (4.10) follows from Lemma 2.11. □

Now, by the definitions of d, $\widetilde{d}_n$, the limits (4.7), (4.10), and a similar argument to that of the proof of Lemma 4.3,

$$\begin{aligned} \widetilde{d}_n &= \phi^{-1}[\widetilde{b}_{1,n} - \widetilde{b}_{0,n} + \mathcal{I}(\widetilde{f}_n)(1)]/\|u_n\|_{1,q} \\ &\to \phi_p^{-1}[b_1(v_\infty;\phi_p) - b_0(v_\infty;\phi_p) + \mathcal{I}\big(h_{(m_+,m_-)}(v_\infty)\big)(1)] \\ &= d(v_\infty;\phi_p,\phi_p,h_{(m_+,m_-)}), \end{aligned}$$

and

$$\widetilde{g}_n = \widetilde{b}_{0,n} - \mathcal{I}(\widetilde{f}_n) \to \widetilde{g}_\infty := b_0(v_\infty; \phi_p) - \mathcal{I}(h_{(m_+,m_-)}(v_\infty)),$$

in $C^0[0,1]$.

Lemma 4.5.

$$\frac{\psi^{\#}_{\tau_n,u_n}(g_n)}{\|u_n\|_{1,q}} \to \phi_p^{-1}(\widetilde{g}_\infty),$$

in $L^q(0,1)$.

Proof. By a similar proof to that of Lemma 4.3,

$$\Big\| \frac{\phi^{-1}(g_n)}{\|u_n\|_{1,q}} - \phi_p^{-1}(\widetilde{g}_\infty) \Big\|_q \to 0.$$

Next, by (2.6), for arbitrary $\epsilon > 0$ and n sufficiently large,

$$\begin{aligned} \Big\| \frac{\psi_{u_n}^{-1}(g_n)}{\|u_n\|_{1,q}} - \phi_p^{-1}(\widetilde{g}_\infty) \Big\|_q &\leqslant \frac{\|\psi_{u_n}^{-1}(g_n) - \phi^{-1}(g_n)\|_q}{\|u_n\|_{1,q}} + \Big\| \frac{\phi^{-1}(g_n)}{\|u_n\|_{1,q}} - \phi_p^{-1}(\widetilde{g}_\infty) \Big\|_q \\ &\leqslant \frac{C_\epsilon + \epsilon\phi^{-1}(|g_n|_0)}{\|u_n\|_{1,q}} + \epsilon < 2\epsilon(1+C). \end{aligned}$$

The result now follows from the definition of $\psi^{\#}_{\tau_n,u_n}(g_n)$. □

Now, letting $n \to \infty$ in (4.5), and combining the above results, yields

$$\begin{aligned} v_\infty &= v_\infty(0) + d(v_\infty; \phi_p, \phi_p, h_{(m_+,m_-)}) + \mathcal{I}\{\phi_p^{-1}[b_0(v_\infty; \phi_p) - \mathcal{I}(h_{(m_+,m_-)}(v_\infty))]\} \\ &= v_\infty(0) + d(v_\infty; \phi_p, \phi_p, h_{(m_+,m_-)}) + G(v_\infty; \phi_p, h_{(m_+,m_-)}). \end{aligned}$$

Lemma 2.8 now shows that $v_\infty \in D(\Delta_p)$ and satisfies

$$-\Delta_p(v_\infty) = m_+\phi_p(v_\infty^+) - m_-\phi_p(v_\infty^-), \tag{4.11}$$

that is, 0 is a half-eigenvalue of (4.11). However, it follows from Theorem 3.2, Lemma 4.4 and the hypothesis (4.1) that $0 \in \Lambda_k^1(m_+, m_-)$, which is a contradiction. This contradiction completes the proof that the constant R exists.

To complete the proof we must show that $\deg(I - H(1,\cdot), B_R, 0) \neq 0$. To show this we use a homotopy to transform the operator $H(1,\cdot)$ to an odd operator, and the result then follows from Borsuk's theorem [9, Theorem 8.3]. We sketch the construction. Set

$$\mu_0 := \tfrac{1}{2}(\mu_+ + \mu_-), \quad \mu_{\pm,\tau} := (1-\tau)\mu_\pm + \tau\mu_0, \quad \tau \in [0,1].$$

Since $0 \in \Lambda_k^1(\mu_+, \mu_-)$ (by Theorem 3.2 and (4.1)), it follows from the continuity properties of the half-eigenvalues (see Theorem 3.2 in [16]) that we can choose a continuous function $\ell : [0,1] \to \mathbb{R}$ such that $\ell(0) = 0$ and

$$\ell(\tau) \in \Lambda_k^1(\mu_{+,\tau}, \mu_{-,\tau}), \quad \tau \in [0,1].$$

Now consider the equation

$$-\Delta_\phi(u) = \mu_{+,\tau}\phi(u^+) - \mu_{-,\tau}\phi(u^-) + \ell(\tau)\phi(u), \quad (\tau, u) \in [0,1] \times W^{1,q}(0,1). \tag{4.12}$$

By converting (4.12) to an integral equation formulation we can construct a homotopy $\widetilde{H} : [0,1] \times W^{1,q}(0,1) \to W^{1,q}(0,1)$, similar to H above, with the following properties: (i) $H(1,\cdot) = \widetilde{H}(0,\cdot)$; (ii) $\widetilde{H}(1,\cdot)$ is odd; (iii) there exists $\widetilde{R} \geqslant R$ such that any solution (τ, u) of equation (4.12) satisfies $\|u\|_{1,q} < \widetilde{R}$ (by a similar argument to the above construction of the corresponding number R for the homotopy H). It now follows from these properties that

$$\deg(I - H(1,\cdot), B_R, 0) = \deg(I - \widetilde{H}(1,\cdot), B_{\widetilde{R}}, 0) = \pm 1,$$

which completes the proof of the theorem. □

Remark 4.6. The above proof of Theorem 4.1 used Leray-Schauder degree, and so required an integral equation formulation of the problem. A somewhat similar result (with a nonresonance condition expressed in terms of the first and second eigenvalues of the ψ-Laplacian) is proved in [1], using the degree theory for operators of type $(S)_+$, as developed in [6] and [7]. This degree theory could also have been used to prove Theorem 4.1 above; indeed, the overall structure of the proof using the two approaches is identical — viz., a homotopy to a simpler problem is constructed, and an a priori bound for all solutions along the homotopy is obtained. The $(S)_+$ degree theory can be applied directly to the operator Δ_ψ so the proof using this degree theory seems slightly more direct than the Leray-Schauder proof. However, the $(S)_+$ approach loses some generality compared with the Leray-Schauder approach, since it requires that the functions $\xi_\pm$, $\Xi_\pm$, ζ in the growth conditions (1.3), (1.4) belong to $L^{p'}(0,1)$ rather than $L^1(0,1)$, see [1]. Since the half-eigenvalue results in [16] are available in the $L^1(0,1)$ setting, and the gain in simplicity of the $(S)_+$ approach is more apparent than real, it seemed advantageous to use the Leray-Schauder approach, and hence obtain the result under the $L^1(0,1)$ hypotheses on the growth conditions.

Remark 4.7. A nonresonance theorem similar to Theorem 4.1 can be obtained for periodic boundary conditions (with a suitable formulation of the periodic boundary condition). The structure of the set of half-eigenvalue of the periodic problem is not fully known, even when $p = 2$, see [3], and when $p \neq 2$ even the eigenvalue structure is not fully known, see [4] and [5]. However, it is known that non-empty intervals of the form $\Lambda^1_k(m_+, m_-)$, $k \geqslant -1$, as used above, exist and contain no half-eigenvalues, see [5] (what is not fully known is what can happen in the intervals $\Lambda^0_k(m_+, m_-)$, $k \geqslant 0$; in the periodic case these intervals can contain half-eigenvalues, or even eigenvalues, see [4] or [5]). Thus, the p-Laplacian results used in the above proof of Theorem 4.1 for separated boundary conditions are also available for the periodic case, so a similar result, with a similar proof, is also available in this case. For brevity, we will not give any further details.

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