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Bundles over Moduli Spaces and the Quantisation of BPS Monopoles

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Abstract

A vector bundle over the moduli space of BPS monopoles is constructed from the zero-modes of the Dirac operator acting on iso-spinor spinors and coupled to BPS monopoles. This bundle of zero-modes is an index bundle and has a natural connection whose curvature is anti-self-dual. For monopoles of charge one or two this information is sufficient for a detailed description of the bundle structure and the connection. Physically, the bundle of zero-modes is interpreted as a model for the quantum dynamics of monopoles coupled to one fermion: the quantum mechanical wavefunction is a section of the bundle and the quantum Hamiltonian is the Laplace-Beltrami operator on the moduli space minimally coupled to the natural connection. Numerical calculations of bound state energies and scattering cross sections are presented and the qualitative implications of the bundle geometry for the quantum theory are emphasised.

1 Introduction

Since their discovery in 1974 by t'Hooft and Polyakov [1] non-abelian monopoles, in particular Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles [2], have featured prominently in a variety of contexts in theoretical and mathematical physics [3] and, more recently, in pure mathematics.

One reason why BPS monopoles have attracted much attention is that they provide a fully 3-dimensional example of topological solitons of Bogomol'nyi type: they are static, finite energy solutions of classical field equations and stable because their energy attains a lower topological bound, the Bogomol'nyi bound. Much progress in the understanding of the dynamics of such solitons has been made over the past decade using the idea of the moduli space approximation [4]. In this scheme one approximates the slow motion of several interacting solitons by geodesic motion on the Riemannian manifold of static multi-soliton solutions. To quantise, one works in the Schrödinger picture: wavefunctions are complex-valued functions on the moduli space and the quantum Hamiltonian is taken to be proportional to the Laplace-Beltrami operator [5]. In general the mathematical analysis of moduli spaces is still a very difficult problem, but in the case of two BPS monopoles one could exploit special properties, in particular the hyperkähler property of the metric, and obtain sufficient information about the moduli space to calculate geodesics and to study the Laplace operator in some detail. The analysis of the moduli space, summarised in the recent book by Atiyah and Hitchin [6], has provided theoretical physicists with a tractable model for soliton interactions and has formed the basis of a "soliton phenomenology", which in turn has suggested approaches to related models, such as Skyrme's soliton model for nucleons.

Because of their topological properties, BPS monopoles have also been studied in the context of anomalies and index theorems. Jackiw and Rebbi first observed that the Dirac operator coupled to a t'Hooft-Polyakov monopole has zero-modes [7] and gave a physical interpretation of them; later Callias [8] rigorously proved an index theorem for the Dirac operator in the background of a general BPS (multi-)monopole. There is another, apparently quite distinct, reason why it is interesting to couple linear operators – in particular the Dirac, Klein-Gordon and Schrödinger operators – to a BPS monopole: the "hedgehog" form of the standard BPS (or t'Hooft-Polyakov) monopole then gives rise to the famous "spin from isospin" phenomenon [9]. This allows one to construct fermions from bosonic fields.

In this paper all these aspects will play a role: we interpret the zero-modes of the Dirac operator as bound states of monopoles and one fermion and use moduli space techniques to study the quantisation and the quantised interaction of such composite objects. The material is organised as follows: In section 2 we consider the zero-modes of the Dirac operator coupled to a fixed BPS (multi-) monopole. Allowing the (multi-) monopole to vary over the moduli space naturally leads to a vector bundle of zero-modes over the moduli space, which is in fact an index bundle. In section 3 we define this bundle and explain why it has a natural connection defined on it; we also note that the curvature of this connection is anti-self-dual in an appropriate sense. To some extent all the material in sections 2 and 3 has been around for some time[1], so our contribution here is mainly an expository one. Exploiting the symmetries of the moduli space for two monopoles, reviewed in section 4, we are able to calculate the connection explicitly for this case and to give a detailed description of the index bundle in section 5. The situation is rather similar to the way in which the hyperkähler property of the Riemannian metric on the moduli space reduces the problem of calculating the metric to a set of ordinary differential equations. In section 6 we show that in an approximate, adiabatic quantisation of the fermion-monopole system one should take the quantum mechanical wavefunction to be a section of the index bundle. The interplay between the geometry of the index bundle and its base space on the one hand and the quantum theory on the other is explored in section 7, where we also give a field theoretical interpretation of our quantum mechanical model. Section 8 is devoted to a detailed study of the quantised interactions of two monopoles in the presence of a fermion and includes calculations of some bound states and scattering cross sections. In section 9 we briefly consider the quantised interactions of two monopoles coupled to two fermions. Section 10 contains our conclusion.

Our model for the interaction of two monopoles is novel in the context of soliton dynamics because it involves a connection as well as a metric on the moduli space. We will therefore pay particular attention to the way in which the bundle structure and the connection determine qualitative aspects of the interacting monopoles, such as their statistics.

2 The Dirac Operator Coupled to BPS Monopoles

Our standard references for background material on BPS monopoles will be the book by Atiyah and Hitchin [6] and the paper by Gibbons and Manton [5]. We follow the notational conventions adopted there and denote by A_i, $i = 1, 2, 3$, the cartesian components of an $SU(2)$ gauge potential $\boldsymbol{A}$ on $\mathbb{R}^3$ and by Φ a Higgs field in the adjoint representation of $SU(2)$. More specifically, writing $su(2)$ for the Lie algebra of $SU(2)$,

[1]We are grateful to N.Hitchin for an enlightening seminar on these topics

we have four maps

$$A_i, \Phi : \mathbb{R}^3 \mapsto su(2).$$

We use the gauge potential to define the covariant derivative

$$D_i = \partial_i + A_i$$

and the curvature

$$F_{ij} = [D_i, D_j] = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

We will often refer to the gauge group $SU(2)$ as the isospin group and use a basis t_a, $(a = 1, 2, 3)$ of $su(2)$ satisfying $[t_a, t_b] = \epsilon_{abc} t_c$, which can be expressed in terms of the Pauli matrices τ_a via $t_a = \frac{1}{2i}\tau_a$. We also require a norm $||\cdot||$ on $su(2)$ which we take to be $||T||^2 = -\frac{1}{2}\mathrm{tr}T^2$. We can then introduce the space $\mathcal{A}$ of pairs $(\boldsymbol{A}, \Phi)$ which satisfy the boundary condition

$$\lim_{|\boldsymbol{x}|\to\infty} ||\Phi(\boldsymbol{x})|| = 1 \tag{2.1}$$

and the base point condition

$$\lim_{x_3\to\infty} \Phi(0, 0, x_3) = -t_3. \tag{2.2}$$

The group $\mathcal{G}$ of gauge transformations is, by definition, the space of maps $g : \mathbb{R}^3 \mapsto SU(2)$ which satisfy the base point condition

$$\lim_{x_3\to\infty} g(0, 0, x_3) = \mathbf{1}_2. \tag{2.3}$$

$\mathcal{G}$ acts freely on $\mathcal{A}$, and dividing by that action we obtain the true configuration space

$$\mathcal{C} = \mathcal{A}/\mathcal{G}. \tag{2.4}$$

Following [6] we sometimes use the shorter notation c for pairs $(\boldsymbol{A}, \Phi)$ and denote the equivalence class of a given element $c \in \mathcal{A}$ by $[c] \in \mathcal{C}$. Finally we introduce a non-abelian magnetic field

$$B_i = \frac{1}{2}\epsilon_{ijk} F_{jk}$$

and define: a BPS monopole is a pair $(\boldsymbol{A}, \Phi) \in \mathcal{A}$ which satisfies the Bogomol'nyi equation

$$B_i = D_i \Phi \tag{2.5}$$

In the Prasad-Sommerfield limit, where the Higgs potential vanishes, all minima of the Yang-Mills-Higgs (YMH) energy functional

$$V[(\boldsymbol{A}, \Phi)] = \int \frac{1}{2}||B_i B_i|| + \frac{1}{2}||D_i \Phi D_i \Phi|| d^3x \tag{2.6}$$

satisfying (2.1) are given by BPS monopoles or by anti-monopoles, which are solutions of $B_i = -D_i\Phi$.

As a result of the boundary conditions the Higgs field defines a map from the 2-sphere at spatial infinity to the unit 2-sphere in $su(2)$. Such a map is characterised by an integer winding number k. By looking at the long-range part of the magnetic field $\boldsymbol{B}$ one can identify k with the magnetic charge of a BPS monopole. To make contact with physics we have to choose units. We employ the Heaviside-Lorentz system [10], where all electromagnetic units are derived from the basic mechanical units for length, time and mass and where the Coulomb potential is $q/4\pi r$. The YMH Lagrangian in the Bogomol'nyi limit really contains two free parameters, the coupling constant e of the gauge field and the vacuum expectation value a of the Higgs field. By setting these, as well as the speed of light, to 1 we have eliminated all units. A consequence of this which will be important later is that $\hbar$ is dimensionless, but not equal to 1. One can convert from our geometrical units to physical units by choosing the unit of energy to be a/e and the unit of length to be $1/ae$, see for example the review article by Goddard and Olive [11]. In geometrical units the magnetic charge is $4\pi k$ and, as a consequence of the Bogomol'nyi equations, the mass of the monopole is $4\pi k$, too. With our definition the integer k is always positive for monopoles and negative for anti-monopoles. In the following we will call monopoles of charge $4\pi k$ k-monopoles but often we simply say "monopole" when referring to a monopole of charge 4π. When $k = 0$, the solutions of the Bogomol'nyi equation are all gauge equivalent to the vacuum field $\boldsymbol{A} = \boldsymbol{0}$, $\Phi = -t_3$.

We introduce the notation X_k for the space of all k-monopoles. X_k is an infinite dimensional manifold on which the group $\mathcal{G}$ acts freely. Quotienting X_k by this action we obtain the moduli space M_k, which is a differentiable manifold of dimension $4k$. It has a natural Riemannian metric induced from the YMH kinetic energy functional and it is explained in [6] why M_k equipped with this metric is a hyperkähler manifold. Since M_k is precisely the subset of $\mathcal{C}$ on which the potential energy functional V is minimal for given k it is a natural candidate for the configuration space of a truncation of the YMH theory in the topological sector with winding number k. For a physical interpretation of the $4k$ parameters consider first the moduli space of a single monopole. M_1 is a flat manifold of the form

$$M_1 = \mathbb{R}^3 \times S^1. \tag{2.7}$$

To interpret the factors occurring in this decomposition we look at a particular solution of (2.5) with charge 4π, the standard BPS monopole:

$$A_i^a = \epsilon_{iab}\frac{x^b}{x}\left(\frac{1}{x} - \frac{1}{\sinh x}\right)$$

$$\Phi^a = -\frac{x^a}{x}\left(\coth x - \frac{1}{x}\right) \tag{2.8}$$

where $x = |\boldsymbol{x}|$. The $\mathbb{R}^3$ part of M_1 is generated by translating this solution in physical 3-space. The circle factor, coordinatised by an angle $\chi \in [0, 2\pi)$, parametrises special "large" gauge transformations of the solution (2.8). These gauge transformations have the form

$$g(\chi) = e^{-\chi\Phi} \tag{2.9}$$

which only satisfies the base point condition if $\chi \in 4\pi\mathbb{Z}$. For $\chi = 2\pi$, however, $\lim_{x_3\to\infty} g(0,0,x_3) = -\mathbf{1}_2$, so $-g(2\pi) \in \mathcal{G}$. The action of $-g(2\pi)$ on $(\boldsymbol{A}, \Phi)$ is the same as the action of $g(2\pi)$, so the fields $(\boldsymbol{A}, \Phi)$ at $\chi = 0$ and at $\chi = 2\pi$ are related by the gauge transformation $-g(2\pi)$ and should be identified. The physical significance of the circle parameter χ becomes apparent when we allow it to vary with time, keeping $A_0 = 0$. This last condition is compatible with Gauss law. One then finds, by looking at the asymptotic electromagnetic field, that the monopole acquires an electric charge $4\pi\dot{\chi}$, thus turning it into a dyon (a particle with both magnetic and electric charge).

Clearly one can define large gauge transformations of the form (2.9) for any k-monopole and then act with them on M_k. This action is a free $U(1)$ action and quotienting M_k by it one obtains a $4k-1$ dimensional manifold N_k. Thus M_k is fibered over N_k with fibre a circle. This generalises the simple product form of M_1. Motion on the fibres of M_k gives a k-monopole a total electric charge $4\pi k\dot{\chi}$. Asymptotically the $4k-1$ parameters of N_k can be interpreted as k position vectors and $k-1$ relative phases of k single monopoles. A more precise statement, proven in [6], is that the asymptotic region of M_k contains k-monopoles which are approximate superpositions of k single monopoles with arbitrary individual positions and phase angles. This justifies the interpretation of k as the monopole number and affirms the suitability of M_k as the truncated configuration space for multi-monopole dynamics. Note, however, that the asymptotic region is not just an (unordered) product of k copies of M_1. Such a decomposition is only possible locally.

How do we couple the Dirac operator to a k-monopole $(\boldsymbol{A}, \Phi)$? For this purpose we interpret the Bogomol'nyi equation as the dimensional reduction of the self-duality equations in four euclidean dimensions [12]. Specifically we consider $SU(2)$ gauge potentials A_μ, $\mu = 1, .., 4$, on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$ which are independent of x_4 and have self-dual curvature

$$F_{\kappa\lambda} = \frac{1}{2}\epsilon_{\kappa\lambda\mu\nu}F_{\mu\nu}. \tag{2.10}$$

One checks that this equation is equivalent to the Bogomol'nyi equations if the fourth component of the gauge potential is identified with the Higgs field, $A_4 = \Phi$. Thus, a natural way to couple the Dirac operator in $1+3$ dimensional Minkowski space to $(\boldsymbol{A}, \Phi)$ is to consider the massless Dirac operator in $1+4$ dimensional Minkowski space coupled to A_μ, independent of x_4 and satisfying (2.10), and then to restrict attention to $SO(1,4)$ spinors which are independent of x_4. We can obtain five 4×4 complex matrices (Γ^0, Γ^μ) satisfying the Clifford algebra $C(1,4)$

$$(\Gamma^0)^2 = \mathbf{1}_4 \qquad (\Gamma^\mu)^2 = -\mathbf{1}_4 \qquad \Gamma^0\Gamma^\mu + \Gamma^\mu\Gamma^0 = 0 \quad \text{and} \quad \Gamma^\mu\Gamma^\nu + \Gamma^\nu\Gamma^\mu = -\delta^{\mu\nu}\mathbf{1}_4 \tag{2.11}$$

from the standard Dirac γ-matrices:

$$\Gamma^0 = \gamma^0, \quad \Gamma^i = \gamma^i, \quad \Gamma^4 = -i\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3. \tag{2.12}$$

Writing $\epsilon(t, \boldsymbol{x})$ for a 4-component spinor which also transforms under the fundamental representation of the $SU(2)$ isospin group, the $1+4$ dimensional Dirac equation in the temporal gauge $A_0 = 0$ reads

$$-i\hbar\left(\Gamma^0 \otimes \partial_t + \Gamma^\mu \otimes D_\mu\right)\epsilon = 0, \tag{2.13}$$

where D_i is as defined above and $D_4 = \Phi$. ϵ really transforms under a spinor representation of $SO(1,4)$ but we can think of it as an $SO(1,3)$ spinor by restricting attention to the Lorentz tranformations in $SO(1,3) \subset SO(1,4)$ respecting the condition $x_4 = 0$. To exhibit the $1+3$ dimensional interpretation of the Dirac equation we rewrite it in terms of the γ matrices

$$\left(-i\hbar(\gamma^0 \otimes \partial_t + \gamma^i \otimes D_i) - \hbar\gamma^5\Phi\right)\epsilon = 0. \tag{2.14}$$

The symmetries of this equation and the Bogomol'nyi equation will be important for us later, so we note a somewhat unusual discrete symmetry here. The Bogomol'nyi equation is not invariant under parity, but instead the parity operation takes a monopole of charge k into an anti-monopole of charge $-k$. However, reversing the sign of the Higgs field also changes the sign of the magnetic charge and both the Bogomol'nyi equation and the Dirac equation (2.14) are invariant under a joint parity transformation and magnetic charge conjugation

$$\begin{aligned} CP : \boldsymbol{x} \mapsto -\boldsymbol{x}, \quad \boldsymbol{A}(t,\boldsymbol{x}) &\mapsto -\boldsymbol{A}(t,-\boldsymbol{x}), \quad \Phi(t,\boldsymbol{x}) \mapsto -\Phi(t,-\boldsymbol{x}) \\ \epsilon(t,\boldsymbol{x}) &\mapsto \gamma^0\epsilon(t,-\boldsymbol{x}). \end{aligned} \tag{2.15}$$

We begin our study of solutions of the Dirac equation (2.14) by considering the vacuum configuration $(\boldsymbol{A}, \Phi) = (\mathbf{0}, -t_3)$. The Dirac equation (2.14) then has plane wave solutions

$$\epsilon(t,\boldsymbol{x}) = u(E,\boldsymbol{p})e^{\frac{i}{\hbar}(Et-\boldsymbol{p}\cdot\boldsymbol{x})} \tag{2.16}$$

where u is a constant 4-spinor and the energy-momentum vector satisfies the dispersion relation

$$E^2 = |\boldsymbol{p}|^2 + \frac{\hbar^2}{4}. \tag{2.17}$$

This shows that the Dirac field has acquired a mass. In the language of perturbative quantum field theory, where one quantises fluctuations around the vacuum, the fundamental excitations of the ϵ-field would therefore represent particles with spin $\hbar/2$, mass $\hbar/2$ and, because t_3 has eigenvalues $\pm\frac{1}{2}$, electric charges $\pm\hbar/2$. Because of the spin-statistics theorem they should be quantised as a fermion, i.e. by imposing anti-commutation relations on the operator corresponding to ϵ. One can similarly analyse the perturbative particle spectrum of the YMH theory (see for example [11]), and we summarise the results in table 1.

	Higgs particle	photon	massive gauge particles (W-bosons)	fermions
mass	0	0	$\hbar$	$\frac{1}{2}\hbar$
spin	0	$\hbar$	$\hbar$	$\frac{1}{2}\hbar$
electric charge	0	0	$\pm\hbar$	$\pm\frac{1}{2}\hbar$

Table 1

Perturbative particles in YMH theory coupled to isospinor-spinors

To proceed with the mathematical analysis of the Dirac equation (2.14) coupled to a general k-monopole we choose a particular representation of the γ matrices, and introduce the spin operator $s_i = \frac{1}{2i}\sigma_i$. Here, in the context of spin (rather than isospin) we write σ_i (rather than τ_i) for the Pauli matrices. Then, in the Dirac representation [13]

$$\gamma^0\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad -i\gamma^0\gamma^5 = \begin{pmatrix} 0 & -i\mathbf{1}_2 \\ i\mathbf{1}_2 & 0 \end{pmatrix}$$

so equation (2.14) becomes

$$\hbar \begin{pmatrix} -i\partial_0 \mathbf{1}_2 \otimes \mathbf{1}_2 & -i\sigma_i \otimes D_i - \mathbf{1}_2 \otimes \Phi \\ -i\sigma_i \otimes D_i + \mathbf{1}_2 \otimes \Phi & -i\partial_0 \mathbf{1}_2 \otimes \mathbf{1}_2 \end{pmatrix} \epsilon = 0. \tag{2.18}$$

We are only interested in stationary solutions, which we write in terms of two 2-spinors ρ and λ

$$\epsilon(t, \boldsymbol{x}) = e^{-iEt} \begin{pmatrix} \rho(\boldsymbol{x}) \\ \lambda(\boldsymbol{x}) \end{pmatrix}. \tag{2.19}$$

The Dirac equation is then equivalent to two coupled equations

$$\begin{aligned} \hbar \left(-i\sigma_i \otimes D_i - \mathbf{1}_2 \otimes \Phi \right) \right) \lambda &= E\rho \\ \hbar \left(-i\sigma_i \otimes D_i + \mathbf{1}_2 \otimes \Phi \right) \right) \rho &= E\lambda. \end{aligned} \tag{2.20}$$

In this paper we will mainly be concerned with the zero-modes of the Dirac operator. For $E = 0$ the above equations decouple and are equivalent to the requirement that ρ lies in the kernel of the operator

$$\not{D} := -i\sigma_i \otimes D_i + \mathbf{1}_2 \otimes \Phi \tag{2.21}$$

and that λ lies in the kernel of

$$\not{D}^\dagger := -i\sigma_i \otimes D_i - \mathbf{1}_2 \otimes \Phi. \tag{2.22}$$

The operators $\not{D}$ and $\not{D}^\dagger$ have been much studied in the mathematics literature. In [14] Taubes showed that, on appropriate Sobolev spaces, they extend to Fredholm operators and are adjoints of each other. We will mainly be concerned with $\not{D}$ but will reserve the term "Dirac operator" for the operator

$$\mathcal{D} = \hbar \begin{pmatrix} 0 & \not{D}^\dagger \\ \not{D} & 0 \end{pmatrix}. \tag{2.23}$$

From a mathematical point of view it is convenient to drop the distinction between spin and isospin and to use quaternionic notation for both. We write e_μ, $\mu = 1, 2, 3, 4$ for the unit quaternions, which satisfy

$$e_4 e_\mu = e_\mu e_4 = e_\mu \qquad \text{and} \qquad e_i e_j = -\delta_{ij} + \epsilon_{ijk} e_k \qquad (i, j = 1, 2, 3), \tag{2.24}$$

and identify

$$e_4 = \mathbf{1}_2, \qquad e_i = -i\sigma_i = -i\tau_i. \tag{2.25}$$

We also denote the real 4-dimensional vector space with basis $\{e_\mu\}$ by $\mathbb{H}$ and write $\bar{e}_\mu$ for the conjugates:

$$\bar{e}_4 = e_4 \qquad \bar{e}_i = -e_i. \tag{2.26}$$

In quaternionic notation we write for example

$$\not{D} = e_i \otimes D_i + e_4 \otimes \Phi. \tag{2.27}$$

Using this notation we summarise the results which are relevant here in a theorem.

Theorem 1

If $(\boldsymbol{A}, \Phi)$ *is a k-monopole, then, on square-integrable spinors,*
i) ker $\not{D}^\dagger = \{0\}$
ii) ker $\not{D}$ *is a k-dimensional real vector space with inner product.*

The notation $V_{(\boldsymbol{A},\Phi)} := \{(\boldsymbol{A}, \Phi)\} \times \ker \not{D}$ will be used to emphasise the dependence on $(\boldsymbol{A}, \Phi)$. We can check some aspects of this theorem explicitly. To see why i) holds, consider the operator $\not{D}\, \not{D}^\dagger$. Using the Bogomol'nyi equation in the form

$$[D_1, D_2] = [D_3, \Phi] \qquad\qquad + cycl.,$$

where $+cycl.$ means we add two further equations obtained by cyclically permuting $1, 2$ and 3, one finds

$$\not{D}\, \not{D}^\dagger = e_4 \otimes (-D_1^2 - D_2^2 - D_3^2 - \Phi^2). \tag{2.28}$$

This is a manifestly positive operator, and therefore

$$\ker \not{D}^\dagger \subset \ker \not{D}\, \not{D}^\dagger = \{0\}.$$

To prove the second part, we recall some definitions: A real structure on a complex vector space is an antilinear map from that vector space into itself which squares to the identity; an operator is said to be real if it commutes with the real structure. We want to show that $\not{D}$ is real with respect to a suitable real structure. First we interpret the quaternions e_a, $a = 1, 2, 3$, as a basis of $su(2)$ and write

$$\begin{aligned} A_i(\boldsymbol{x}) &= \frac{1}{2} A_i^a(\boldsymbol{x}) e_a \\ \Phi(\boldsymbol{x}) &= \frac{1}{2} \Phi^a(\boldsymbol{x}) e_a \end{aligned} \tag{2.29}$$

where A_i^a and Φ^a are real-valued functions of $\boldsymbol{x}$. Then

$$\not{D} = e_i \otimes e_4 \partial_i + \frac{1}{2} e_i \otimes e_a A_i^a + \frac{1}{2} e_4 \otimes e_a \Phi^a. \tag{2.30}$$

If we arrange the components of the isospinor-spinor ρ into a, generally complex, 2×2 matrix, the action of the tensor product of two quaternions is given by

$$e_\mu \otimes e_\nu \rho = e_\mu \rho e_\nu^t. \tag{2.31}$$

The operations on the right hand side are standard matrix multiplication and transposition, using the identification (2.25). Any 2×2 complex matrix can be expressed as a linear combination of the e_μ so we can write

$$\rho(\boldsymbol{x}) = b_\mu(\boldsymbol{x}) e_\mu.$$

with complex valued functions b_μ. We claim that $\not{D}$ is real with respect to the real structure which acts on ρ via complex conjugation of the b_μ. By the expansion (2.30) this follows if the action (2.31) is real. But this is clearly the case, since the quaternion multiplication laws (2.24) are real operations, and $e_\mu^t = e_\mu$ for $\mu \neq 2$ and $e_2^t = -e_2$. The reality of $\not{D}$ permits us to assume, without loss of generality, that the b_μ are real, and so ρ can be written

$$\rho = \begin{pmatrix} b_4 + ib_3 & -b_2 + ib_1 \\ b_2 - ib_1 & b_4 - ib_3 \end{pmatrix}. \tag{2.32}$$

A consequence of this is that we can think of $\ker \not{D}$ as a subspace of the Hilbert space $L^2(\mathbb{R}^3, \mathbb{H})$ of square-integrable quaternion valued functions on $\mathbb{R}^3$. An inner product on this Hilbert space compatible with the quaternionic structure is

$$\langle \rho_1, \rho_2 \rangle = \int \frac{1}{2} \mathrm{tr}(\rho_1^\dagger \rho_2)\, d^3x = \int b_{1\mu} b_{2\mu}\, d^3x,$$

and this induces an inner product on $V_{(\boldsymbol{A},\Phi)}$.

Finally, to show that $\ker \not{D}$ is k dimensional, we invoke an index theorem. In [8] it is shown that

$$\dim \ker \not{D} - \dim \ker \not{D}^\dagger = k. \tag{2.33}$$

Together with the vanishing theorem i) this justifies the remainder of theorem 1.

Here we have only used the fact that $V_{(\boldsymbol{A},\Phi)}$ is naturally embedded in an infinite dimensional quaternionic vector space to define an inner product, but in the next section the quaternionic structure will be crucial. Note also an important generalisation: If we do not restrict attention to zero-modes of the Dirac operator $\mathcal{D}$ (2.23) but consider general eigenspinors $\epsilon = \begin{pmatrix} \rho \\ \lambda \end{pmatrix}$ then it is consistent to assume $\epsilon \in L^2(\mathbb{R}^3, \mathbb{H} \oplus \mathbb{H})$, with inner product

$$\langle\langle \epsilon_1, \epsilon_2 \rangle\rangle = \langle \rho_1, \rho_2 \rangle + \langle \lambda_1, \lambda_2 \rangle \tag{2.34}$$

by the same argument as given above.

It is instructive to find the (up to normalisation) unique zero-mode of the Dirac operator coupled to the standard BPS monopole (2.8) explicitly, which is a special case of the zero-mode for the t'Hooft-Polyakov monopole found by Jackiw and Rebbi [7]. The standard BPS monopole is spherically symmetric in the sense that it is invariant under a combined spatial and isospin rotation. As a result $\not{D}$ commutes with

$$\mathbf{J} = \mathbf{1}_2 \otimes \mathbf{1}_2 \ell + \mathbf{s} \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{t},$$

where ℓ is the usual orbital angular momentum operator. Thus we look for a solution $\rho = b_\mu e_\mu$ which is annihilated by $\mathbf{J}$. The simplest way in which this can happen is if ρ is annhilated by ℓ, so that the functions b_μ depend only on $x = |\boldsymbol{x}|$, and also by $\mathbf{s} \otimes \mathbf{1}_2 + \mathbf{1}_2 \otimes \mathbf{t}$, which implies that ρ is proportional to e_2. The ansatz

$$\rho(x) = \begin{pmatrix} 0 & -b(x) \\ b(x) & 0 \end{pmatrix} \tag{2.35}$$

leads to the ordinary differential equation

$$\frac{db}{dx} + \left(\frac{1}{2} \coth x + \frac{1}{\sinh x} - \frac{3}{2x} \right) b = 0 \tag{2.36}$$

which is solved by

$$b(x) = b_0 \sqrt{\frac{x^3 \cosh \frac{x}{2}}{\sinh^3 \frac{x}{2}}} \tag{2.37}$$

where b_0 is a real number chosen so that $\langle \rho, \rho \rangle = 1$. Asymptotically b decays like

$$b(x) \sim x^{3/2} e^{-x/2}. \tag{2.38}$$

3 The Bundle of Zero-modes and its Connection

Recall that a monopole is determined, up to $SU(2)$ gauge transformation, by its position and the circle parameter χ: we obtain a general monopole, with angle χ and centred at $\boldsymbol{X}$, by acting on the standard BPS monopole with the translation $\boldsymbol{X}$ and with the "large" gauge transformation $g(\chi)$. Thus it is straightforward to write down the zero-mode of the Dirac operator coupled to such a general monopole. It can be obtained, up to $SU(2)$ gauge transformation, from the zero-mode ρ of the Dirac operator coupled to the standard BPS monopole as

$$\rho_{\boldsymbol{X},\chi}(\boldsymbol{x}) = g(\chi) \rho(\boldsymbol{x} - \boldsymbol{X}). \tag{3.1}$$

For $k > 1$ there are no explicit formulae for the zero-modes of the Dirac operator, but the existence of a k-dimensional real vector space of zero-modes for every k-monopole

$(\boldsymbol{A}, \Phi) \in X_k$ is assured by theorem 1 for all $k \geq 1$. Quotienting out by $SU(2)$ gauge transformations one obtains a k-dimensional real vector space for every equivalence class $[c] \in M_k$. We denote this vector space by $V_{[c]}$. For a k-monopole $(\boldsymbol{A}, \Phi)$ which consists approximately of k well-separated monopoles there is, up to sign, a canonical basis of $V_{(\boldsymbol{A},\Phi)}$. It consists of k zero-modes, determined only up to sign, which are each of the form (3.1) and localised at one of the k monopoles. Thus, the physical interpretation of the result $\dim V_{(\boldsymbol{A},\Phi)} = k$ is that the isospinor fermion can be bound to any of the k monopoles. This simple picture is useful for k well-separated monopoles, but it is not appropriate for a generic k-monopole. In general there is no canonical basis of zero-modes of the Dirac operator, so the union

$$\bigcup_{(\boldsymbol{A},\Phi)\in X_k} V_{(\boldsymbol{A},\Phi)}, \tag{3.2}$$

topologised by its embedding in $X_k \times L^2(\mathbb{R}, \mathbb{H})$ is some rank k vector bundle over X_k. Using the inner product on each fibre we can choose the transition functions of that bundle to be elements of the orthogonal group $O(k)$. The group $\mathcal{G}$ of gauge transformations acts via bundle isomorphisms on this bundle and this action is free. Quotienting by it we obtain an $O(k)$ vector bundle over the moduli space M_k which we denote by Ind_k. One may also define Ind_k more formally as the index bundle of the continuous family of (equivalence classes of) operators $\not{D}$ parametrised by elements $[c] \in M_k$. This definition, given by Taubes in [14], justifies the notation we have chosen. Cohen and Jones discuss some topological properties of Ind_k in [15] and relate them to representations of the braid group.

The bundle (3.2) is a subbundle of the trivial bundle

$$X_k \times L^2(\mathbb{R}^3, \mathbb{H})$$

and as a result a natural covariant derivative on it is given by standard differentiation followed by orthogonal projection (in the same way in which the Levi-Civita connection on a hypersurface arises from the embedding of the surface in $\mathbb{R}^n$). The natural connection on (3.2) is invariant under the action of $\mathcal{G}$ and thus well-defined on the bundle Ind_k. An important result is that the curvature of the natural connection on Ind_k is anti-self-dual in an appropriate sense. Since M_k is not in general 4-dimensional we cannot use the standard notion of anti-self-duality for 2-forms based on the Hodge $*$ operator. However, M_k is hyperkähler, which means that its Riemannian metric is Kähler with respect to three complex structures I, J and K satisfying the quaternion algebra written down for e_1, e_2 and e_3 in (2.24). We use the hyperkähler property to define

Definition 3.1 . *A 2-form on a hyperkähler manifold M is anti-self-dual if it is of type*

$(1,1)$ *for the complex structures I, J and K.*

It is shown in [16] that for $M = \mathbb{R}^4$ this notion of anti-self-duality is equivalent to the usual one. The proof also applies to an arbitrary 4-dimensional hyperkähler manifold since it only involves linear algebra in the tangent space. To illustrate the equivalence, identify $\mathbb{R}^4$ with the real vector space of quaternions $\mathbb{H}$. Then we can represent the action of the complex structures on $x = x_\mu e_\mu$ explicitly by left-multiplication with e_1, e_2 and e_3. The 2-form $d\bar{x} \wedge dx$ constructed from the quaternionic 1-form $dx = e_\mu dx_\mu$ is of type $(1,1)$ for the complex structures $e_i \, i = 1,2,3$ because $d\overline{e_i x} \wedge de_i x = d\bar{x} \wedge dx$. On the other hand one sees, by carrying out the quaternion multiplication, that its coefficients are precisely a basis for the anti-self-dual 2-forms in the standard sense:

$$\begin{aligned} d\bar{x} \wedge dx &= 2e_1(dx_4 \wedge dx_1 - dx_2 \wedge dx_3) \\ &+ 2e_2(dx_4 \wedge dx_2 - dx_3 \wedge dx_1) \\ &+ 2e_3(dx_4 \wedge dx_3 - dx_1 \wedge dx_2) \end{aligned} \tag{3.3}$$

We can now formulate

Theorem 3.2 . *The natural connection on Ind_k has anti-self-dual curvature.*

The theorem and its proof are due to N. Hitchin. The proof is based on an infinite-dimensional analogue of the ADHM construction for anti-self-dual Yang-Mills instantons on S^4 [17] and can be found in [18]. Later we will require a formula for a local gauge potential of the natural connection. Given an open cover $\{U_r\}$ of M_k which trivialises Ind_k we find the gauge potential on U_r by picking, for every $[c] \in U_r$ an orthonormal basis $\{\rho_m\}_{m=1,..,k}$ of the real vector space $V_{[c]}$ which varies smoothly with $[c]$. The gauge potential, a 1-form on U_r taking values in the Lie algebra of $O(k)$, has matrix elements

$$\omega_{mn} = \langle \rho_m, d\rho_n \rangle \tag{3.4}$$

where d is differentiation with respect to the coordinates of M_k.

As an illustration of these general results consider the bundle Ind_1 over M_1. It is an $O(1)$ bundle, so it is necessarily flat (i.e. it has no curvature), but it may have non-trivial holonomy. At the beginning of this section we saw how we can obtain (modulo $SU(2)$ gauge transformations) any monopole and the zero-mode of the corresponding Dirac operator from the zero-mode (2.35) of the standard BPS-monopole via translations and the action of the "large" gauge transformations $g(\chi)$. The zero-modes $g(\chi)\rho(\boldsymbol{x}-\boldsymbol{X})$ (3.1) trivialise the bundle Ind_1 over the $\mathbb{R}^3$ part of M_1. Since ρ is an isospinor, however, the holonomy around the circle factor is non-trivial. Recall that the gauge transformation identifying $(\boldsymbol{A}, \Phi)$ at $\chi = 0$ and $\chi = 2\pi$ is $-g(2\pi)$, which satisfies the base point condition (2.3). Acting on $\rho_{\chi=0}$, this gauge transformation gives $-\rho_{\chi=2\pi}$. Thus Ind_1 is twisted once over S^1 and we have established that

$$\mathrm{Ind}_1 = \mathbb{R}^3 \times \text{Möb} \tag{3.5}$$

where Möb is the Möbius bundle over S^1.

Before we can similarly describe the geometry of the bundle Ind_2 we need to review some basic properties of the moduli space M_2.

4 Geometry and Symmetries of Moduli Spaces

It is explained in [6] that the moduli space for k-monopoles decomposes isometrically into

$$M_k = \mathbb{R}^3 \times \frac{S^1 \times M_k^0}{\mathbb{Z}_k}. \tag{4.1}$$

The $\mathbb{R}^3$ part parametrises the centre-of-mass position of the monopoles: the asymptotic form of the Higgs field allows one to associate a centre to every k-monopole and $\boldsymbol{X} = (X_1, X_2, X_3) \in \mathbb{R}^3$ are the cartesian coordinates of that centre. S^1 is the overall phase factor explained in section 2. M_k^0 is $(4k-4)$-dimensional, simply connected and also hyperkähler [2]. Writing $\tilde{M}_k$ for the k-fold (universal) covering space of M_k we also have the decomposition $\tilde{M}_k = \mathbb{R}^3 \times S^1 \times M_k^0$.

The symmetries of M_k are particularly important for us . Elements of M_k can be identified (up to gauge transformation) with fields on euclidean 3-space and are therefore naturally acted upon by the euclidean group $E_3 = SO(3) \times \mathbb{R}^3$. Furthermore the YMH dynamics is invariant under the action of E_3, and therefore E_3 acts on M_k via isometries. The translations act on a k-monopole by shifting its centre

$$(\boldsymbol{A}(\boldsymbol{x}), \Phi(\boldsymbol{x})) \mapsto (\boldsymbol{A}(\boldsymbol{x} - \mathbf{a}), \Phi(\boldsymbol{x} - \mathbf{a})). \tag{4.2}$$

In terms of the coordinates X_i the generators of their action on M_k are simply $T_i = \partial/\partial X_i$. We think of the $SO(3)$ action as a left action, so acting first with $\mathcal{O}_1 \in SO(3)$ and then with $\mathcal{O}_2 \in SO(3)$ is the same as acting with $\mathcal{O}_2\mathcal{O}_1$. There is no simple expression for the action of rotations on the actual fields because every rotation needs

[2] M_k^0 is denoted $\tilde{M}_k^0$ in [6]

to be accompanied by a gauge transformation to preserve the asymptotic conditions. Nevertheless one can write down a formula for the generators L_i of the $SO(3)$ action on M_1 and M_2 and we will do so further below. Another isometry, a circle action on the S^1 part of M_k has its physical origin in the unbroken $U(1)$ subgroup of the full isospin group. We call the generating vector field U. Finally there is a discrete symmetry, the combined charge-conjugation and parity transformation CP introduced in section 2. After identifying fields related by an $SU(2)$ gauge transformation this defines a map

$$CP : M_k \mapsto M_k$$

Of course all these symmetries can be combined, so we introduce the direct product

$$G = E_3 \times U(1) \times CP. \tag{4.3}$$

As an illustration consider the action of G on M_1. The generators of translations are given by the general formula above and for the other generators we find

$$\begin{aligned} L_i &= \epsilon_{ijk} X_j \frac{\partial}{\partial X_k} \\ U &= \frac{\partial}{\partial \chi}. \end{aligned} \tag{4.4}$$

The CP transformation is the four dimensional reflection

$$CP : (\boldsymbol{X}, \chi) \mapsto (-\boldsymbol{X}, -\chi) \tag{4.5}$$

To prepare for the following sections we need to describe M_2 in detail. The manifold M_2^0 is called the Atiyah-Hitchin (AH) manifold and is the interesting part of M_2: it models the relative motion of two monopoles. The generic orbit of the $SO(3)$ action on M_2^0 is 3-dimensional and moreover $SO(3)$ acts on M_2^0 by isometries. Thus we coordinatise M_2^0 by Euler angles and a radial coordinate r, and write the metric in terms of the left-invariant 1-forms η_i, $i = 1, 2, 3$ on $SO(3)$, which satisfy $d\eta_i = \epsilon_{ijk}\eta_j \wedge \eta_k$ (explicit formulae for the 1-forms η_i can be found in [5] where they are denoted by σ_i),

$$ds^2 = f(r)^2 dr^2 + a(r)^2 \eta_1^2 + b(r)^2 \eta_2^2 + c(r)^2 \eta_3^2 \,. \tag{4.6}$$

In four dimensions a metric is hyperkähler if and only if its Riemann tensor is anti-self-dual, and for a rotationally invariant metric of the above form anti-self-duality implies

$$\frac{2bc}{f}\frac{da}{dr} = (b-c)^2 - a^2, \qquad\qquad +cycl. \tag{4.7}$$

where $+cycl.$ means we add the two further equations obtained by cyclic permutation of a, b, c. It is also known that the metric is complete and finite and this condition together with the fact that the generic $SO(3)$ orbit is three dimensional selects an essentially unique solution of (4.7) which is plotted in [5]. There the function $f(r)$, which defines the radial coordinate r, is taken to be $f = -b/r$, and we follow that convention. With this choice the range of r is $[\pi, \infty)$.

We are particularly interested in two regions of the manifold

$$\frac{S^1 \times M_2^0}{\mathbb{Z}_2} : \tag{4.8}$$

the asymptotic region, which describes well-separated monopoles, and the interior region which models coincident or near-coincident monopoles.

For large r, the functions a, b and c tend to simple asymptotic expressions

$$a \sim b \sim r\sqrt{1 - \frac{2}{r}} \qquad \text{and} \quad c \sim -\frac{2}{\sqrt{1 - 2/r}}. \tag{4.9}$$

Replacing a, b and c by their asymptotic expressions in (4.6) one obtains the euclidean Taub-NUT (TN) metric. It is also anti-self-dual, but singular at $r = 2$. It is shown in [19] that the TN space can be interpreted as a model for the relative motion of two point-like monopoles with suitable magnetic, scalar and electric charges. Note that, since $a = b$, the TN metric has an extra $SO(2)$ symmetry. In the asymptotic region of the AH manifold, where the TN metric is a good approximation, it is convenient to introduce a set of Euler angles (ϕ, θ, ψ) , $0 \leq \theta < \pi$, $0 \leq \phi < 2\pi$, $0 \leq \psi < 2\pi$ such that (θ, ϕ) gives the direction in space of the line joining the monopoles and ψ is the rotation angle about this line. We also coordinatise the S^1 factor by an angle $0 \leq \chi < 2\pi$.

For $r = \pi$ the monopoles become coincident and the field configuration is axially symmetric. This is reflected in the metric by the fact that $b(\pi) = -c(\pi) = \pi$. Moreover $a(\pi) = 0$ and as a result the $SO(3)$ orbit in M_2^0 collapses to a 2-sphere called a bolt by gravity theorists. Following [5] we introduce new Euler angles $(\tilde{\phi}, \tilde{\theta}, \tilde{\psi})$ such that $(\tilde{\theta}, \tilde{\phi})$ are polar coordinates for the bolt and $\tilde{\psi}$, which is only well-defined away from the bolt, is the rotation angle around the direction given by $(\tilde{\theta}, \tilde{\phi})$.

For $r > \pi$ the orbit of the $SO(3)$ action on $M_2^0/\mathbb{Z}_2$ is $SO(3)/\mathrm{V}$, where V is the viergruppe of diagonal $SO(3)$ matrices. This reduces the actual range of the Euler angles just introduced, but it is often useful to retain the ranges as given above and think of the elements of V as discrete symmetries. Acting on M_2 some of the elements of V should be combined with a transformation on χ. We denote the non-trivial elements of this viergruppe by I_1,I_2 and I_3 and only specify and interpret I_1 and I_3, because $I_2 = I_3 I_1$. In terms of the Euler angles appropriate to the asymptotic region they are the maps

$$\begin{array}{llllll} I_1: & \theta \mapsto \pi - \theta, & \phi \mapsto \phi + \pi, & \psi \mapsto -\psi, & \chi \mapsto \chi, \\ I_3: & \theta \mapsto \theta, & \phi \mapsto \phi, & \psi \mapsto \psi + \pi, & \chi \mapsto \chi + \pi. \end{array} \tag{4.10}$$

For large r , the mapping $\theta \mapsto \pi - \theta, \quad \phi \mapsto \phi + \pi$ exchanges the positions of the monopoles and $\psi \mapsto -\psi$ corresponds, as we shall see, to a reversal of their relative electric charge. Thus at least in the asymptotic region, where one can treat the monopoles as distinct, particle-like objects, the interpretation of the symmetry I_1 is that monopoles of equal electric charge cannot be distinguished, even classically. The symmetry I_3 appears explicitly as division by $\mathbb{Z}_2$ in (4.8). It is best to interpret it in the set of Euler angles appropriate to the bolt. Here we find

$$\begin{array}{llllll} I_1: & \tilde{\theta} \mapsto \tilde{\theta}, & \tilde{\phi} \mapsto \tilde{\phi}, & \tilde{\psi} \mapsto \tilde{\psi} + \pi, & \chi \mapsto \chi, \\ I_3: & \tilde{\theta} \mapsto \pi - \tilde{\theta}, & \tilde{\phi} \mapsto \tilde{\phi} + \pi, & \tilde{\psi} \mapsto \pi - \tilde{\psi}, & \chi \mapsto \chi + \pi, \end{array} \tag{4.11}$$

which shows that, as a result of the symmetry I_3, the bolt in $M_2^0/\mathbb{Z}_2$ is really the projective plane $\mathbb{R}P^2$.

Finally we specify the action of G on M_2. The generators of translations and the $U(1)$ transformations are as for M_1, but the generators of the $SO(3)$ action are

$$L_i = \epsilon_{ijk} X_j \frac{\partial}{\partial X_k} + \xi_i^L. \tag{4.12}$$

Here the ξ_i^L are the generators of the left-action of $SO(3)$ on itself. We use the superscript L to distinguish them from the left-invariant vector fields ξ_i which are dual to the one forms η_i, but which generate right-actions. The ξ_i will be used in the quantum theory later. The action of the CP transformation is reflection of the centre-of-mass coordinates

$$CP : (\boldsymbol{X}, \chi) \mapsto (-\boldsymbol{X}, -\chi). \tag{4.13}$$

CP acts trivially on M_2^0 because, in the centre-of-mass frame, it corresponds to exchanging the monopoles' positions and reversing the relative phase ψ. Points in M_2^0 related by such a transformation are already identified by I_1.

5 Fibre Bundles over the Two-Monopole Moduli Space

Ind_2 is a real rank 2 vector bundle over M_2 with structure group $O(2)$. By the same argument that we used for Ind_1 in section 3 we find that the holonomy around any closed loop lying entirely in the $\mathbb{R}^3$ part is $\mathbf{1}_2 \in O(2)$ and the holonomy around the S^1 factor is $-\mathbf{1}_2 \in O(2)$. Hence the bundle is trivial over $\mathbb{R}^3$ and we can concentrate on the vector bundle over $(S^1 \times M_2^0)/\mathbb{Z}_2$, which we also call Ind_2. Since the base space of this bundle is not simply connected, the bundle cannot, in general, be oriented, i.e. it is not always possible to reduce the structure group to $SO(2)$. To see why Ind_2 is in fact not orientable we exhibit a holonomy involving a reflection by operating on well-separated monopoles. Namely, we rotate the phase of one of the monopoles by 2π and leave the other unchanged. This transformation is a closed path in M_2 which connects two points related by the identification map I_3. It follows from our discussion of Ind_1 that the holonomy for such a path is

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.1}$$

We can get an orientable bundle by pulling Ind_2 back to a bundle over $S^1 \times M_2^0$. Because of the S^1 factor it is still not obvious that the pulled-back bundle can be oriented, but in our case this is possible because we have already seen that the holonomy around S^1 does not involve reflections. Thus we can pick an orientation (we will make this choice later) and call the resulting oriented vector bundle $\widetilde{\mathrm{Ind}}_2$. Identifying $\mathbb{R}^2$ with $\mathbb{C}$ and $SO(2)$ with $U(1)$, $\widetilde{\mathrm{Ind}}_2$ can be viewed as a complex line bundle over $S^1 \times M_2^0$. More specifically we may think of it as a family of complex line bundles over M_2^0 parametrised by $\chi \in S^1$, such that the bundles at $\chi = 0$ and $\chi = 2\pi$ are identified via the bundle map that acts on fibres by multiplication with -1. Ultimately we want to study sections of Ind_2, but our strategy will be to investigate the structure of $\widetilde{\mathrm{Ind}}_2$ in detail and then obtain sections of Ind_2 by imposing an equivariance condition on sections of $\widetilde{\mathrm{Ind}}_2$.

A gauge potential of the connection on $\widetilde{\mathrm{Ind}}_2$ can be written $\omega = -iw$, where w is a real 1-form on $\tilde{M}_2$. From section 3 we have, at a point $[c] \in M_2$, the expression

$$w = \langle \rho_1, d\rho_2 \rangle \tag{5.2}$$

where $\{\rho_1, \rho_2\}$ is a basis of $V_{(\boldsymbol{A},\Phi)}$ and $(\boldsymbol{A}, \Phi)$ is a representative of $[c]$. The symmetry group G (4.3) also acts naturally on the isospinor-spinors ρ_l, $l = 1, 2$. Thus, writing generically g for the action of an element of G on ρ_l, $\boldsymbol{A}$ and Φ, it follows that $\{g\rho_1, g\rho_2\}$ is a basis of $V_{(g\boldsymbol{A},g\Phi)}$. This implies that w, though not necessarily invariant under the action of G, changes only by an $O(2)$ gauge transformation. The curvature 2-form

$$\Omega = dw \tag{5.3}$$

is gauge invariant and therefore truly invariant under the symmetry group G. We will now show that this invariance and the anti-self-duality requirement determines it uniquely. First we exploit the anti-self-duality condition. It is shown in [6] that the decomposition

$$\tilde{M}_2 = (\mathbb{R}^3 \times S^1) \times M_2^0 \tag{5.4}$$

is a decomposition into a product of hyperkähler manifolds. Hence there is an action of the complex structures I, J and K introduced in section 3 on both the cotangent bundle of $(\mathbb{R}^3 \times S^1)$ and on the cotangent bundle of M_2^0. This action can be represented by left-multiplication of a suitably chosen quaternionic 1-form by the quaternions e_1, e_2 and e_3. For $(\mathbb{R}^3 \times S^1)$ this quaternionic 1-form is

$$dx = e_1 dX_1 + e_2 dX_2 + e_3 dX_3 + e_4 d\chi \tag{5.5}$$

and on M_2^0 we write it as

$$dq = e_1 dq_1 + e_2 dq_2 + e_3 dq_3 + e_4 dq_4. \tag{5.6}$$

The 1-forms dq_μ are not known explicitly (see, however, [20]) but for our purposes their existence is sufficient. The real 2-forms appearing as coefficients of e_μ in the quaternionic 2-forms

$$d\bar{x} \wedge dx \qquad d\bar{x} \wedge dq \qquad d\bar{q} \wedge dx \quad \text{and} \quad d\bar{q} \wedge dq, \tag{5.7}$$

form a basis of the 2-forms of type $(1,1)$ with respect to all three complex structures, so we can expand Ω in this basis. Since the holonomy around any loop lying entirely in the $\mathbb{R}^3$ part of M_2 is trivial, the real 2-froms in $d\bar{x} \wedge dx$ cannot occur in this expansion. Further, it follows from the invariance of Ω under G that the coefficient functions in the expansion do not depend on $\boldsymbol{X}$ and χ. Since the discrete symmetry CP changes the sign of dx but acts trivially on dq we conclude that Ω does not contain the real 2-forms in $d\bar{x} \wedge dq$ or $d\bar{q} \wedge dx$. Hence Ω can be constructed out of the real 2-forms in $d\bar{q} \wedge dq$ and lives entirely on M_2^0. M_2^0 is 4-dimensional, so all 2-forms of type (1,1) are anti-self-dual in the standard sense. In terms of the tetrad $(f\,dr, a\eta_1, b\eta_2, c\eta_3)$ the three basic anti-self-dual 2-forms are

$$\begin{aligned}
&f\,dr \wedge a\eta_1 - b\eta_2 \wedge c\eta_3 \\
&f\,dr \wedge b\eta_2 - c\eta_3 \wedge a\eta_1 \\
&f\,dr \wedge c\eta_3 - a\eta_1 \wedge b\eta_2.
\end{aligned} \tag{5.8}$$

Of these only the first respects the identification I_1 (which changes the sign of η_2 and η_3 but not of η_1). Hence the most general anti-self-dual 2-form on M_2^0 which is also invariant under the $SO(3)$ action is

$$\Omega = A(r)(f\,dr \wedge a\eta_1 - b\eta_2 \wedge c\eta_3). \tag{5.9}$$

To determine the function A we make use of the fact that Ω is closed. Thus we have reduced the problem of finding the curvature of the natural connection on $\widetilde{\mathrm{Ind}_2}$ to the ordinary differential equation

$$\frac{d}{dr}(Abc) = -Aaf. \tag{5.10}$$

Using the equations for a, b and c, this is equivalent to

$$\frac{1}{A}\frac{dA}{dr} = -f\left(\frac{2a}{bc} - \frac{1}{b} - \frac{1}{c}\right). \tag{5.11}$$

This equation can easily be solved numerically, but the most important features of the function A can be read off immediately. The logarithmic derivative of A vanishes at the bolt and is negative for $r > \pi$. For large r, A falls off like $e^{-r/2}$.

It is convenient to introduce a new name for the combination Abc which occurs in Ω. We write $\alpha(r) = -Abc(r)$ so that

$$\Omega = d\alpha \wedge \eta_1 + \alpha\eta_2 \wedge \eta_3 \tag{5.12}$$

We still have to determine the normalisation of α. The most direct way to calculate $\alpha(\pi)$ is to parallelly transport a basis $\{\rho_1, \rho_2\}$ of zero-modes around a suitable path in M_2^0. The path should correspond to exchanging two well-separated monopoles in physical space in such a way that they remain well-separated at every point. If the internal phases of the monopoles are suitably adjusted this path is closed in the moduli space. An example of such a path is a loop on the AH cone, a 2-dimensional geodesic submanifold of M_2^0 described in [6], which winds round the cone "at infinity". We can work out the holonomy around this loop directly as follows. Consider a 2-monopole in the asymptotic region of M_2 which is approximately a superposition of two single monopoles. Suppose that initially one monopole has the position $\boldsymbol{Y}$ and the internal phase χ_1 and the other position $-\boldsymbol{Y}$ and phase χ_2 and choose a basis $\{\rho_1, \rho_2\}$ of zero-modes such that ρ_1 is bound to the monopole at $\boldsymbol{Y}$ and ρ_2 to the one at $-\boldsymbol{Y}$. So, in terms of the zero-modes of a single monopole $\rho_1(\boldsymbol{x}) \approx g(\chi_1)\rho(\boldsymbol{x} - \boldsymbol{Y})$ and $\rho_2(\boldsymbol{x}) \approx g(\chi_2)\rho(\boldsymbol{x} + \boldsymbol{Y})$. Since the connection is exponentially small away from the bolt each fermion will follow the monopole to which it was bound initially, and after the monopoles are exchanged (which involves an exchange of phase as well as position), $\rho_1(\boldsymbol{x}) \approx \pm g(\chi_2)\rho(\boldsymbol{x}+\boldsymbol{Y})$ and $\rho_2(\boldsymbol{x}) \approx \pm g(\chi_1)\rho(\boldsymbol{x} - \boldsymbol{Y})$. The exact determination of the sign is tricky because each fermion is also influenced by the long-range magnetic field of the monopole to which it isn't bound and even asymptotically this effect cannot be neglected. But it is sufficient for us to know that the holonomy is of the form

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}. \tag{5.13}$$

We compare this with the holonomy calculated by integrating the curvature over the surface spanned by the loop

$$\exp(i\int_{\text{AH cone}}\Omega) = \exp(i\int_{\text{AH cone}} d\alpha\wedge\eta_1) = e^{-i\pi\alpha(\pi)} \tag{5.14}$$

which corresponds to the real 2×2 matrix

$$\begin{pmatrix} \cos\alpha(\pi)\pi & \sin\alpha(\pi)\pi \\ -\sin\alpha(\pi)\pi & \cos\alpha(\pi)\pi \end{pmatrix}.$$

Comparing with (5.13) we see that $\alpha(\pi)\in\mathbb{Z}+1/2$ The explicit construction of the holonomy showed that the basis of zero-modes gets permuted only once, so we conclude $\alpha(\pi)=\pm 1/2$. The normalisation of α is related to the first Chern number of the complex line bundle $\widetilde{\text{Ind}}_2$, which we can calculate by integrating $\Omega/2\pi$ over the (double-covered) bolt

$$c_1 = -\frac{1}{2\pi}\int_{\widetilde{\text{bolt}}}\Omega = -\frac{\alpha(\pi)}{2\pi}\int_{\widetilde{\text{bolt}}}\eta_2\wedge\eta_3 = -2\alpha(\pi) \tag{5.15}$$

Thus $c_1=\mp 1$. In fact the two possible choices of orientation for $\widetilde{\text{Ind}}_2$ lead to opposite first Chern classes, and since the orientation is not determined by the structure of Ind_2, we are only interested in c_1 up to sign. Everything we say about Ind_2 will be independent of the choice of sign and we can fix $c_1=-1$ and $\alpha(\pi)=1/2$.

The 2-form Ω is closed and, since it is anti-self-dual, also co-closed. Hence it is harmonic and can be interpreted as a solution of the free Maxwell equations on the AH manifold. Since M_2^0 is homotopic to S^2 the second homology group is $H_2(M_2^0,\mathbb{Z})=\mathbb{Z}$ and the double covered bolt is the generator of this homology group. Our calculation of c_1 shows that the "magnetic flux" through the double covered bolt is non-vanishing. Thus Ω is not exact and can be thought of as a "Dirac monopole in configuration space". This motivated Gibbons and Ruback to consider Ω in [21] and to conjecture that it might arise as the curvature of some connection. However, they overlooked the subtleties concerning the identification I_3 and claimed incorrectly, as we shall see, that the connection leads to fermionic quantum scattering if its first Chern number is ± 1.

The connection on $\widetilde{\text{Ind}}_2$ is a $U(1)$ connection and it is convenient to construct local $U(1)$ gauge potentials, for later use. For a fixed value of $\chi\in S^1$ the bundle $\widetilde{\text{Ind}}_2$ is homotopy equivalent to the Hopf line bundle L over S^2. Recall that the Hopf line bundle can be defined as the complex line bundle associated to the principal $U(1)$ bundle

$$\begin{tikzcd} U(1) \arrow[r] & S^3 \arrow[d, "\pi^H"'] \\ & S^2 \end{tikzcd} \tag{5.16}$$

This is the celebrated Hopf fibration of S^3 by circles. We can exploit the diffeomorphism $S^3 \cong SU(2)$ to specify the injection $U(1) \to S^3$ in terms of the Pauli matrices τ_i

$$e^{i\alpha} \to e^{i\alpha\tau_3} \tag{5.17}$$

and also write down the projection map π^H:

$$\pi^H : U \in SU(2) \to \hat{x} \in S^2 \quad \text{where} \quad U\tau_3 U^\dagger = \hat{x} \cdot \boldsymbol{\tau}. \tag{5.18}$$

The Hopf bundle is familiar in the context of Dirac monopoles, and we can exhibit the relationship between Ω and the magnetic field of a Dirac monopole by writing the restriction of Ω to the double covered bolt in terms of the Euler angles $\tilde{\phi}, \tilde{\theta}, \tilde{\psi}$:

$$\Omega_{r=\pi} = \frac{1}{2}\eta_2 \wedge \eta_3 = \frac{1}{2}\sin\tilde{\theta}\, d\tilde{\theta} \wedge d\tilde{\phi}. \tag{5.19}$$

Away from the bolt the radial dependence is, of course, different from the magnetic field of a Dirac monopole. Moreover, Ω is actually exact for $r > \pi$:

$$\Omega = d(\alpha\eta_1). \tag{5.20}$$

This formula does not apply at $r = \pi$ because $\eta_1 = \cos\tilde{\theta}\, d\tilde{\phi} + d\tilde{\psi}$: on the bolt $\tilde{\psi}$, and hence η_1, is ill-defined. We can, however, use $\alpha\eta_1$ as a gauge potential for $r > \pi$. We refer to it as the singular gauge and write

$$w^s = \alpha\eta_1. \tag{5.21}$$

When we apply the gauge transformations $e^{-\frac{i}{2}(\tilde{\psi}\pm\tilde{\phi})}$, which are defined everywhere except at $\theta = \pi$ (upper sign) or $\theta = 0$ (lower sign), we get the gauge potentials

$$w^\pm = (\alpha\cos\tilde{\theta} \mp \frac{1}{2})d\tilde{\phi} + (\alpha - \frac{1}{2})d\tilde{\psi}. \tag{5.22}$$

On the bolt they reduce to

$$w^{\pm}_{r=\pi} = \frac{1}{2}(\cos\tilde{\theta} \mp 1)d\tilde{\phi} \tag{5.23}$$

which are commonly used in the discussion of the Dirac monopole. Since the gauge potentials (5.23) are well-defined on the northern hemisphere or southern hemisphere of the double covered bolt for the upper and lower sign respectively, the gauge potentials w^+ and w^- are well-defined on the patches U^+ and U^- of the following trivialising cover of the bundle $\widetilde{\mathrm{Ind}}_2$ over $S^1 \times M_2^0$

$$\begin{aligned} U^+ &= (S^1 - \{\chi = \pi\}) \times (M_2^0 - \{\theta = \pi\}) \\ U^- &= (S^1 - \{\chi = 0\}) \times (M_2^0 - \{\theta = 0\}). \end{aligned} \tag{5.24}$$

Note that the overlap $U^+ \cap U^-$ has two disjoint components. We denote the component where $0 < \chi < \pi$ by U^R and the component where $\pi < \chi < 2\pi$ by U^L. Local functions

$$\Psi^{\pm} : U^{\pm} \to \mathbb{C} \tag{5.25}$$

can be patched together to a section of $\widetilde{\mathrm{Ind}}_2$ if they are related by the transition functions

$$\begin{aligned} \Psi^+(\chi, r, \tilde{\phi}, \tilde{\theta}, \tilde{\psi}) &= e^{-i\tilde{\phi}}\Psi^-(\chi, r, \tilde{\phi}, \tilde{\theta}, \tilde{\psi}) \quad \text{on} \quad U^R \\ \Psi^+(\chi, r, \tilde{\phi}, \tilde{\theta}, \tilde{\psi}) &= -e^{-i\tilde{\phi}}\Psi^-(\chi, r, \tilde{\phi}, \tilde{\theta}, \tilde{\psi}) \quad \text{on} \quad U^L. \end{aligned} \tag{5.26}$$

Since there is no gauge potential on the S^1 factor the different transition functions on the two components are necessary to encode the holonomy of -1 around this circle. In the following we will refer to this trivialisation and the corresponding gauge potentials $w^{\pm}$ as the regular gauge.

We still have to characterise the bundle Ind_2 and describe its relationship to the bundle $\widetilde{\mathrm{Ind}}_2$. This is done in appendix A. In particular it is explained there how one can obtain sections of Ind_2 from sections of $\widetilde{\mathrm{Ind}}_2$ by imposing an equivariance condition under the map I_3. In practical applications we will represent sections in terms of the local functions Ψ^+ and Ψ^- associated to the trivialisation $\{U^+, U^-\}$. The equivariance condition for these is a special case of the condition (A.19) given in appendix A:

$$*\Psi^{\pm} \circ I_3 = \Psi^{\mp} \tag{5.27}$$

where $*$ denotes complex conjugation. Note that $I_3 U^+ = U^-$, so this equation is well-defined. On the component U^R it is equivalent to

$$*\Psi^{\pm}(\chi + \pi, r, \tilde{\phi} + \pi, \pi - \tilde{\theta}, \pi - \tilde{\psi}) = e^{\pm i\tilde{\phi}}\Psi^{\pm}(\chi, r, \tilde{\phi}, \tilde{\theta}, \tilde{\psi}). \tag{5.28}$$

Away from the bolt we can go to the singular gauge via

$$\begin{aligned}\Psi^s(\chi,r,\tilde\phi,\tilde\theta,\tilde\psi) &= e^{\frac{i}{2}(\tilde\psi\pm\tilde\phi)}\Psi^\pm(\chi,r,\tilde\phi,\tilde\theta,\tilde\psi) \quad \text{on} \quad U^R\\ \Psi^s(\chi,r,\tilde\phi,\tilde\theta,\tilde\psi) &= -e^{\frac{i}{2}(\tilde\psi\pm\tilde\phi)}\Psi^\pm(\chi,r,\tilde\phi,\tilde\theta,\tilde\psi) \quad \text{on} \quad U^L.\end{aligned} \tag{5.29}$$

If the $\Psi^\pm$ obey the equivariance condition (5.27) it follows that

$$\Psi^s \circ I_3 \;=\; *\Psi^s. \tag{5.30}$$

Conversely, if Ψ^s satisfies this condition, $\Psi^\pm$ automatically satisfy (5.27). Thus (5.30) is the most concise and useful way of characterising those section of $\widetilde{\mathrm{Ind}}_2$ which are also sections of Ind_2.

One can construct new vector bundles from Ind_2 by various natural operations. From the point of view of physics we are particularly interested in the tensor product of Ind_2 with itself. $\mathrm{Ind}_2\otimes\mathrm{Ind}_2$ is the vector bundle over M_2 with fibres $\mathbb{R}^2\otimes_{\mathbb{R}}\mathbb{R}^2$ and transition functions $t_{rs}\otimes t_{rs}$ if t_{rs} are the transition functions of Ind_2. If, as we will argue in the next section, sections of Ind_2 are possible quantum mechanical states for two monopoles and one fermion then the possible states for two fermions interacting with two monopoles should, by the spin-statistics theorem, be sections of the antisymmetrised tensor product $\mathrm{Ind}_2\wedge\mathrm{Ind}_2$. Now $\mathbb{R}^2\wedge\mathbb{R}^2=\mathbb{R}$ and elements $\mathcal{O}\otimes\mathcal{O}\in O(2)\otimes O(2)$ act on this space via multiplication by $\det\mathcal{O}$. Thus $\mathrm{Ind}_2\wedge\mathrm{Ind}_2$ is a non-trivial real line bundle over M_2 whose sections we can characterise equivariantly as functions

$$\Psi : \tilde M_2 \to \mathbb{R} \tag{5.31}$$

obeying

$$\Psi\circ I_3 = -\Psi. \tag{5.32}$$

6 Adiabatic Quantisation of Monopoles Coupled to One Fermion

In [5] it was argued that one can approximate the quantum theory of monopoles by doing quantum mechanics on the moduli space: the quantum mechanical wavefunction is assumed to be a complex valued function on the moduli space and the Hamiltonian is taken to be proportional to the Laplace-Beltrami operator on M_k, equipped with its natural metric. Details of the quantum theory for $k=1$ and $k=2$ were worked out in [5],[22] and [23].

From a geometric point of view it is natural to generalise this quantisation scheme to the present situation by requiring the wavefunction to be a section of the index bundle. The

Hamiltonian and all other observables should be covariant with respect to the choice of coordinates on M_k and the choice of gauge in Ind_k, so that the Laplace-Beltrami operator minimally coupled to the $O(k)$ gauge field is a natural candidate for the Hamiltonian. This is in fact the quantisation scheme that we will adopt, but to exhibit the physical approximations underlying it we will "derive" it from three basic assumptions which are modelled on the Born-Oppenheimer approximation in molecular physics. Throughout this section we will restrict attention to $k = 1$ or $k = 2$ and consider only stationary states.

1. At a point $[c] \in M_k$ the wavefunction is a linear combination of isospinor-spinors. The coefficients vary over the moduli space and are, in general, complex. More precisely, recalling that the domain of the Dirac operator is $L^2(\mathbb{R}^3, \mathbb{H} + \mathbb{H})$, we assume that in every open contractable subset U of a covering of M_k the wavefunction is an element of $L^2(U, \mathbb{C}) \otimes L^2(\mathbb{R}^3, \mathbb{H}+\mathbb{H})$. We introduce coordinates ζ_α, $\alpha = 1, ..., 4k$ for U and write $\bar{\nabla}$ for the covariant derivative associated with the Levi-Civita connection of M_k. Denoting the identity maps on $L^2(U, \mathbb{C})$ and $L^2(\mathbb{R}^3, \mathbb{H} + \mathbb{H})$ by id we take the Hamiltonian to be

$$H = -\frac{\hbar^2}{8\pi}\bar{\nabla}_\alpha \bar{\nabla}^\alpha \otimes \mathrm{id} + \mathrm{id} \otimes \hbar \begin{pmatrix} 0 & \not{D}^\dagger \\ \not{D} & 0 \end{pmatrix}. \tag{6.1}$$

For $k = 1$ the operator $-\frac{\hbar^2}{8\pi}\bar{\nabla}_\alpha \bar{\nabla}^\alpha$ can be written more explicitly in terms of the coordinates $(\boldsymbol{X}, \chi)$ on M_1 as

$$-\frac{\hbar^2}{8\pi}(\frac{\partial^2}{\partial \boldsymbol{X}^2} + \frac{\partial^2}{\partial \chi^2}), \tag{6.2}$$

and for $k = 2$ it can be written as as

$$-\frac{\hbar^2}{16\pi}(\frac{\partial^2}{\partial \boldsymbol{X}^2} + \frac{\partial^2}{\partial \chi^2}) - \frac{\hbar^2}{4\pi}\bar{\nabla}_\mu \bar{\nabla}^\mu \tag{6.3}$$

where $\mu = 1, .., 4$ labels coordinates on the AH manifold M_2^0. $\bar{\nabla}_\mu \bar{\nabla}^\mu$ is the Laplace-Beltrami operator on the AH manifold. We denote it by Δ_{AH} and refer to $-\Delta_{AH}$ as the AH Hamiltonian.

In standard applications of the Born-Oppenheimer approximation in molecular physics one distinguishes between "slow variables", which are usually the coordinates for the atomic nuclei, and "fast variables", which label the electronic degrees of freedom. We will similarly refer to monopole coordinates ζ as the "slow variables" and to the fermionic degrees of freedom as "fast variables". This is justified because the monopole motion can be arbitrarily slow. Then the above Hamiltonian is the sum of the slow and the fast Hamiltonians.

2. We make the "adiabatic ansatz" for the wavefunction: at a point $[c] \in M_k$ the wavefunction is a linear combination of the zero-modes of the Dirac operator coupled to

a representative of $[c]$. On the open set U it can be written

$$\boldsymbol{\Psi}(\zeta, \boldsymbol{x}) = \sum_{n=1}^{k} \Psi_n(\zeta)\epsilon_n(\zeta, \boldsymbol{x}). \tag{6.4}$$

Here

$$\epsilon_n(\zeta, \boldsymbol{x}) = \begin{pmatrix} \rho_n(\zeta, \boldsymbol{x}) \\ 0 \end{pmatrix} \tag{6.5}$$

and the ρ_n are an orthonormal basis of $V_{[c]}$, if $[c]$ is the element of M_k with coordinate ζ. The Dirac operator is unbounded from below, so we have to prevent the zero-modes from decaying into negative energy states by the usual postulate that all negative energy states are filled.

We will study both bound states and scattering by looking at stationary states of H, so we will be concerned with the Schrödinger equation

$$H\boldsymbol{\Psi} = E\boldsymbol{\Psi}. \tag{6.6}$$

One further assumption is needed to bring this equation into the promised form. To exhibit it we multiply (6.6) on the left by $(\rho_m(\zeta, \boldsymbol{x})^\dagger, 0)$ and integrate over $\boldsymbol{x}$. Exploiting the orthonormality of the ρ_m we get

$$\bar{\nabla}^\alpha \bar{\nabla}_\alpha \Psi_m + 2\sum_{n=1}^{k} \bar{\nabla}^\alpha \Psi_n \langle \rho_m, \bar{\nabla}_\alpha \rho_n \rangle + \sum_{n=1}^{k} \Psi_n \langle \rho_m, \bar{\nabla}^\alpha \bar{\nabla}_\alpha \rho_n \rangle = -\frac{4\pi E}{\hbar^2} \Psi_m \tag{6.7}$$

Now, writing ω_α for the components of the $o(k)$-valued gauge potential defined in (3.4), i.e. $\omega_{\alpha mn} = \langle \rho_m, \bar{\nabla}_\alpha \rho_n \rangle$. we find

$$\langle \rho_m, \bar{\nabla}^\alpha \bar{\nabla}_\alpha \rho_n \rangle = \bar{\nabla}^\alpha \omega_{\alpha mn} - \langle \bar{\nabla}^\alpha \rho_m, \bar{\nabla}_\alpha \rho_n \rangle.$$

To interpret the last term, assume that there exists a basis of orthonormal eigenspinors of the Dirac operator $\mathcal{D}$ (2.23). We write this basis in the form $\{\epsilon_1, ..., \epsilon_k, \epsilon_{k+1}, ..., \epsilon_N, ...\}$ consisting of zero-modes $\epsilon_1, ..., \epsilon_k$ and eigenspinors

$$\epsilon_N = \begin{pmatrix} \rho_N \\ \lambda_N \end{pmatrix}$$

with eigenvalues $e_N \neq 0$. Then, using the notation (2.34),

$$\begin{aligned} &-\langle \bar{\nabla}^\alpha \rho_m, \bar{\nabla}_\alpha \rho_n \rangle \\ &= -\sum_{l=1}^{k} \langle\langle \bar{\nabla}^\alpha \epsilon_m, \epsilon_l \rangle\rangle \langle\langle \epsilon_l, \bar{\nabla}_\alpha \epsilon_n \rangle\rangle - \sum_{N=k+1}^{\infty} \langle\langle \bar{\nabla}^\alpha \epsilon_m, \epsilon_N \rangle\rangle \langle\langle \epsilon_N, \bar{\nabla}_\alpha \epsilon_n \rangle\rangle \\ &= \sum_{l=1}^{k} \omega^\alpha_{ml} \omega_{\alpha ln} + \sum_{N=k+1}^{\infty} \langle\langle \epsilon_m, \bar{\nabla}^\alpha \epsilon_N \rangle\rangle \langle\langle \epsilon_N, \bar{\nabla}_\alpha \epsilon_n \rangle\rangle \end{aligned} \tag{6.8}$$

The infinite sum can be rewritten as follows

$$\sum_{N=k+1}^{\infty} \frac{1}{e_N^2} \langle\langle \epsilon_m, (\bar{\nabla}^\alpha \mathcal{D})\epsilon_N \rangle\rangle \langle\langle \epsilon_N, (\bar{\nabla}^\alpha \mathcal{D})\epsilon_n \rangle\rangle \tag{6.9}$$

which shows that the contribution from eigenspinors with large eigenvalues e_N is suppressed. The Born-Oppenheimer approximation consists of neglecting this sum altogether. It is a good approximation if the level spacing of the the slow variables is much smaller than for the fast variables [24]. We will be able to check this for the monopole-fermion system at the end of this section, but for now we simply note it as our third assumption:

3. In terms of the orthonormal basis given above,

$$\sum_{N=k+1}^{\infty} \langle\langle \bar{\nabla}^\alpha \epsilon_m, \epsilon_N \rangle\rangle \langle\langle \epsilon_N, \bar{\nabla}_\alpha \epsilon_m \rangle\rangle \approx 0.$$

Then, arranging the Ψ_m into a column vector Ψ, equation (6.7) is equivalent to

$$-(\bar{\nabla}_\alpha \mathbf{1}_k + \omega_\alpha)^2 \Psi = \frac{4\pi E}{\hbar^2} \Psi \tag{6.10}$$

which is the desired form. Note that from the point of view of adiabatic quantisation ω is the gauge potential of a Berry connection.

Although we assumed initially that Ψ is a complex k-component vector we see now that solutions of (6.10) can be chosen to be real. Such real solutions can then be patched together to form a section of Ind_k. Conversely if we have two real solutions f_1 and f_2 then $\Psi = f_1 + if_2$ can be thought of as a section of the complexified index bundle. We use the simplest example, the index bundle over M_1, to illustrate why it is useful to take the quantum mechanical wavefunction to be complex although from a geometric point of view real sections are more natural.

First recall the situation for the single monopole without fermions. The Hamiltonian is given by (6.2). Eigenfunctions can be taken to be of the form

$$\Psi(\boldsymbol{X}, \chi) = e^{i\boldsymbol{K}\cdot\boldsymbol{X}} e^{is\chi}, \qquad \boldsymbol{K} \in \mathbb{R}^3, \quad s \in \mathbb{Z}. \tag{6.11}$$

Such states have energy

$$E = \frac{\hbar^2}{8\pi}(\boldsymbol{K}^2 + s^2) \tag{6.12}$$

and are eigenstates of the linear momentum operator $\boldsymbol{P}$ and of the electric charge operator Q

$$P_i = -i\hbar \frac{\partial}{\partial X_i} \qquad \text{and} \qquad Q = -i\hbar \frac{\partial}{\partial \chi} \tag{6.13}$$

with eigenvalues $\hbar \boldsymbol{K}$ and $q = \hbar s$ respectively. Note that requiring the wavefunction to be an eigenfunction of $\boldsymbol{P}$ and Q forces it to be complex. States with $s \neq 0$ are dyons with quantised electric charge. Since s is an integer the product of magnetic charge $g(= 4\pi)$ and electric charge satisfies the Schwinger quantisation condition

$$gq = 4\pi\hbar s \qquad s \in \mathbb{Z}. \tag{6.14}$$

The field configurations parametrised by M_1 are all spherically symmetric and therefore the quantum states (6.11) represent spin zero states. Moreover it was shown in [5] that the map I_1 acting on M_2^0 (4.10) forces one to quantise two monopoles or dyons of the same electric charge as identical bosons. Thus we see that, in the moduli space approximation, quantum states of a single monopole without fermions are spinless bosons with electric charge satisfying the Schwinger quantisation condition.

Now we turn to the index bundle over M_1. The Hamiltonian is still (6.2) because the connection on Ind_1 is flat. However, the bundle is not trivial and the wavefunction (6.11) needs to be modified so that both its real and imaginary part are sections of it. We saw in section 3 that the non-trivial part of Ind_1 is a Möbius bundle over the S^1 factor of M_1. It is useful to think of sections of that bundle as periodic functions f on the extended range $[0, 4\pi)$ satisfying

$$f(\chi + 2\pi) = -f(\chi).$$

Sections of Ind_1 which are eigenstates of the Hamiltonian (6.2) are functions like $\cos(\boldsymbol{K} \cdot \boldsymbol{X}) \sin(s\chi)$, where s is now a half-odd integer. The disadvantage of real wavefunctions like this is that they are not eigenstates of linear momentum or electric charge (6.13). To obtain such eigenstates we need to consider sections of the complexified bundle. The required wavefunctions are of the same form as for the monopole without fermions, but now the condition on s is changed:

$$e^{i\boldsymbol{K}\cdot\boldsymbol{X}} e^{is\chi} \quad \boldsymbol{K} \in \mathbb{R}^3 \quad s \in \mathbb{Z} + 1/2 \tag{6.15}$$

Correspondingly the energy eigenvalues are still given by formula (6.12) and the eigenvalues of the electric charge operator are still of the form $q = \hbar s$, but both are now labelled by half-odd integers. The effect of the non-trivial topology of Ind_1 on quantum states representing dyons is thus to replace the Schwinger quantisation condition by the Dirac quantisation condition

$$gq = 4\pi\hbar s, \qquad s \in \mathbb{Z} + \frac{1}{2}. \tag{6.16}$$

For the rest of this paper, quantum states which are eigenstates of the electric charge operator will be called pure monopoles if they have no electric charge, and dyons if they have non-vanishing electric charge (satisfying the Dirac or the Schwinger condition). If we simply say monopoles, this may refer to pure monopoles or dyons.

The quantum states of the monopole-fermion system are still spinless, because the field configurations parametrised by Ind_1 are all spherically symmetric. At first sight this is surprising because we have coupled spin 1/2 fermions to a spinless boson. But the fermions also carry isospin 1/2 and this combines with the spin (recall from section 2 that $\mathbf{s}+\mathbf{t}=0$) to form a composite object of spin 0. The situation is just the reverse of the "spin from isospin" phenomenon mentioned in the introduction: the isospin "cancels" the spin degrees of freedom and the resulting composite object has no spin.

To finish this section we estimate the validity of the Born-Oppenheimer approximation. From (6.12) we find the energy difference between a dyon of charge $\hbar/2$ and a dyon of charge $3\hbar/2$: $\Delta E_{\text{slow}} = \hbar^2/4\pi$. On the other hand we saw in section 2 that the continuum of the Dirac operator starts at $\hbar/2$. There could be discrete energy levels between the zero-modes and the continuum, but they will not accumulate at $E=0$. Thus the level spacing for the "fast coordinates" is of the order $\Delta E_{\text{fast}} \approx \hbar/2$ and we have the ratio

$$\frac{\Delta E_{\text{slow}}}{\Delta E_{\text{fast}}} \approx \frac{\hbar}{2\pi} \tag{6.17}$$

If we identify the smallest allowed electric charge $q_0 = \hbar/2$ with the charge of the electron, the fine structure constant is

$$\frac{q_0^2}{4\pi\hbar} \approx \frac{1}{137} \quad \Leftrightarrow \quad \frac{\hbar}{16\pi} \approx \frac{1}{137}$$

Then the ratio (6.17) is approximately 1/17.

7 The Schrödinger Equation for Two Monopoles Coupled to One Fermion

The description of the geometry of Ind_2 in section 5 together with the quantisation prescription of the previous section leads to the following Hamiltonian for 2-monopole dynamics in the presence of a fermion:

$$-\frac{\hbar^2}{16\pi}\left(\frac{\partial^2}{\partial \boldsymbol{X}^2} + \frac{\partial^2}{\partial \chi^2}\right)\mathbf{1}_2 - \frac{\hbar^2}{4\pi}(\bar{\nabla}_\mu \mathbf{1}_2 + \omega_\mu)^2 \tag{7.1}$$

where $\bar{\nabla}_\mu$ is defined as in (6.3) and ω_μ is a gauge potential of the connection on Ind_2 viewed as a 1-form taking values in $o(2)$. If we choose the basis element

$$\boldsymbol{i}_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

of $o(2)$ we can identify the Lie algebras of $U(1)$ and of $O(2)$ via $i \leftrightarrow \boldsymbol{i}_2$. We separate the centre-of-mass motion by writing the wavefunction in the form

$$\Psi(\chi, \zeta_\mu) e^{i\boldsymbol{K}\cdot\boldsymbol{X}} \tag{7.2}$$

and concentrate on the Schrödinger equation on $(S^1 \times M_2^0)/\mathbb{Z}_2$:

$$\left(-\frac{1}{4}\frac{\partial^2}{\partial \chi^2}\mathbf{1}_2 - (\bar{\nabla}_\mu \mathbf{1}_2 + \omega_\mu)^2 \right) \Psi = \varepsilon \Psi. \tag{7.3}$$

Here ε is related to the total energy E via

$$E = \frac{\hbar^2}{4\pi}\varepsilon + \frac{\hbar^2}{16\pi}\boldsymbol{K}^2. \tag{7.4}$$

Ψ is locally an $\mathbb{R}^2$-valued function on $(S^1 \times M_2^0)/\mathbb{Z}_2$, but globally it should be patched together to a section of Ind_2.

Following the strategy mapped out in section 5 we first study sections of $\widetilde{\mathrm{Ind}_2}$ and then impose equivariance conditions. We also work with the radial coordinate r and the Euler angles (ϕ, θ, ψ). The Hamiltonian can then be expressed in terms of the left-invariant vector fields ξ_i on $SO(3)$ (for explicit formulae see [5]). We write

$$\omega_i = \omega(\xi_i) = -w_i \boldsymbol{i}_2$$

and find that, in a gauge where $\omega_r = \omega(\partial_r) = 0$, the Hamiltonian on $(S^1 \times M_2^0)/\mathbb{Z}_2$ is, after dividing by $\hbar^2/4\pi$,

$$\hat{H} = -\frac{1}{4}\frac{\partial}{\partial \chi}^2 \mathbf{1}_2 - \frac{1}{abcf}\frac{\partial}{\partial r}\frac{abc}{f}\frac{\partial}{\partial r}\mathbf{1}_2 - \frac{(\xi_1 \mathbf{1}_2 - w_1 \boldsymbol{i}_2)^2}{a^2} - \frac{(\xi_2 \mathbf{1}_2 - w_2 \boldsymbol{i}_2)^2}{b^2} - \frac{(\xi_3 \mathbf{1}_2 - w_2 \boldsymbol{i}_2)^2}{c^2} \tag{7.5}$$

For the remainder of this paper we will be concerned with the Schrödinger equation

$$\hat{H}\Psi = \varepsilon \Psi. \tag{7.6}$$

We have already made use of the possibility of combining the real functions Ψ_1 and Ψ_2 into one complex function $\Psi_1 + i\Psi_2$ in section 5. We will do so again for the purpose of studying the Hamiltonian $\hat{H}$ near the bolt, hopefully without creating confusion with the complexification of Ind_2, where Ψ_1 and Ψ_2 both become complex. For small $h := r - \pi$ we can use the approximations $a \approx 2h$ and $b \approx -c \approx \pi$. Then $\hat{H}$ becomes, on the open set U^+ with the gauge potential ω^+,

$$-\tfrac{1}{4}\tfrac{\partial}{\partial \chi}^2 - \tfrac{1}{h}\tfrac{\partial}{\partial h}h\tfrac{\partial}{\partial h} - \tfrac{1}{4h^2}\tfrac{\partial^2}{\partial \tilde{\psi}^2} -$$
$$\tfrac{1}{\pi^2}\left[\tfrac{1}{\sin\tilde{\theta}}\tfrac{\partial}{\partial \tilde{\theta}}\sin\tilde{\theta}\tfrac{\partial}{\partial \tilde{\theta}} + \tfrac{1}{\sin^2\tilde{\theta}}\left(\cos^2\tilde{\theta}\tfrac{\partial^2}{\partial \tilde{\psi}^2} + (\tfrac{\partial}{\partial \tilde{\phi}} - \tfrac{i}{2}(\cos\tilde{\theta} - 1))^2 - 2\cos\tilde{\theta}\tfrac{\partial}{\partial \tilde{\psi}}(\tfrac{\partial}{\partial \tilde{\phi}} - \tfrac{i}{2}(\cos\tilde{\theta} - 1))\right)\right].$$

The part of the operator in square brackets can be cast into the form

$$\frac{1}{\sin\tilde{\theta}}\frac{\partial}{\partial\tilde{\theta}}\sin\tilde{\theta}\frac{\partial}{\partial\tilde{\theta}}+\frac{1}{\sin^2\tilde{\theta}}\left(\frac{\partial}{\partial\tilde{\phi}}-\frac{\partial}{\partial\tilde{\psi}}-(\frac{\partial}{\partial\tilde{\psi}}+\frac{i}{2})(\cos\tilde{\theta}-1)\right)^2 \tag{7.7}$$

To find eigenfunctions of this operator we make the ansatz

$$e^{i\tilde{s}\tilde{\psi}}e^{i\tilde{m}\tilde{\phi}}\Theta(\tilde{\theta}) \tag{7.8}$$

where $\tilde{m}\in\mathbb{Z}$ and, to satisfy invariance under I_1 in the form (4.11), $\tilde{s}\in 2\mathbb{Z}$. The resulting eigenvalue equation for Θ is

$$\left[\frac{1}{\sin\tilde{\theta}}\frac{d}{d\tilde{\theta}}\sin\tilde{\theta}\frac{d}{d\tilde{\theta}}-\frac{1}{\sin^2\tilde{\theta}}\left((\tilde{m}-\tilde{s})-(\tilde{s}+\frac{1}{2})(\cos\tilde{\theta}-1)\right)^2\right]\Theta=\lambda\Theta \tag{7.9}$$

This is precisely the equation one needs to solve to find eigenfunctions of the Laplacian on S^2 coupled to a Dirac monopole of charge $2\tilde{s}+1$! Its solutions are given by monopole harmonics, which can be constructed from Wigner functions of half-integer spin (we follow the conventions and normalisations of [25])

$$D^j_{sm}(\tilde{\phi},\tilde{\theta},\tilde{\psi})=e^{is\tilde{\psi}}d^j_{sm}(\tilde{\theta})e^{im\tilde{\phi}}.$$

They satisfy

$$\begin{aligned}
-\Delta_{S^3}D^j_{sm} &= j(j+1)D^j_{sm}\\
-i\frac{\partial}{\partial\tilde{\phi}}D^j_{sm} &= mD^j_{sm}\\
-i\frac{\partial}{\partial\tilde{\psi}}D^j_{sm} &= sD^j_{sm}
\end{aligned} \tag{7.10}$$

where

$$\begin{aligned}
\Delta_{S^3} &= \xi_1^2+\xi_2^2+\xi_3^2\\
&= \frac{1}{\sin\tilde{\theta}}\frac{\partial}{\partial\tilde{\theta}}\sin\tilde{\theta}\frac{\partial}{\partial\tilde{\theta}}+\frac{1}{\sin^2\tilde{\theta}}(\frac{\partial^2}{\partial\tilde{\phi}^2}+\frac{\partial^2}{\partial\tilde{\psi}^2}-2\cos\tilde{\theta}\frac{\partial}{\partial\tilde{\psi}}\frac{\partial}{\partial\tilde{\phi}})
\end{aligned} \tag{7.11}$$

is the standard Laplacian on the 3-sphere. The solutions of (7.9) are

$$\Theta(\tilde{\theta})=d^j_{\tilde{s}+\frac{1}{2},\bar{m}}(\tilde{\theta}) \tag{7.12}$$

where $\bar{m}=(\tilde{m}-\tilde{s})+(\tilde{s}+1/2)=\tilde{m}+1/2$ and necessarily $j\in\mathbb{Z}+1/2$, $j\geq\tilde{s}+1/2$ and $j\geq\tilde{m}+1/2$. The eigenvalue λ is then

$$\lambda=j(j+1)-(\tilde{s}+\frac{1}{2})^2$$

Thus we see that our original ansatz (7.8) leads to functions

$$e^{i\tilde{s}\tilde{\psi}} d^j_{\tilde{s}+\frac{1}{2},\tilde{m}+\frac{1}{2}}(\tilde{\theta}) e^{i\tilde{m}\tilde{\phi}}. \tag{7.13}$$

These functions look quite complicated, but when we apply the gauge transformation $e^{i\frac{1}{2}(\tilde{\psi}+\tilde{\phi})}$ which takes the gauge potential ω^+ into the singular gauge potential ω^s they become simply

$$D^j_{\tilde{s}+\frac{1}{2},\tilde{m}+\frac{1}{2}}(\tilde{\phi},\tilde{\theta},\tilde{\psi}). \tag{7.14}$$

It is not surprising that we can separate variables in the singular gauge by expressing the $SO(3)$-dependent part of the wavefunction in terms of Wigner functions: in this gauge the Hamiltonian (7.5) commutes with Δ_{S^3}, because $w_2 = w_3 = 0$ and w_1, the coefficient of η_1 in (5.21), depends on r only. The crucial lesson that we have learnt from studying the Schrödinger equation near the bolt is that the effect of the connection on $\widetilde{\mathrm{Ind}}_2$ is to add 1/2 to the "body-fixed angular momentum" $\tilde{s}$. As a result we need to consider Wigner functions of half-odd integral spin. Our main motivation for using the singular gauge is a practical one: our method for studying quantum scattering in the next section will be to compare scattering states of $\hat{H}$ with scattering states of the TN Hamiltonian, which is minus the Laplace-Beltrami operator on the TN space

$$-\Delta_{TN} = -(1-\frac{2}{r})^{-1}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{\Delta_{S^3}}{r^2} + \xi_3^2(\frac{1}{4}-\frac{1}{r})\right) \tag{7.15}$$

In the singular gauge $\hat{H}$ simplifies, for large r, to

$$-\frac{1}{4}\frac{\partial}{\partial\chi}^2\mathbf{1}_2 - \Delta_{TN}\mathbf{1}_2 \tag{7.16}$$

so that the asymptotic form of scattering states of $\hat{H}$ can easily be compared with the scattering states of $-\Delta_{TN}$.

Thus, to summarise our strategy for solving (7.6), we will look for a $\mathbb{R}^2$-valued function Ψ and work in the singular gauge, but we only allow solutions that are mapped into regular solutions at the bolt by the gauge transformation

$$\begin{pmatrix} \cos(\frac{\tilde{\psi}\pm\tilde{\phi}}{2}) & -\sin(\frac{\tilde{\psi}\pm\tilde{\phi}}{2}) \\ \sin(\frac{\tilde{\psi}\pm\tilde{\phi}}{2}) & \cos(\frac{\tilde{\psi}\pm\tilde{\phi}}{2}) \end{pmatrix}. \tag{7.17}$$

This transformation is not invariant under I_1, and hence the invariance of the solution in the regular gauge $\pm$ implies a non-trivial action of I_1 on solutions in the singular gauge

$$I_1\Psi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\Psi. \tag{7.18}$$

Finally, to get sections of Ind_2, we impose the condition (5.30), which in $O(2)$ notation reads

$$I_3\Psi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi. \tag{7.19}$$

Our characterisation of the invariance properties of the wavefunction is clearly independent of the set of Euler angles we choose as coordinates, and for the scattering theory it is best to use the coordinates (ϕ, θ, ψ), which have a physical interpretation in the asymptotic region. Thus we work with Wigner functions $D^j_{sm}(\phi, \theta, \psi)$ which are simultaneous eigenstates of the operator Δ_{S^3} for the squared total angular momentum, of the operator $-i\partial/\partial\phi$ representing the angular momentum about the space-fixed three-axis and of the operator
$-i\partial/\partial\psi = -i\xi_3$ representing the angular momentum about the body-fixed three-axis. The eigenvalue s is particularly important to us because it represents the relative electric charge of the monopoles. The requirement that j (and hence s and m) be half-odd integers implies the following transformation rules for the Wigner functions under the discrete symmetries

$$\begin{aligned} I_1 D^j_{sm} &= -i^{2j} D^j_{-sm} && (7.20)\\ I_3 D^j_{sm} &= i^{2s} D^j_{sm}. && (7.21) \end{aligned}$$

We combine the Wigner functions with functions of χ of the form $e^{iS\chi}$ to construct the part of the wavefunction which depends on $(\chi, \phi, \theta, \psi)$. I_1 does not affect χ but

$$I_3 e^{iS\chi} = i^{2S} e^{iS\chi} \tag{7.22}$$

so that for $S \in \mathbb{Z} + 1/2$

$$\hat{W}_{jsmS} = \left\{ \begin{array}{lcl} \binom{D^j_{-sm} e^{iS\chi}}{i^{2j} D^j_{sm} e^{iS\chi}} & \text{if} & s + S \in 2\mathbb{Z} + 1 \\ & & \\ \binom{D^j_{sm} e^{iS\chi}}{i^{2j} D^j_{-sm} e^{iS\chi}} & \text{if} & s + S \in 2\mathbb{Z} \end{array} \right\} \tag{7.23}$$

satisfies the conditions (7.18) and (7.19). The distinction between $s + S$ even or odd is necessary so that I_3 is always represented by $\mathrm{diag}(1, -1)$ rather than by $\mathrm{diag}(-1, 1)$. We have constructed a complex doublet, but since the matrices representing I_1 and I_3 are both real, the real and imaginary part of $\hat{W}_{jsmS}$ separately satisfy the symmetry requirements, thus making it a section of the complexified bundle. As in the single monopole case, this is desirable for a physical wavefunction because it can then be an eigenstate of physically interesting first order differential operators, such as $-i\partial/\partial\chi$, the operator for the total electric charge, or $-i\partial/\partial\phi$. Physically, the fact that both s and

S are half-odd integers can be understood as follows: in the asymptotic region, where individual charges are well-defined, simultaneous eigenstates of the individual electric charge operators are states where the fermion is bound to one of the monopoles. The electric charge of that monopole must obey the Dirac condition and the electric charge of the other the Schwinger condition, but this is equivalent to both the relative electric charge s and the total electric charge S being half-odd integers.

A possible ansatz for the full wavefunction is

$$\Psi_{jmS} = \sum_{s=1/2}^{j} u_{js} \hat{W}_{jsmS} \tag{7.24}$$

where the sum is over all half-odd integers in the given range and the u_{js} are real-valued functions of r. Furthermore we can assume without loss of generality that $s > 0$ because it follows from the definition that $\hat{W}_{j-smS} = \hat{W}_{jsmS}$. Inserting this ansatz into (7.6) will lead to $(2j+1)/2$ coupled second order ordinary differential equations for the u_{js}. To find these equations one needs to evaluate the matrix elements of ξ_1, ξ_1^2, ξ_2^2 and ξ_3^2 in the states $\hat{W}_{jsmS}$. This is best done by combining ξ_1 and ξ_2 into ladder operators

$$J_{\pm} = -i(\xi_1 \pm i\xi_2)$$

and using

$$J_{\pm} D^j_{sm} = -\sqrt{j(j+1) - s(s+1)} D^j_{s\pm 1,m}. \tag{7.25}$$

We will solve the equations for $j = 1/2$ and $j = 3/2$ in the next section, so we give them here as an illustration. For $j = 1/2$ only $s = 1/2$ is allowed and the equation for $u_{\frac{1}{2}\frac{1}{2}}$ is

$$\left(\frac{1}{abcf} \frac{d}{dr} \frac{abc}{f} \frac{d}{dr} + \varepsilon - \frac{(\alpha - 1/2)^2}{a^2} - \frac{1}{4b^2} - \frac{1}{4c^2} \right) u_{\frac{1}{2}\frac{1}{2}} = 0. \tag{7.26}$$

When $j = 3/2$, one obtains a coupled set of equations

$$\begin{aligned} &\left(\frac{1}{abcf} \frac{d}{dr} \frac{abc}{f} \frac{d}{dr} + \varepsilon - \frac{(\alpha^2 + 2\alpha + 7/4)}{a^2} - \frac{7}{4b^2} - \frac{1}{4c^2} \right) u_{\frac{3}{2}\frac{1}{2}} - \frac{\sqrt{3}}{2} \left(\frac{(2\alpha+1)}{a^2} - \frac{1}{b^2} \right) u_{\frac{3}{2}\frac{3}{2}} = 0 \\ &\left(\frac{1}{abcf} \frac{d}{dr} \frac{abc}{f} \frac{d}{dr} + \varepsilon - \frac{(\alpha^2 + 3/4)}{a^2} - \frac{3}{4b^2} - \frac{9}{4c^2} \right) u_{\frac{3}{2}\frac{3}{2}} - \frac{\sqrt{3}}{2} \left(\frac{(2\alpha+1)}{a^2} - \frac{1}{b^2} \right) u_{\frac{3}{2}\frac{1}{2}} = 0 \end{aligned}$$
$$. \tag{7.27}$$

For computational purposes, and also to convince ourselves that the equations are regular at the bolt, we perform a linear transformation

$$\begin{aligned} v_{\frac{3}{2},0} &= -\frac{1}{2} u_{\frac{3}{2}\frac{1}{2}} + \frac{\sqrt{3}}{2} u_{\frac{3}{2}\frac{3}{2}} \\ v_{\frac{3}{2},2} &= \frac{\sqrt{3}}{2} u_{\frac{3}{2}\frac{1}{2}} + \frac{1}{2} u_{\frac{3}{2}\frac{3}{2}} \end{aligned} \tag{7.28}$$

which decouples the equations at the bolt:

$$\left(\frac{1}{abcf}\frac{d}{dr}\frac{abc}{f}\frac{d}{dr}+\varepsilon-\frac{(\alpha-1/2)^2}{a^2}-\frac{7}{4b^2}-\frac{7}{4c^2}\right)v_{\frac{3}{2},0}+\frac{\sqrt{3}}{2}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)v_{\frac{3}{2},2}=0$$
$$\left(\frac{1}{abcf}\frac{d}{dr}\frac{abc}{f}\frac{d}{dr}+\varepsilon-\frac{(\alpha+3/2)^2}{a^2}-\frac{3}{4b^2}-\frac{3}{4c^2}\right)v_{\frac{3}{2},2}+\frac{\sqrt{3}}{2}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)v_{\frac{3}{2},0}=0. \quad (7.29)$$

Near the bolt $b^2 \approx c^2$ and $\alpha \approx 1/2$, and two linearly independent solutions of these equations can be given in terms of Bessel functions of integer order

$$\begin{pmatrix} v_{\frac{3}{2},0} \\ v_{\frac{3}{2},2} \end{pmatrix}(\pi+h) \approx \begin{pmatrix} J_0(\sqrt{\varepsilon-\frac{7}{2\pi^2}}h) \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} v_{\frac{3}{2},0} \\ v_{\frac{3}{2},2} \end{pmatrix}(\pi+h) \approx \begin{pmatrix} 0 \\ J_1(\sqrt{\varepsilon-\frac{3}{2\pi^2}}h) \end{pmatrix}$$

For large r, equations (7.27) and indeed all the coupled equations that occur for $j > 1/2$ decouple. This happens because in the asymptotic region the relative electric charge s becomes a good quantum number. In the language of scattering theory s labels channels which are coupled by the full Hamiltonian $\hat{H}$. This is analogous to the scattering theory associated with the AH Hamiltonian, but there only channels with s differing by an even integer can couple to each other. The physical interpretation is that the interaction is mediated by a W-boson carrying one Schwinger unit of electric charge. Here channels whose electric charges differ by any integer multiple of the Schwinger unit can couple to one another. The mathematical reason is that $\hat{H}$ contains a term linear in the rank 1 vector operator ξ_1, but the physical interpretation is that a particle carrying a Dirac unit (i.e. half a Schwinger unit) is exchanged in the scattering process. Furthermore, the radial function multiplying ξ_1 in $\hat{H}$ decays like $e^{-r/2}$ for large r, so that the exchanged particle should have mass $\hbar/2$. We saw in table 1 that this is precisely the charge and mass of the isospinor fermion. This leads us to the interpretation of $\hat{H}$ as a model for monopole interactions via the exchange of isospinor fermions as well as W-bosons.

8 Two Monopole Scattering and Bound States

The study of the scattering and bound states of $\hat{H}$ is particularly instructive when it is done with reference to similar investigations of the TN Hamiltonian and the AH Hamiltonian. The TN Hamiltonian is a model for the quantised interaction of two monopoles which only takes into account the long range forces between monopoles. It was first studied in [5]. In the limit of large monopole separation the TN and the AH Hamiltonian become identical, but the AH Hamiltonian also models the short range forces between monopoles which are not included in the TN approximation. Some bound states of the AH Hamiltonnian were calculated in [22] and the scattering theory was studied in [23]. The techniques developed in those papers will be used in the remaining sections of this paper to study bound and scattering states of $\hat{H}$. While we have tried to state our results clearly we have omitted most of the technical details. Thus a familiarity with the work on TN and the AH Hamiltonian is helpful for filling in intermediate steps in our calculations and also for appreciating comparisons between the AH Hamiltonian and $\hat{H}$.

First consider the bound state problem. From equation (7.16) we know that the TN Hamiltonian is the asymptotic version of $\hat{H}$, so as a first approximation we take the Hamiltonian to be the TN Hamiltonian but require the wavefunction to obey the invariance conditions (7.18) and (7.19). Inserting the ansatz (7.24) into the TN Schrödinger equation

$$-\Delta_{TN}\Psi = \varepsilon\Psi \tag{8.1}$$

yields the following differential equations for the radial functions $u_{js}(r)$, where r lies now in $[0, \infty)$.

$$\left(\frac{1}{r}\frac{d^2}{dr^2}r - \frac{j(j+1)}{r^2} - \frac{2\varepsilon - s^2}{r} + (\varepsilon - \frac{s^2}{4})\right) u_{js} = 0. \tag{8.2}$$

This is formally the same equation as found in [5], but now j and s are half-odd integral and as a result "regular" solutions, which behave at the origin like r^j, are still square integrable, but only finitely often differentiable. For fixed s (recall that $s > 0$) and $\varepsilon < s^2/4$, (8.2) has bound state solutions, discussed in [5], with energies labelled by s and a half-odd integer $n > j$

$$\varepsilon_{sn} = \frac{1}{2}(n^2 - s^2)(n - \sqrt{n^2 - s^2}). \tag{8.3}$$

Since $j \geq s$, we have $n = s+1, s+2....$, and, since ε_{sn} does not depend on j, the degeneracy of the bound state of monopoles of relative electric charge s and label n is

$$\sum_{j=s}^{n-1}(2j+1) = (n - \frac{1}{2})(n + \frac{1}{2}) - (s - \frac{1}{2})(s + \frac{1}{2}).$$

Now turn to the full Schrödinger equation (7.6). In the previous section we wrote down the only uncoupled radial equation that results from the ansatz (7.24). It is characterised by $j = s = 1/2$ and for large r it becomes identical to (8.2) with those values for j and s. In the interior region, however, the equations differ substantially: they are defined on different domains, (7.26) involves the gauge potential and the more complicated functions a, b and c. We have calculated the lowest bound states of (7.26) numerically and in table 2 we compare the bound state energies with the TN estimates.

n	$\varepsilon_{\frac{1}{2},n}$	$\varepsilon^{TN}_{\frac{1}{2},n}$
$\frac{3}{2}$	0.06066	0.06053
$\frac{5}{2}$	0.06186	0.06187

Table 2

Bound state energies of two monopoles in the presence of a fermion. The $\varepsilon_{\frac{1}{2},n}$ are numerically calculated eigenvalues of $\hat{H}$, the $\varepsilon^{TN}_{\frac{1}{2},n}$ are eigenvalues of the TN Hamiltonian obtained from formula (8.3)

The TN estimates are remarkably accurate, echoing a similar surprise in the bound state calculations in [22]. The value of $\varepsilon_{\frac{1}{2}\frac{3}{2}}$ is also interesting because it is the energy of the lowest lying bound state of two monopoles. More precisely, by choosing $S = 1/2$ we may think of it as a bound state of a pure monopole and a dyon of charge 1/2. Note finally that, as a consequence of $s \geq 1/2$ the Hamiltonian has a purely discrete spectrum below $\varepsilon = 1/16$. The continuous spectrum is $[1/16, \infty)$ and presumably none of the bound states of the TN Hamiltonian in this range survive as bound states of $\hat{H}$. The physical basis for this conjecture is that bound states of two monopoles with relative electric charge $s \geq 3/2$ can always exchange half a (Schwinger) unit of electric charge and decay into a bound state with $s = 1/2$. This is in marked contrast to bound states of monopoles without fermions, which are all embedded in the continuum but cannot decay because of discrete symmetries.

To understand the scattering associated with $\hat{H}$ it is essential to understand the scattering associated with the TN Hamiltonian first. Our strategy for studying the scattering theory of $\hat{H}$, already used in [23], is to perform a partial wave analysis of $\hat{H}$ to obtain elements of the S-matrix of $\hat{H}$ relative to the TN Hamiltonian. Thus the TN Hamiltonian plays a role similar to that of the standard Coulomb Hamiltonian in modified Coulomb scattering in atomic and nuclear physics. We describe the scattering associated with H_{TN} in spherical coordinates (r, ϕ, θ, ψ) which are also suitable for a partial wave analysis of $\hat{H}$. For fixed s, scattering takes place at energies above $s^2/4$. Adapting the results of [23] to our situation we can construct scattering solutions by taking suitable linear combinations of the regular solutions u_{js} of (8.2). Introducing parameters k and η defined via $k^2 = \varepsilon - s^2/4$ and $2\eta k = 2\varepsilon - s^2$ we normalise these solutions so that for large r

$$u_{js}(r) \sim \frac{1}{r}\sin(kr - \eta\ln kr - \frac{1}{2}j\pi + \sigma_{js}) \tag{8.4}$$

where

$$\sigma_{js} = \arg\Gamma(j+1+i\eta)$$

We then find that

$$\Psi^{TN}_{sS} := \sum_{j\geq s}(2j+1)i^j e^{i\sigma_{js}} u_{js}\hat{W}_{jssS} \tag{8.5}$$

has the asymptotic form ($z = r\cos\theta$)

$$e^{iS\chi}\left\{\begin{pmatrix} i^{2s}e^{is(\phi-\psi)}e^{i(kz+\eta\ln k(r-z))} \\ e^{is(\phi+\psi)}e^{i(-kz+\eta\ln k(r+z))}\end{pmatrix} + \begin{pmatrix} i^{2s}f^-_s(\phi,\theta,\psi) \\ f^+_s(\phi,\theta,\psi)\end{pmatrix}e^{i(kr-\eta\ln 2kr)}\right\}. \tag{8.6}$$

This is the TN scattering solution we are after: it consists of a plane wave (the logarithmic distortion is due to the long-range Coulomb forces) and a radially outgoing scattered wave with scattering amplitude

$$\begin{pmatrix} i^{2s}f^-_s(\phi,\theta,\psi) \\ f^+_s(\phi,\theta,\psi)\end{pmatrix} = \begin{pmatrix} \sum_{j\geq s}(2j+1)\frac{e^{2i\sigma_{js}}-1}{2ik}D^j_{-ss}(\phi,\theta,\psi) \\ \sum_{j\geq s}(2j+1)i^{2j}\frac{e^{2i\sigma_{js}}-1}{2ik}D^j_{ss}(\phi,\theta,\psi)\end{pmatrix}. \tag{8.7}$$

Closed expressions for the scattering amplitudes f^-_s and f^+_s can be found in [23]. They describe elastic scattering because the relative electric charge of the incoming and the outgoing monopoles is the same. For us the differential cross sections are more interesting. They don't depend on the phase of the scattering amplitudes and are

$$\begin{aligned}\left(\frac{d\sigma}{d\Omega}\right)^-_s(\theta) &= |f^-_s|^2(\theta) = \frac{1}{4}\left(1+\frac{s^2}{4k^2}\right)\frac{1}{\sin^4\frac{\theta}{2}} \\ \left(\frac{d\sigma}{d\Omega}\right)^+_s(\theta) &= |f^+_s|^2(\theta) = \frac{1}{4}\left(1+\frac{s^2}{4k^2}\right)\frac{1}{\cos^4\frac{\theta}{2}}.\end{aligned} \tag{8.8}$$

To interpret these cross sections consider the scattering wavefunction $\Psi^{TN}_{\frac{1}{2}\frac{1}{2}}$, which describes scattering of pure monopoles and dyons with electric charge 1/2. We need to set up a coordinate system for physical 3-space and to do this we go to the centre-of-mass frame of the two colliding monopoles and choose cartesian axes so that the monopoles enter the collision along the 3-axis. We also choose spherical coordinates (θ,ϕ) in such a way that they can be identified with the spherical coordinates (θ,ϕ) in the asymptotic region of the moduli space M^0_2. $\Psi^{TN}_{\frac{1}{2}\frac{1}{2}}$ is an eigenstate of the operator $-i\partial/\partial\phi$ for the angular momentum about the space-fixed 3-axis with eigenvalue 1/2. Since the incoming monopoles have no orbital angular momentum about the 3-axis this eigenvalue corresponds to their relative electric charge. Thus, in the scattering process described by $\Psi^{TN}_{\frac{1}{2}\frac{1}{2}}$ pure monopoles enter the collision from $x_3 = -\infty$ and dyons of charge 1/2 from $x_3 = +\infty$. The lower component of $\Psi^{TN}_{\frac{1}{2}\frac{1}{2}}$ is also an eigenstate of the operator $-i\partial/\partial\psi$ for the angular momentum about the body-fixed axis (whose direction is given by (θ, ϕ)) with eigenvalue 1/2. This eigenvalue gives the difference between the electric charges of the monopoles scattered in the direction (θ, ϕ) and the monopoles scattered in the direction $(\pi - \theta, \phi + \pi)$. Thus the TN approximation to the differential cross section for dyons of charge 1/2 to be scattered into the interval $(\theta, \theta + d\theta)$ can be calculated from the lower component of the wavefunction. It is

$$\left(\frac{d\sigma}{d\Omega}\right)^{+}_{\frac{1}{2}}(\theta) = \frac{1}{4}\left(1 + \frac{1}{16k^2}\right)\frac{1}{\cos^4\frac{\theta}{2}}. \tag{8.9}$$

Similarly we use the upper component of $\Psi^{TN}_{\frac{1}{2}\frac{1}{2}}$, which is an eigenstate of $-i\partial/\partial\psi$ with eigenvalue $-1/2$, to calculate the differential cross section for pure monopoles to be scattered into $(\theta, \theta + d\theta)$. We find

$$\left(\frac{d\sigma}{d\Omega}\right)^{-}_{\frac{1}{2}}(\theta) = \frac{1}{4}\left(1 + \frac{1}{16k^2}\right)\frac{1}{\sin^4\frac{\theta}{2}}. \tag{8.10}$$

In fact we have for any s

$$\left(\frac{d\sigma}{d\Omega}\right)^{-}_{s}(\pi - \theta) = \left(\frac{d\sigma}{d\Omega}\right)^{+}_{s}(\theta) \tag{8.11}$$

and this relation is a consequence of the invariance of Ψ^{TN}_{sS} under I_1. Applied to our example it simply means that whenever a pure monopole is scattered into $(\theta, \theta + d\theta)$ a dyon of charge 1/2 is scattered into $(\pi - \theta, (\pi - \theta) - d\theta)$.

Note that we have characterised the incoming and outgoing particles only by their electric charges and have not labelled them in any other way. In the TN approximation, where individual electric charges are conserved, we can, in a collision of a pure monopole and a dyon of charge 1/2, identify the outgoing pure monopole with the incoming pure monopole. In the full Hamiltonian $\hat{H}$, however, the relative electric charge is no longer conserved and consequently such an identification is no longer possible. We will argue that one can nevertheless calculate the cross section for monopoles of a given electric charge to be scattered in a given direction. First recall the physical interpretation of $\hat{H}$ in terms of W-boson and fermion exchange. On this basis we expect there to be inelastic scattering and corrections to the elastic scattering amplitudes calculated in the TN approximation. As a result, the form of the scattering wavefunction (8.5) is too restrictive: we have to allow for inelastic scattering which involves several radially outgoing scattered waves with different values of s. For a scattering process in which the incoming monopoles have relative electric charge s we consider a superposition of scattered waves with relative electric charges $\tilde{s}$. Asymptotically the scattering wavefunction should therefore be

$$e^{iS\chi}\left\{\begin{pmatrix} i^{2s}e^{is(\phi-\psi)}e^{i(kz+\eta\ln k(r-z))} \\ e^{is(\phi+\psi)}e^{i(-kz+\eta\ln k(r+z))}\end{pmatrix} + \sum_{\tilde{s}} e^{i(\tilde{k}r-\tilde{\eta}\ln 2\tilde{k}r)}\begin{pmatrix} i^{2s}f^-_{s\tilde{s}}(\phi,\theta,\psi) \\ f^+_{s\tilde{s}}(\phi,\theta,\psi)\end{pmatrix}\right\} \tag{8.12}$$

where $s=\tilde{s}$ is allowed and $\tilde{k}=k(\tilde{s})$, $\tilde{\eta}=\eta(\tilde{s})$. Thus, the scattering associated with $\hat{H}$ is described in terms of scattering amplitudes $f^-_{s\tilde{s}}$ and $f^+_{s\tilde{s}}$ for the upper and lower component of the wavefunction respectively. They can be expressed in terms of the elements $S^j_{s\tilde{s}}$ of the S-matrix of $\hat{H}$ relative to the TN Hamiltonian as follows (S^j denotes the restriction of the S-matrix to the space of scattering states with total angular momentum j):

$$i^{2\tilde{s}}f^-_{s\tilde{s}} = \frac{1}{2i\sqrt{k\tilde{k}}}\sum_{j=s}^{\infty}(2j+1)(e^{i\sigma_{js}}S^j_{s\tilde{s}}e^{i\sigma_{j\tilde{s}}}-1)D^j_{-\tilde{s}s} \quad \text{if} \quad s=\tilde{s} \tag{8.13}$$

$$i^{2\tilde{s}}f^-_{s\tilde{s}} = \frac{1}{2i\sqrt{k\tilde{k}}}\sum_{j=s}^{\infty}(2j+1)e^{i\sigma_{js}}S^j_{s\tilde{s}}e^{i\sigma_{j\tilde{s}}}D^j_{-\tilde{s}s} \quad \text{if} \quad s\neq\tilde{s} \tag{8.14}$$

and $f^+_{s\tilde{s}} = i^{2\tilde{s}}f^-_{s\tilde{s}}\circ I_1$. The matrix elements $S^j_{s\tilde{s}}$ can be calculated from the asymptotic behaviour of the solutions u_{js} of the radial equations generalising (7.26) and (7.27) and we will discuss this further below. In calculating scattering cross sections we should take care to obtain expressions which are invariant under $O(2)$ gauge transformation. The simplest way to obtain a gauge invariant differential cross section for the scattering of monopoles of relative electric charge s into monopoles of relative electric charge $\tilde{s}$ is to combine $f^+_{s\tilde{s}}$ and $f^-_{s\tilde{s}}$ into

$$\frac{\tilde{k}}{k}\left(|f^+_{s\tilde{s}}|^2+|f^-_{s\tilde{s}}|^2\right). \tag{8.15}$$

This cross section should be compared with an experimental one where both kinds of scattered monopoles (recall that $\tilde{s}$ cannot be zero here) are counted as part of the scattered current. In the example of a collision discussed above one could, however, conduct a more detailed (thought) experiment by counting only, say, the pure monopoles in the scattered current. To calculate the corresponding differential cross section we again use the body-fixed angular momentum operator to split the scattering amplitude into two pieces, one with eigenvalue $\tilde{s}=1/2$ and the other with eigenvalue $\tilde{s}=-1/2$. To do this covariantly we have to use the covariant version of $\partial/\partial\psi$ which is

$$\frac{\partial}{\partial\psi}-\boldsymbol{i}_2\omega_\psi, \tag{8.16}$$

where $\omega_\psi = \omega(\partial/\partial\psi)$ and ω is a gauge potential of the connection on Ind_2. In the singular gauge $\omega^s_\psi = 0$, so the desired decompostion into eigenstates of (8.16) with eigenvalues $1/2$ and $-1/2$ is simply

$$\begin{pmatrix} if^-_{\frac{1}{2}\frac{1}{2}}(\phi,\theta,\psi) \\ f^+_{\frac{1}{2}\frac{1}{2}}(\phi,\theta,\psi) \end{pmatrix} = \begin{pmatrix} 0 \\ f^+_{\frac{1}{2}\frac{1}{2}}(\phi,\theta,\psi) \end{pmatrix} + \begin{pmatrix} if^-_{\frac{1}{2}\frac{1}{2}}(\phi,\theta,\psi) \\ 0 \end{pmatrix}. \tag{8.17}$$

Cross sections should now be calculated from the $O(2)$ invariant norms of each of the terms in this decomposition. Thus, in the collision considered above the differential cross section for pure monopoles to be scattered into the interval $(\theta, \theta + d\theta)$ is

$$\left(\frac{d\sigma}{d\Omega}\right)_{\frac{1}{2}\frac{1}{2}} = |f^-_{\frac{1}{2}\frac{1}{2}}|^2. \tag{8.18}$$

More generally, consider a collision where the incoming monopoles have relative electric charge s and those with the lower electric charge enter the collision from $x_3 = -\infty$. The differential cross section for the outgoing monopoles to have relative electric charge $\tilde{s}$ and for those with the lower electric charge to be scattered into $(\theta, \theta + d\theta)$ is

$$\left(\frac{d\sigma}{d\Omega}\right)_{s\tilde{s}} = \frac{\tilde{k}}{k}|f^-_{s\tilde{s}}|^2. \tag{8.19}$$

Solving the radial equations (7.26) and (7.27) numerically we can now calculate some parameters of the S-matrix of $\hat{H}$ relative to the TN Hamiltonian. For scattering at low energies the leading corrections to the TN approximation can be calculated from the partial waves with $j = 1/2$ and $j = 3/2$. For energies $\varepsilon > 1/16$ the single-channel characterised by $j = 1/2$ determines the diagonal element $S^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}$ of the S-matrix for the Hamiltonian $\hat{H}$ relative to the TN Hamiltonian. $S^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}}$ is best parametrised by a phase shift $\delta_{\frac{1}{2}\frac{1}{2}}$ via $S^{\frac{1}{2}}_{\frac{1}{2}\frac{1}{2}} = \exp(2i\delta_{\frac{1}{2}\frac{1}{2}})$. We calculate this phase shift by comparing the solutions $u_{\frac{1}{2}\frac{1}{2}}$ of (7.26) asymptotically with the regular and irregular solutions of the TN radial equation with $(j, s) = (\frac{1}{2}, \frac{1}{2})$:

$$u_{\frac{1}{2}\frac{1}{2}}(r) \propto \frac{1}{r}\sin(kr - \eta\ln kr - \frac{\pi}{4} + \sigma_{\frac{1}{2}\frac{1}{2}})\ \cos\delta_{\frac{1}{2}\frac{1}{2}} + \frac{1}{r}\cos(kr - \eta\ln kr - \frac{\pi}{4} + \sigma_{\frac{1}{2}\frac{1}{2}})\ \sin\delta_{\frac{1}{2}\frac{1}{2}}. \tag{8.20}$$

In figure 1 we plot the phase shift $\delta_{\frac{1}{2}\frac{1}{2}}$ as a function of energy. The graph shows a maximum at $\varepsilon \approx 1/8$ which is the energy at which the coefficient η of the Coulomb potential in (8.2) vanishes. Thus, as one might expect, the short range forces encoded in the connection ω and the metric coefficients a, b and c of the AH manifold are most noticable when the long range forces vanish.

The coupled differential equations (7.27) for the $j = 3/2$ partial wave allow us to study details of one inelastic process: incoming monopoles of relative electric charge 1/2 can turn into monopoles of relative electric charge 3/2 provided the energy is above the production threshold $\varepsilon = 9/16$. To analyse this two-channel problem in the formalism explained in [23] we compute two linearly independent solutions of the radial equations (7.27) and then calculate the four matrix elements $S^{\frac{3}{2}}_{s\tilde{s}}$, $s, \tilde{s} = 1/2, 3/2$. The matrix $S^{\frac{3}{2}}$ can be parametrised by two eigenphase shifts δ^+ and δ^- and one mixing parameter ϵ defined in [23]. We have calculated these and we plot them in figure 2. Also shown there is the "bar" parameter $\bar{\epsilon}$, also defined in [23], which is a good measure of the inelastic scattering: The probability of producing monopoles of relative charge 3/2 when monopoles of relative charge 1/2 collide at energies just above the threshold is

$$\frac{4\pi}{(\varepsilon - 1/16)} \sin^2 2\bar{\epsilon}. \tag{8.21}$$

The parameter $\bar{\epsilon}$ decreases as one approaches the threshold energy from above, but it tends to a non-zero constant at the threshold. This is the threshold behaviour already encountered in the study of the AH Hamiltonian in [23] and it is related to the scattering below the production threshold. There we require $u_{\frac{3}{2}\frac{3}{2}}$ to be exponentially decaying for large r and define a single phase shift $\delta_{\frac{3}{2}\frac{1}{2}}$ by comparing $u_{\frac{3}{2}\frac{1}{2}}$ with the corresponding solution of the TN radial equation (8.2). This phase shift is also plotted in figure 2. It displays Coulomb resonance behaviour similar to that already found in monopole scattering without fermions: $\delta_{\frac{3}{2}\frac{1}{2}}$ increases in steps of π as one approaches the threshold from below and crosses the values $(n + 1/2)\pi$, $n = 0, 1, \ldots$, at approximately the bound state energies of the TN radial equation for $j = s = 3/2$. Since there are infinitely many such bound states, there should be infinitely many resonances below the threshold and we therefore expect that $\delta_{\frac{3}{2}\frac{1}{2}} \to \infty$ as $\varepsilon \to 9/16$. Moreover, since these resonances are quasi-bound states of monopoles with relative electric charge 3/2 whose lifetime becomes arbitrarily long as one approaches the threshold energy from below, it is to be expected that the probability of producing monopoles of relative electric charge 3/2 *at* the threshold is non-vanishing.

We will not pursue the detailed study of the quantum dynamics of 2-monopoles further, although there are a number of interesting questions remaining. The three coupled channels with $j = 5/2$ would allow one to study, though with considerable computational effort, scattering processes which involve, in the language of perturbative quantum field theory, the exchange of fermions *and* W-bosons, which could lead to qualitatively new phenomena. Also we haven't considered the large k, semiclassical limit, which proved so instructive in [23]. To finish this section we briefly compare our results for $\hat{H}$ with the discussion of the AH Hamiltonian in [23]. We have seen that quantum mechanics on moduli space with a connection shares many of the qualitative phenomena found without the connection: elastic, inelastic and resonance scattering and Coulomb-like threshold behaviour. On the other hand we found differences in our brief study of bound states: the Laplace-Beltrami operator on the AH manifold has only a continuous spectrum, but embedded bound states. When coupled to the connection on the index bundle it has a discrete spectrum below the continuous spectrum and probably no bound states embedded in the continuum. A further, qualitative difference is that in the case discussed here the two interacting monopoles never have the same electric charge and therefore they can, at least outside the interaction region, always be distinguished by their electric charges.

9 Two Monopoles Coupled to Two Fermions

We briefly describe the adiabatic quantisation of two fermions coupled to two monopoles. Asymptotically, each monopole is now coupled to a fermion. The exclusion principle forbids the two fermions both being attached to one monopole. Wavefunctions should be such that both their real and imaginary parts are sections of the real line bundle $\mathrm{Ind}_2 \wedge \mathrm{Ind}_2$. From our remarks at the end of section 5 it follows that such wavefunctions are just functions

$$\Psi : M_2 \mapsto \mathbb{C} \tag{9.1}$$

satisfying

$$\Psi \circ I_3 = -\Psi. \tag{9.2}$$

Since the connection is flat the Hamiltonian is the Laplace-Beltrami operator on M_2, which is the same Hamiltonian as for two monopoles without fermions. Thus, in the adiabatic approximation the presence of two fermions affects the quantum dynamics of two monopoles only by changing the Hilbert space of allowed states. For wavefunctions which are simultaneous eigenstates of the relative and total electric charge operators with eigenvalues s and S, the equivariance condition (9.2) means that $s, S \in \mathbb{Z}$ and $s + S \in 2\mathbb{Z} + 1$. This selection rule reflects the Dirac condition for a single monopole coupled to a fermion: it is equivalent to requiring that both the individual electric charges are half-odd integral. Since the total electric charge only contributes to the total energy but otherwise does not affect the dynamics, the quantum scattering and bound states of two monopoles coupled to two fermions and of two monopoles without fermions differ by their total electric charge but are otherwise identical. In particular, two monopoles each coupled to one fermion scatter like bosons if their individual electric charge are equal. This is consistent with our earlier observation that a single monopole coupled to an isospinor-spinor has spin 0. The fact that the interaction of two monopoles is unchanged if we attach one isospinor fermion to each monopole can be understood physically as follows. Our low-energy approximation takes into account interactions mediated by the exchange of single particles, but does not allow for the simultaneous exchange of two particles. Since the two fermions cannot both be attached to the same monopole they are therefore effectively excluded from the interaction. Hence the monopoles can only exchange single Higgs particles, photons and W-bosons, which are already fully accounted for by the AH Hamiltonian.

10 Conclusion

In table 3 we summarise the essential qualitative features of quantised monopole interactions in the presence of fermions and compare them with corresponding properties of other models for monopole interactions, namely the TN and the AH Hamiltonian.

interacting particles:	point-like monopoles	BPS monopoles	BPS monopoles + 1 fermion	BPS monopoles +2 fermions
model for interaction:	TN-Hamiltonian	AH-Hamiltonian	AH-Hamiltonian coupled to connection	AH-Hamiltonian
condition on j:	$j \in \mathbb{Z}$ or $j \in \mathbb{Z} + \frac{1}{2}$	$j \in \mathbb{Z}$	$j \in \mathbb{Z} + \frac{1}{2}$	$j \in \mathbb{Z}$
conditions on s and S:	$s \in \mathbb{Z}$ or $s \in \mathbb{Z} + \frac{1}{2}$	$s, S \in \mathbb{Z}$, $s + S \in 2\mathbb{Z}$	$s, S \in \mathbb{Z} + \frac{1}{2}$	$s, S \in \mathbb{Z}$, $s + S \in 2\mathbb{Z} + 1$
statistics:	distinguishable	indistinguishable bosons	indistinguishable	indistinguishable bosons

Table 3

Here "point-like monopoles" means point-particles with scalar, magnetic and electric charges whose (quantised) interaction is modelled by the TN Hamiltonian. In the last row we have interpreted the behaviour of the partial differential cross sections under the exchange map I_1 in terms of the statistics of the interacting particles. We call two monopoles indistinguishable if they cannot be labelled consistently in a scattering process. Two monopoles coupled to one fermion are indistinguishable in this sense although they can asymptotically be distinguished by their electric charge. Two monopoles without fermions and two monopoles coupled to two fermions may have the same electric charge and in that case the scattering cross section contains the characteristically bosonic interference term.

From the point of view of soliton phenomenology it is natural to ask whether one can construct a model for monopole interactions where the monopoles scatter like fermions. Fermionic scattering actually occurs in a bundle which arises if one quantises the isospinor-spinors in violation of the spin-statistics theorem by imposing commutation relations on the corresponding operators in quantum field theory. Then the wavefunction for two monopoles coupled to two isospinor-spinors would be a section of the symmetrised tensor product bundle

$$\mathrm{Ind}_2 \otimes_{\mathrm{sym}} \mathrm{Ind}_2 = \mathrm{Triv} \oplus \mathrm{Ind}_2^2.$$

Here Triv is the trivial real line bundle $M_2 \times \mathbb{R}$ and Ind_2^2 is a rank 2 real vector bundle whose pull-back to $\tilde{M}_2$ has first Chern number $c_1 = 2$ (when viewed as a complex line bundle). The behaviour under the map I_1 of a function Ψ representing a section of Ind_2^2 in the singular gauge is determined by the behaviour of the transition function (7.17) that connects the singular gauge with the regular gauge. Thus, for $c_1 = 2$, we have

$$\Psi \circ I_1 = -\Psi$$

which leads to fermionic scattering. Although Ind_2^2 is physically the wrong model, its structure underscores the consistency between adiabatic quantum mechanics and the quantum field theory it approximates: violation of the spin-statistics theorem at the level of quantum field theory, which presumably leads to an inconsistency when monopole pair production is considered, automatically leads to a quantum mechanical model where spinless particles scatter fermionically.

One expects fermionic monopoles to arise in a physically consistent way when an iso*vector* fermion field is coupled to a k-monopole. The Dirac operator acting on isovector-spinors has a $2k$-complex dimensional kernel for every k-monopole [8], and a construction analogous to the one for isospinors would lead to a 4-complex dimensional vector bundle over the 2-monopole moduli space. There is a natural connection on this bundle, too, and its curvature is again anti-self-dual. In principle one could calculate this connection with our methods. However, there is also a different approach, based on the observation that isovector-spinors occur naturally in supersymmetric YMH theory. Work in this direction [26] suggests that one should quantise the dynamics of two such monopoles by studying the Laplace-Beltrami operator acting on forms on the AH manifold. It would clearly be interesting to understand the relationship between the more formal supersymmetry arguments and our geometrical approach.

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A The Relationship between $\mathbf{Ind}_2$ and $\widetilde{\mathbf{Ind}}_2$

To what extent is the bundle structure of Ind_2 determined by the pulled-back bundle $\widetilde{\mathrm{Ind}}_2$? We claim that there is a unique (up to isomorphism) $O(2)$ vector bundle on $(S^1 \times M_2^0)/\mathbb{Z}_2$ whose pull-back to $S^1 \times M_2^0$ is, after orientation, $\widetilde{\mathrm{Ind}}_2$. We will demonstrate this at the level of principal bundles and we will continue to use complex notation, so we think of $O(2)$ as acting on $\mathbb{C}$ by identifying

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} &\leftrightarrow \text{complex conjugation } * \\ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} &\leftrightarrow e^{i\alpha}. \end{aligned} \tag{A.1}$$

The total space of the Hopf bundle (5.16) is $S^3 \cong SU(2)$, and the total space P of the principal $U(1)$ bundle whose line bundle is, up to homotopy equivalence, $\widetilde{\mathrm{Ind}}_2$ is $[0, 2\pi] \times_{-1} S^3$. Here the subscript -1 means that the bundles at 0 and 2π should be identified, as for the line bundles, by the map that multiplies every fibre by -1. This map can now be written explicitly as simply $U \in SU(2) \mapsto -U \in SU(2)$. The projection map π^H is compatible with this identification, so we can define a projection

$$\pi : (\chi, U) \in [0, 2\pi] \times_{-1} S^3 \to (\chi, \pi^H(U)) \in S^1 \times S^2. \tag{A.2}$$

Then $\widetilde{\mathrm{Ind}}_2$ is homotopy equivalent to the complex line bundle associated with the principal bundle

$$\begin{tikzcd} U(1) \arrow[r] & P \arrow[d, "\pi"'] \\ & S^1 \times S^2 \end{tikzcd} \tag{A.3}$$

In every principal fibre bundle there is a free right action of the structure group on the total space of the bundle which maps each fibre onto itself. For the Hopf bundle (5.16) the right-action of $U(1)$ on an element $U \in SU(2)$ is given by

$$R_\alpha : U \mapsto U e^{i\alpha\tau_3} \tag{A.4}$$

where $\alpha \in [0, 4\pi)$. Clearly $\pi^H \circ R_\alpha = \pi^H$. We can extend the action of R_α to P via

$$R_\alpha : (\chi, U) \mapsto (\chi, U e^{i\alpha\tau_3}) \tag{A.5}$$

which satisfies $\pi \circ R_\alpha = \pi$. We stress the right-action of the structure group because it gives an alternative way of constructing principal fibre bundles: we could also define (A.3) by giving the total space P together with the $U(1)$ action R_α. The base space of the bundle is then defined to be $P/U(1)$. Using $\pi \circ R_\alpha = \alpha$ one checks this definition is equivalent to our original one and that π can be identified with the canonical projection $P \mapsto P/U(1)$.

The restriction ι_3 of I_3 to $S^1 \times S^2$ is the antipodal map

$$\iota_3 : (\hat{y}, \hat{x}) \in S^1 \times S^2 \to (-\hat{y}, -\hat{x}). \tag{A.6}$$

We aim to construct an $O(2)$ bundle over $(S^1 \times S^2)/\mathbb{Z}_2$, where the action of $\mathbb{Z}_2$ identifies points related by ι_3, whose pull-back is (A.3). The basic idea is to lift the anitpodal map ι_3 from the base space to the total space P. To do this we require a map $\star : P \to P$ such that

1. The following diagram is commutative

$$\begin{tikzcd}
P \arrow[r, "\star"] \arrow[d, "\pi"'] & P \arrow[d, "\pi"'] \\
S^1 \times S^2 \arrow[r, "\iota_3"] & S^1 \times S^2
\end{tikzcd} \tag{A.7}$$

and
2. The action of $\star$ together with the right action R_α on P define a free $O(2)$ action on P.

Given this data, we can compose π with the canonical projection

$$\mathrm{pr} : S^1 \times S^2 \to (S^1 \times S^2)/\mathbb{Z}_2$$

and obtain a well-defined projection map

$$\Pi = \mathrm{pr} \circ \pi : P \mapsto (S^1 \times S^2)/\mathbb{Z}_2. \tag{A.8}$$

The $O(2)$ action of $\{R_\alpha, \star\}$ on P then allows us to define the principal $O(2)$ bundle $P \mapsto P/\{R_\alpha, \star\}$. Further, since $\Pi \circ R_\alpha = \Pi$ and $\Pi \circ \star = \Pi$, the map Π can be identified with the canonical projection map of this principal $O(2)$ bundle, which we can therefore write as

$$\begin{tikzcd}
O(2) \arrow[r] & P \arrow[d, "\Pi"'] \\
& (S^1 \times S^2)/\mathbb{Z}_2
\end{tikzcd} \tag{A.9}$$

To construct $\star$ explicitly we exploit again the identification of S^3 with $SU(2)$. One checks that the map

$$\star : (\chi, U) \in P \mapsto (\chi + \pi, U\tau_1) \tag{A.10}$$

satisfies the requirements 1. and 2. The geometrical interpretation of $\star$ is that it copies the fibre at a point in $S^1 \times S^2$ to the antipodal point and complex conjugates it. We also explicitly see the gauge freedom in choosing $\star$ because we could have used some linear combination $a_1\tau_1 + a_2\tau_2$, with $a_1^2 + a_2^2 = 1$, instead of τ_1 in defining it.

Sections of the vector bundles $\widetilde{\mathrm{Ind}}_2$ and Ind_2 can be characterised as equivariant maps

$$F : P \to \mathbb{R}^2 \cong \mathbb{C}. \tag{A.11}$$

The equivariance condition for $\widetilde{\mathrm{Ind}}_2$ is

$$F \circ R_\alpha = e^{-i\alpha} F \tag{A.12}$$

and for sections of Ind_2 we have to impose, in addition to this,

$$F \circ \star = *F \tag{A.13}$$

This is the extra condition that we have to impose on sections of $\widetilde{\mathrm{Ind}}_2$ to get sections of Ind_2. In practical applications we patch together sections from functions defined on open subsets of the base space. How does (A.13) translate into a condition for such locally defined functions? Consider an open covering $\{U_r\}$ of $S^1 \times S^2$ and a trivialisation of the $U(1)$ bundle P (A.3)

$$f_r : \pi^{-1}(U_r) \subset P \to U_r \times U(1) \tag{A.14}$$

Associated to this there are local sections

$$s_r : U_r \to P$$

defined via

$$s_r(u) = f_r^{-1}(u, 1). \tag{A.15}$$

We now impose that, if U_r is an element of the open covering, then so is $U_{r'} = \iota_3 U_r$ (we write down an example of such an open covering in the main text) and require that the f_r satisfy the compatibility condition

$$s_{r'} \circ \iota_3 = \star \circ s_r. \tag{A.16}$$

Then it follows from (A.13) that the local functions

$$\Psi_r = F \circ s_r \tag{A.17}$$

must satisfy

$$\Psi_{r'} \circ \iota_3 = F \circ s_{r'} \circ \iota_3 = F \circ \star \circ s_r = *F \circ s_r = *\Psi_r. \tag{A.18}$$

So far we have considered the bundles $\widetilde{\mathrm{Ind}}_2$ and Ind_2 only up to homotopy equivalence and worked over the base spaces $S^1 \times S^2$ and $(S^1 \times S^2)/\mathbb{Z}_2$. However, the condition (A.18) can easily be extended to an open cover $\{U_r\}$ of $S^1 \times M_2^0$ which trivialises $\widetilde{\mathrm{Ind}}_2$ and is such that $U_{r'} = I_3 U_r$ is contained in it if U_r is. Now the condition for local functions of an equivariant section is

$$\Psi_{r'} \circ I_3 = *\Psi_r. \tag{A.19}$$

This is the form of the equivariance condition that is used in the main text.

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Figure Headings

Figure 1

Phase shift $\delta_{\frac{1}{2}\frac{1}{2}}$ as a function of the energy ε ; in the plot on the left the energy scale is chosen so that the maximum near $\varepsilon = 1/8$ is clearly visible.

Figure 2

$j = 3/2$ coupled problem. Parameters $\delta^+, \delta^-, \epsilon$ and $\bar{\epsilon}$ for the S-matrix above threshold, and phase shift $\delta_{\frac{3}{2}\frac{1}{2}}$ for scattering of monopoles of relative electric charge 1/2 below threshold.