Nuclear Physics B 367 (1991) 177–214
North-Holland

NUCLEAR
PHYSICS B

Quantum scattering of BPS monopoles at low energy

B.J. Schroers

Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, UK

Received 4 March 1991
(Revised 1 August 1991)
Accepted for publication 5 August 1991

The quantum scattering of non-relativistic BPS monopoles is investigated by supposing that the hamiltonian is proportional to the covariant laplacian on the space of collective coordinates of the monopoles equipped with the Atiyah–Hitchin metric. Using a partial-wave analysis and numerical methods we find a rich quantum mechanical structure, including inelastic and resonance scattering. Quantitative estimates of certain cross sections are given. It is pointed out that a similar method could be used to discuss nucleon–nucleon scattering in the Skyrme model.

1. Introduction

The so-called geodesic approximation has proven to be a very useful tool for understanding the classical dynamics of slowly moving BPS monopoles. The idea that the motion of monopoles might be well described by geodesic motion on the manifold of static multimonopole solutions, called the moduli space, was formulated in ref. [1]. Atiyah and Hitchin found the relevant metric for the two-monopole case by an indirect method which is described in some detail in their recent book [2]. The moduli space M_2 for two-monopoles is eight dimensional and decomposes into $\mathbb{R}^3 \times (S^1 \times M_2^0)/\mathbb{Z}_2$. Its metric is block diagonal with respect to this decomposition. The flat $\mathbb{R}^3 \times S^1$ part parametrizes the centre of mass position and an overall phase angle. M_2^0 is the interesting part of the metric: it describes the relative motion of the monopoles. The riemannian metric on M_2^0 has four crucial properties: it is finite, geodesically complete, SO(3) symmetric and hyperkähler. Asymptotically (in a sense that will be made precise later) the metric on M_2^0 equals a modified euclidean Taub-NUT metric. The Taub-NUT metric is also hyperkähler but it is not finite and is acted on naturally by SU(2) rather than SO(3). In four dimensions hyperkähler is equivalent to anti-self-dual, so both the Atiyah–Hitching and the Taub-NUT metric are examples of gravitational instantons.

The classical motion of monopoles in the geodesic approximation has been investigated in a number of publications. Atiyah and Hitchin discuss geodesics on certain two-dimensional geodesic submanifolds of M_2^0 in ref. [2]. These include the 90 degree scattering of pure monopoles in a head-on collision and processes in which orbital angular momentum is transformed into internal rotation, thus turning pure monopoles into electrically charged dyons. In ref. [2] more general geodesics were considered and it was shown that this transfer of orbital angular momentum into electric charge is in fact generic. Furthermore it was argued that the geodesic motion is not integrable and that there are regions of chaotic behaviour. Motion in the asymptotic region of M_2^0, where the Atiyah–Hitchin metric can be approximated by the Taub-NUT metric, models the dynamics due to the long-range forces between BPS monopoles. These forces can also be derived by assuming that the monopoles are point particles with suitable scalar, magnetic and electric charges [4]. Thus one may think of the Taub-NUT metric as a point-particle approximation to monopole motion. The Taub-NUT metric has an additional SO(2) symmetry leading to the conservation of the relative electric charge. As a result, the geodesic motion in Taub-NUT space is completely integrable with a remarkably close analogy to the Coulomb problem [5]. Of course the moduli space picture cannot replace a full field-theoretic treatment of classical monopole motion. The authors of ref. [6] argued that the main correction to the geodesic picture is energy loss to the long-wavelength modes of the massless fields. They also showed that this radiation as a fraction of the total energy is $O(v^3)$ in a typical scattering process, where v is the initial speed of each monopole.

In the spirit of the geodesic approximation, the quantum dynamics of two monopoles is approximated by supposing that the Schrödinger operator is proportional to the covariant laplacian on M_2^0. Gibbons and Manton first discussed the quantum problem in ref. [5] and their paper will be our standard reference throughout. They were able to solve the quantum problem completely in the Taub-NUT limit. However, in the region of M_2^0 which models two monopoles close together, the Atiyah–Hitchin metric differs substantially from the Taub-NUT metric. In particular the relative electric charge, which is exactly conserved in Taub-NUT space, is no longer conserved in close encounters of monopoles. It was therefore difficult to estimate how good the Taub-NUT approximation to the quantum dynamics on M_2^0 really is. In ref. [7] Manton studied bound states in the Atiyah–Hitchin metric numerically, but avoided coupled problems. He found the bound-state energies to be in very good agreement with those obtained in the Taub-NUT approximation. Here we study the quantum scattering in the Atiyah–Hitchin manifold at low total angular momentum: taking the laplacian on the Atiyah–Hitchin manifold as the generator of the "interacting" dynamics and the laplacian of the Taub-NUT manifold as the generator of the "free" dynamics we use numerical and WKB methods to calculate elements of the S-matrix that relates the two dynamics. In the quantum theory it is more difficult to estimate the range of validity of the moduli space approximation. As in the classical theory one expects a truncation of the field theory to be sensible at low energies. But there are now additional complications, pointed out in ref. [5]. In the quantised field theory the scalar field acquires a small mass, so that static monopoles repel each other. One also expects the zero-point energy of the field fluctuations orthogonal to M_2 to vary over M_2. Both these effects suggest that we should add a potential of order $\hbar$ to the hamiltonian. But this is very hard to estimate quantitatively. We will instead adopt the point of view of ref. [5]: the geometry of the moduli space alone gives rise to interesting physical phenomena. The concrete aim of this paper is a detailed understanding of the low-energy scattering of monopoles. But in presenting a careful case study we also want to illustrate the importance of the topological and geometrical structure of the space of collective coordinates for an adequate description of slowly moving solitons. In particular we will highlight some implications of this insight for the physically interesting case of skyrmions.

2. Quantum scattering in the Atiyah–Hitchin metric

We follow ref. [5] in deriving the Schrödinger equation (6) on the Atiyah–Hitchin manifold M_2^0.

Since almost all the orbits of the SO(3) action on M_2^0 are three dimensional, M_2^0 can be coordinatized by a radial coordinate r and Euler angles $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, $0 \leqslant \psi \leqslant 2\pi$. Introducing the standard right invariant forms on SO(3)

$$\begin{aligned} \sigma_1 &= -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi, \\ \sigma_2 &= \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi, \\ \sigma_3 &= d\psi + \cos\theta\, d\phi, \end{aligned} \tag{1}$$

the metric on M_2^0 can be written

$$ds^2 = f(r)^2\, dr^2 + a(r)^2\sigma_1^2 + b(r)^2\sigma_2^2 + c(r)^2\sigma_3^2. \tag{2}$$

The self-duality of the metric implies

$$\frac{2bc}{f}\frac{da}{dr} = (b-c)^2 - a^2, \quad +\text{cycl.}, \tag{3}$$

where "cycl." means we add the two further equations obtained by cyclic permutation of a, b, c. Atiyah and Hitchin discuss various solutions of eq. (3) in ref. [2]. They find the essentially unique solution that leads to a complete manifold in which the generic orbit of the SO(3) action is three dimensional and show that it can be expressed in terms of elliptic integrals. A plot of the numerical solution is presented in ref. [5]. We follow the convention adopted there for the choice of the radial coordinate by setting

$$f = -b/r.$$

The range of r is then $[\pi, \infty)$, and for large values r can be thought of as the distance in physical space between the centres of the monopoles. More generally the kinetic energy for the relative motion of two monopoles derived from (2) is similar to that for an asymmetric body with principal moments of inertia proportional to a^2, b^2 and c^2. Since these vary with r (and hence with time) the body is not rigid but has a one-parameter family of shapes. When $r = \pi$ the monopoles coincide and the field configuration becomes axially symmetric. The moments of inertia about the axes orthogonal to the axis of symmetry then become equal and as a result $b = |c| = \pi$. Moreover a vanishes at $r = \pi$ showing that there is zero distance in field configuration space between configurations related by a rotation about the axis of symmetry. As a result, the three-dimensional orbit of the SO(3) collapses to an $\mathbb{RP}^2$ called a "bolt" by relativists. For a qualitative understanding of monopole scattering it will be useful to recall that near the bolt $r - \pi$ is approximately equal to te proper radial distance from the bolt and that a, b and c have the following form to lowest order in $\mathrm{r} - \pi$:

$$a \approx 2(r - \pi), \qquad b \approx \pi + \tfrac{1}{2}(r - \pi), \quad c \approx -\pi + \tfrac{1}{2}(r - \pi). \tag{4}$$

For large r one finds

$$a = b \approx r\sqrt{1 - \frac{2}{r}}, \quad c \approx -\frac{2}{\sqrt{1 - 2/r}} \tag{5}$$

with corrections of order e^{-r}. The metric obtained from (2) by replacing a, b and c with the asymptotic expressions is a modified euclidean Taub-NUT metric. Because $a = b$, it has the additional SO(2) symmetry mentioned in sect. 1. The radial coordinate r, again defined by $f = -b/r$, now has the range $(2, \infty)$.

For our partial-wave analysis of the quantum scattering of monopoles we will need an estimtate of the range R of the core region of the Atiyah–Hitchin space, where its meric differs significantly from the Taub-NUT metric. To define R we first note that naively one might expect the moments of inertia a^2 and b^2 of the two-monopole system about axes perpendicular to the line joining the monopoles to be proportional to r^2 for r sufficiently large. Actually this proportionality is modified even in the Taub-NUT limit by the factor $(1 - 2/r)^{1/2}$ which is a reflection of the long-range forces between monopoles. For small r, the deviation of the Atiyah–Hitchin metric coefficients from the asymptotic form (5) is due to the short-range forces between monopoles. We therefore define R to be that value of the radial coordinate r at which the difference between the Atiyah–Hitchin and the Taub-NUT expressions for a and b is "small" compared to the long-range correction $r - r(1-2/r)^{1/2}$. As in the definition of the range of potentials, the vagueness of "small" makes this definition somewhat arbitrary. If we take "small" to mean "half as big" we find $R \approx 5$.

To discuss the quantum scattering of monopoles in the geodesic approximation we study the scattering solutions of the Schrödinger equation

$$-\frac{1}{abcf}\frac{\partial}{\partial r}\left(\frac{abc}{f}\frac{\partial \Psi}{\partial r}\right) - \left(\frac{\xi_1^2}{a^2} + \frac{\xi_2^2}{b^2} + \frac{\xi_3^2}{c^2}\right)\Psi = \varepsilon \Psi. \tag{6}$$

The operator on the left-hand side is minus the covariant laplacian on M_2^0 and ε is a rescaled dimensionless energy, $\varepsilon = 4\pi\ E/\hbar^2$. As in ref. [5] we choose units in which a single monopole has both mass and magnetic charge 4π and the Higgs field of the single monopole solution given in ref. [5] has a range of 1. The range of the Higgs field will henceforth be referred to as the size of the monopole. We also fix the speed of light to be 1. ξ_1, ξ_2, ξ_3 are the vector fields on SO(3) dual to σ_1, σ_2, σ_3

$$\xi_1 = -\cot\theta\cos\psi\frac{\partial}{\partial\psi} - \sin\psi\frac{\partial}{\partial\theta} + \frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi},$$

$$\xi_2 = -\cot\theta\sin\psi\frac{\partial}{\partial\psi} + \cos\psi\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi},$$

$$\xi_3 = \frac{\partial}{\partial\psi}. \tag{7}$$

In order to separate variables we introduce Wigner functions following the convention of ref. [8],

$$D^j_{sm}(\phi, \theta, \psi) = e^{is\psi} d^j_{sm}(\theta)\ e^{im\phi},$$

which form an orthogonal set and are normalised so that

$$\int |D^j_{sm}|^2\ d\psi\ d\phi\ d\cos\theta = \frac{8\pi^2}{2j+1}.$$

They have the symmetry property

$$d^j_{sm}(\theta) = (-1)^{j+m} d^j_{-sm}(\pi - \theta)$$

and satisfy

$$-\left(\xi_1^2+\xi_2^2+\xi_3^2\right)D^j_{sm}=j(j+1)D^j_{sm},$$

$$-i\frac{\partial}{\partial\phi}D^j_{sm}=mD^j_{sm},$$

$$-i\frac{\partial}{\partial\psi}D^j_{sm}=sD^j_{sm}. \tag{8}$$

Here the operator $-\hbar^2(\xi_1^2+\xi_2^2+\xi_3^2)$ represents the squared total angular momentum (in the center-of-mass frame), $-i\hbar\partial/\partial\phi$ the angular momentum about the space-fixed 3-axis, and $-i\hbar\partial/\partial\psi$ the angular momentum about the body-fixed 3-axis, which corresponds to the relative electric charge of the two monopoles. Quantum states which are eigenstates of the electric charge operator will be called pure monopoles if $s=0$, and dyons of relative electric charge s if $|s|\geqslant 1$. If we simply say monopoles, this may refer to pure monopoles or dyons.

Both $\partial/\partial\phi$ and $-(\xi_1^2+\xi_2^2+\xi_3^2)$ commute with the hamiltonian, but in our ansatz for a scattering solution we fix only m and consider a sum over the total angular momenta $j\geqslant|m|$,

$$\Psi_m(r,\phi,\theta,\psi)=\sum_{j=|m|}^{\infty}\sum_{s=-j}^{s=j}u^j_{sm}(r)D^j_{sm}(\phi,\theta,\psi). \tag{9}$$

For fixed j and m the $2j+1$ radial functions u^j_{sm} have to satisfy the $2j+1$ coupled ordinary differential equations

$$-\frac{1}{abcf}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{abc}{f}\frac{du^j_{sm}}{dr}\right)+\sum_{\bar{s}=-j}^{\bar{s}=j}M_s^{\bar{s}}(j,r)u^j_{\bar{s}m}=\varepsilon u^j_{sm}. \tag{10}$$

The matrix elements $M_s^{\bar{s}}(r,j)$ are the expectation values of the rigid-body hamiltonian

$$M_s^{\bar{s}}=-\frac{2j+1}{8\pi^2}\left\langle D^j_{\bar{s}m}\left|\frac{\xi_1^2}{a^2}+\frac{\xi_2^2}{b^2}+\frac{\xi_3^2}{c^2}\right|D^j_{sm}\right\rangle.$$

Since the ξ_i are cartesian tensor operators of rank 1, ξ_i^2 has non-vanishing matrix elements only if $|s-\bar{s}|=0$ or 2. As a result, states with odd s do not mix with states where s is even. The equations are further simplified by discrete symmetries. The rigid-body hamiltonian and the commutation relations of the ξ_i are invariant with respect to a simultaneous change in sign of any two of the ξ_i. The full group corresponding to this symmetry is the vierergruppe but we want to consider specifically the transformation which, at least for large r, corresponds to the exchange of the position of the two monopoles and the simultaneous reversal of their relative electric charge:

$$x \to x, \qquad y \to -y, \qquad z \to -z, \tag{11}$$

in body-fixed cartesian coordinates, or

$$\theta \to \pi - \theta, \qquad \phi \to \pi + \phi, \qquad \psi \to -\psi. \tag{12}$$

in terms of the Euler angles. It follows from the symmetries of the Wigner functions that under this transformation

$$D^j_{sm}(\pi + \phi, \pi - \theta, -\psi) = (-1)^j D^j_{-sm}(\phi, \theta, \psi).$$

We can therefore assume s to be non-negative and consider positive "parity" states

$$\psi^+_{jsm} = \begin{cases} \left(D^j_{-sm} + (-1)^j D^j_{sm}\right) & \text{if } s \neq 0, \\ \sqrt{2}\, D^j_{0m} & \text{if } s = 0 \text{ and } j \text{ even}, \\ \text{not defined} & \text{if } s = 0 \text{ and } j \text{ odd}, \end{cases}$$

separately from the negative "parity" states

$$\psi^-_{jsm} = \begin{cases} \left(D^j_{-sm} - (-1)^j D^j_{sm}\right) & \text{if } s \neq 0, \\ \sqrt{2}\, D^j_{0m} & \text{if } s = 0 \text{ and } j \text{ odd}, \\ \text{not defined} & \text{if } s = 0 \text{ and } j \text{ even}. \end{cases}$$

In the Atiyah–Hitchin manifold points related by (12) are identified. This is a consequence of the fact that even classically one cannot consistently identify and label individual monopoles. While it does make sense, for well-separated monopoles, to say "a monopole of electric charge s_1 is at position $\boldsymbol{r}_1$ and another of electric charge s_2 is at $\boldsymbol{r}_2$", it is, strictly speaking, meaningless to say "monopole A with electric charge s_A is at $\boldsymbol{r}_A$ and monopole B with electric charge s_B is at $\boldsymbol{r}_B$". The use of such individual labels for the monopoles corresponds to using the coordinates ϕ, θ, ψ without making the identification (12). Such labelling is often helpful and allowed as long as we make sure that all our wave functions are invariant under the map (12) and hence well defined on the true Atiyah–Hitchin space. This requirement implies that the angular dependence of any permissible wave function must be expressible purely in terms of the ψ^+_{jsm}. Note that this means in particular that pure monopoles cannot exist in a state of odd total angular momentum (which equals the orbital angular momentum in this case).

We should mention another consequence of discrete symmetries of M_2, which is explained in detail in ref. [5]. The conserved total electric charge is an integer S which can differ from s only by an even number. The value of S does not affect the dynamics, so we set $S=0$ if s even, and $S=1$ if s odd.

In order to evaluate the matrix elements $M_s^{\bar{s}}(j, r)$ we first notice that they are independent of m. Physically this is due to the fact that the energy of a free top does not depend on its orientation in space. More formally it is evident because we can compute $M_s^{\bar{s}}(j, r)$ just using the algebraic properties of the ξ_i and their action on states with well defined j and s. First we rewrite the rigid-body hamiltonian as follows:

$$\frac{\xi_1^2}{a^2}+\frac{\xi_2^2}{b^2}+\frac{\xi_3^2}{c^2}=\frac{\xi_1^2+\xi_2^2+\xi_3^2}{b^2}+\xi_1^2\left(\frac{1}{a^2}-\frac{1}{b^2}\right)+\xi_3^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right).$$

Writing $|js\rangle$ for $\sqrt{(2j+1)/8\pi^2}\,D^j_{sm}$ and $|js^+\rangle$ for $\sqrt{(2j+1)/16\pi^2}\,\psi^+_{jsm}$, we only need to calculate $\langle j\bar{s}|\xi_1^2|js\rangle$. From simple angular momentum theory one finds

$$\begin{aligned}
\langle js|\xi_1^2|js\rangle &= -\tfrac{1}{2}(j(j+1)-s^2),\\
\langle js|\xi_1^2|j(s+2)\rangle &= \langle j(s+2)|\xi_1^2|js\rangle\\
&= -\tfrac{1}{4}\sqrt{(j-s)(j-s-1)(j+s+1)(j+s+2)}\,,\\
\langle js^+|\xi_1^2|js^+\rangle &= \langle js|\xi_1^2|js\rangle, \quad s\neq 1,\\
\langle j1^+|\xi_1^2|j1^+\rangle &= \langle j1|\xi_1^2|j1\rangle+(-1)^j\langle j1|\xi_1^2|j(-1)\rangle,\\
\langle js^+|\xi_1^2|j(s+2)^+\rangle &= \langle js|\xi_1^2|j(s+2)\rangle, \quad s\neq 0,\\
\langle j0^+|\xi_1^2|j2^+\rangle &= \sqrt{2}\langle j0|\xi_1^2|j2\rangle.
\end{aligned} \tag{13}$$

Finally we have to choose the quantum number m. If we were looking for the bound state energies as in ref. [7] the value of m would be immaterial, but in a scattering problem the angular dependence of the wave function is crucial. Physically we envisage a scattering situation where two monopoles are very far apart long before and long after the scattering, thus having well-defined initial relative (and hence individual) electric charge, which we shall momentarily denote by q. In our model this corresponds to q units of relative angular momentum about the body-fixed 3-axis, which asymptotically points in the direction of the straight line joining the two monopoles. If we now choose our space-fixed coordinate system for the relative motion so that the monopoles travel towards each other along the space-fixed 3-axis, we see that space-fixed and body-fixed 3-axis are parallel or antiparallel for the incoming beam in the asymptotic region, and we must set $m = \pm q$. At least for dyon scattering the choice of the sign allows us to fix the orientation of the z-axis relative to the beam. By setting $m = q$ we choose the z-axis so that, in the center-of-mass system, the monopole with positive electric charge comes in from the positive z-direction and the one with negative or no electric charge enters the collision from the negative z-direction. Since the space-fixed angular momentum is conserved in the scattering process, the outgoing particles will be in an eigenstate of $-i\partial/\partial\phi$ with the same eigenvalue q but in a superposition of eigenstates of $-i\partial/\partial\psi$ with possible eigenvalues s determined by the above selection rules.

Using all this we can modify our ansatz (9) and try instead

$$\Psi_q(r, \phi, \theta, \psi) = \sum_{j=q}^{\infty} \sum_{s} u_{js}(r) \psi^{+}_{jsq}(\phi, \theta, \psi). \tag{14}$$

We are then led to the slightly simplified set of coupled differential equations for the u_{js},

$$-\frac{1}{abcf}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{abc}{f}\frac{\mathrm{d}u_{js}}{\mathrm{d}r}\right) + \sum_{\tilde{s}} M_s^{\tilde{s}}(j, r) u_{j\tilde{s}} = \varepsilon u_{js}, \tag{15}$$

where now

$$M_s^{\tilde{s}} = -\left\langle j\tilde{s}^{+} \left| \frac{\xi_1^2}{a^2} + \frac{\xi_2^2}{b^2} + \frac{\xi_3^2}{c^2} \right| js^{+} \right\rangle$$

and the sums over s and $\tilde{s}$ are over either all even or all odd non-negative integers $\leqslant j$, depending on whether q is even or odd.

It is clear that, as j increases, the size of the above system of coupled ordinary differential equations will become arbitrarily large. In the language of scattering theory: for fixed j we will have to consider two separate systems of $\frac{1}{2}(j+2)[\frac{1}{2}(j+1)]$ and $\frac{1}{2}j[\frac{1}{2}(j-1)]$ coupled channels if j is even [odd]. Here we will restrict ourselves to the single- and two-channel problems. These occur for $j \leqslant 5$. Representing single-channels by ●, channels which are part of a doublet by ★, and channels which are part of a triplet by ▲, we arrive at table 1.

If, in eq. (6), we replace a, b, c and $f = -b/r$ by the asymptotic expressions (5) we obtain the Schrödinger equation corresponding to the Taub-NUT metric. Due to the extra SO(2) symmetry the ansatz (14) leads to decoupled ordinary differential equations for the u_{js}. Setting $h_{js} = ru_{js}$ we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{j(j+1)}{r^2} - \frac{(2\varepsilon - s^2)}{r} + \left(\varepsilon - \frac{s^2}{4}\right)\right) h_{js}(r) = 0. \tag{16}$$

TABLE 1
Single and coupled channels at low angular momentum j. Explanations in text

s	j					
	0	1	2	3	4	5
0	●		★		▲	
1		●	●	★	★	▲
2			★	●	▲	★
3				★	★	▲
4					▲	★
5						▲

This equation is formally identical to the radial equation for the standard Coulomb problem, but note that in eq. (16) the strength of the Coulomb potential depends on the energy. It is also remarkable that the point $r=2$, where the Taub-NUT metric is singular and which classical geodesics cannot cross, is a regular point of the radial equation (16). The quantum problem in the Taub-NUT space was completely solved in ref. [5] using parabolic coordinates. In particular it was found that for each $s \geqslant 1$ there are infinitely many bound states with energy below $\varepsilon = s^2/4$. Scattering takes place above that energy. We want to study quantum scattering in M_2^0 by comparing the solution of (15) with the partial wave solutions u_{js} of (16) at large r.

In order to do this, we have to carry out a partial wave analysis of the scattering problem in Taub-NUT space. This will differ from the standard Coulomb problem because we expand the incoming plane wave in terms of the angular states ψ^+_{jss} rather than in Legendre polynomials (which would correspond to $m=0$). In fact, other authors [9,10] have derived a partial-wave expansion of the Taub-NUT scattering amplitude purely algebraically, exploiting the existence of a Runge–Lenz type conserved quantity in this problem. We are interested in both a partial wave decomposition of the scattering amplitude and of the entire scattering wave function found in ref. [5] (the two being intimately connected, of course). We will compute these by an elementary method, which will also be useful for fixing our notation and normalisation conventions.

For $\varepsilon \geqslant s^2/4$ it is convenient to introduce the momentum $k = \sqrt{\varepsilon - s^2/4}$, and a parameter $\eta = (\varepsilon - s^2/2)/k = k - s^2/4k$ characterizing the strength of the Coulomb potential. Eq. (16) then becomes

$$\left(\frac{d^2}{dr^2} - \frac{j(j+1)}{r^2} - \frac{2k\eta}{r} + k^2\right) h_{js} = 0. \tag{17}$$

A solution which is regular at the origin is

$$F_{js}(kr) = C_{js}\, e^{ikr}(kr)^{j+1} F(j+1+i\eta, 2j+2, -2ikr).$$

Here $F(a, b, u)$ is the confluent hypergeometric function and the constant C_{js} is chosen so that asymptotically

$$F_{js}(kr) \approx \sin\left(kr - \eta \ln 2kr - \tfrac{1}{2}j\pi + \sigma_{js}\right), \tag{18}$$

where

$$\sigma_{js} = \arg \Gamma(j + 1 + i\eta). \tag{19}$$

One can also obtain a solution G_{js} of the radial equation which is irregular at the origin. We normalise it so that we have asymptotically

$$G_{js}(kr) \approx -\cos\left(kr - \eta \ln 2kr - \tfrac{1}{2}j\pi + \sigma_{js}\right). \tag{20}$$

Having solved the radial equation, we can write the general regular solution of eq. (6) (with Taub-NUT coefficients a, b, c, f) for a fixed s, and respecting the identification (12), as

$$\Psi_s^{\mathrm{TN}}(r, \phi, \theta, \psi) = \frac{1}{kr} \sum_{j=s}^{\infty} B_{js} \psi_{jss}^{+}(\phi, \theta, \psi) F_{js}(kr). \tag{21}$$

We want to determine the constants B_{js} so that the above solution agrees with the scattering solution found in ref. [5] for fixed s. We digress briefly to describe this solution, which we write as

$$\Phi_s = C_s\left(\Phi_s^+ + (-1)^s \Phi_s^-\right),$$

where $(z = r \cos \theta)$

$$\Phi_s^+(r, \phi, \theta, \psi) = \mathrm{e}^{is(\phi+\psi)} k^s (r+z)^s \, \mathrm{e}^{-ikz} F(x - i\eta, 2s+1, ik(r+z)),$$

$$\Phi_s^-(r, \phi, \theta, \psi) = \mathrm{e}^{is(\phi-\psi)} k^s (r-z)^s \, \mathrm{e}^{ikz} F(s - i\eta, 2s+1, ik(r-z)) \tag{22}$$

and C_s is a normalisation constant chosen so that Φ_s^- has the asymptotic form

$$\mathrm{e}^{is(\phi-\psi)} \, \mathrm{e}^{i(kz + \eta \ln k(r-z))} \left(1 + \frac{s^2 + \eta^2}{ik(r-z)}\right) + f_s^-(\phi, \theta, \psi) \frac{\mathrm{e}^{i(kr - \eta \ln 2kr)}}{r}. \tag{23}$$

Here f_s^- is the scattering amplitude found in ref. [5],

$$f_s^-(\phi, \theta, \psi) = \mathrm{e}^{is(\phi-\psi)} \frac{(s - i\eta)}{2ik} \frac{\mathrm{e}^{-i\eta \ln \sin^2\theta/2}}{\sin^2\theta/2} \mathrm{e}^{2i\sigma_{ss}}(-1)^s.$$

The corresponding quantities Φ_s^+ and f_s^+ are determined by

$$\Phi^+(r, \phi, \theta, \psi) = (-1)^s \Phi^-(r, \phi + \pi, \pi - \theta, -\psi).$$

The scattering amplitude in Taub-NUT space is therefore

$$f_s^{\mathrm{TN}} = \left(f_s^+ + (-1)^s f_s^-\right) \tag{24}$$

which is invariant under (12), as required. The interpretation of this scattering amplitude requires some care. Formally one would calculate the Taub-NUT approximation to the differential elastic cross section for monopole scattering by taking the square of the modulus of f_s^{TN} and averaging over ψ. This average is required because the internal angle ψ, being conjugate to relative electric charge, is not measured in the initial or final state. This would give a symmetrized cross section which corresponds to counting *all* outgoing monopoles as part of the scattered current. For pure monopoles, this is satisfactory because one could not tell experimentally which of the outgoing pure monopoles originated from the "incoming beam" and which from the "target" anyway. The resulting cross section, first derived in ref. [5], has the form characteristic for scattering of identical bosons:

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_0^{\mathrm{TN}} = \frac{1}{4}\left[\operatorname{cosec}^4\frac{\theta}{2} + \sec^4\frac{\theta}{2} + 8\operatorname{cosec}^2\theta\cos\left(2k\ln\tan\frac{\theta}{2}\right)\right]. \tag{25}$$

For dyons, the interference term $2Re(f_s^-(f_s^+)^*)$ is proportional to $\cos s\psi$, and vanishes when averaged over ψ. The resulting symmetrized differential cross section is

$$|f_s^-|^2(\theta) + |f_s^+|^2(\theta). \tag{26}$$

But now a symmetrized differential cross section is less satisfactory: If we wanted to measure the cross section for elastic scattering of dyons of relative charge 2, say, we could very well set up a (thought) experiment such that, in the center of mass frame, negatively (positively) charged dyons enter the collision along the negative (positive) z-axis, and only the scattered negatively charged dyons are counted in measuring the scattered current. This gives more detailed information about the scattering than counting *all* scattered dyons, and since it is available experimentally, we should be able to calculate it from the theory. To do this we go back to the coordinates (ϕ, θ, ψ) without the identification (12) and, for large separation, label the monopoles A and B, their position vectors $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ and their charges s_A and s_B. We also introduce a (directed) relative position vector $\boldsymbol{r} = \boldsymbol{r}_A - \boldsymbol{r}_B$ and a relative electric charge (with sign) $s = s_A - s_B$. In fig. 1 we use this notation to illustrate the "different" scattering processes described by the scattering wavefunc-

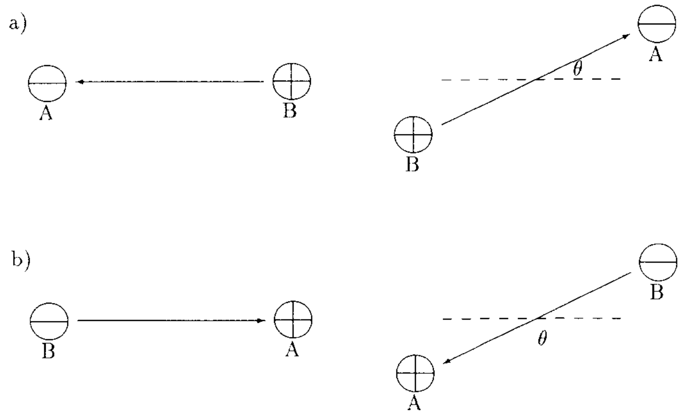

Fig. 1. The situation before and after the scattering described by (a) Φ_2^- and (b) Φ_2^+. The processes are related by the simultaneous reversal of the relative position coordinate $\boldsymbol{r} = \boldsymbol{r}_A - \boldsymbol{r}_B$, whose direction is indicated by the arrows, and of the relative electric charge $s = s_A - s_B$. Physically they are indistinguishable.

tions Φ_2^- and Φ_2^+. The point is that Φ^- and Φ^+ describe the same physical scattering process using different labels for the monopoles and hence different relative coordinates. In particular, the direction of the outgoing negatively charged dyon is given by θ in fig. 1a but by $\pi - \theta$ in fig. 1b. But clearly

$$|f_2^-|^2(\theta) = |f_2^+|^2(\pi - \theta).$$

Thus we see that in the case of dyon scattering the amplitudes f_s^- and f_s^+, when properly interpreted, give the same cross section and do not interfere. Hence the Taub-NUT approximation to the elastic cross section for dyon scattering that should be compared with one measured in the hypothetical experiment described above is not (26) but

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_s^{\mathrm{TN}}(\theta) = |f_s^-|^2(\theta) = |f_s^+|^2(\pi - \theta) = \frac{1}{4}\left(1 + \frac{s^2}{4k^2}\right)^2 \operatorname{cosec}^4\frac{\theta}{2}, \quad (27)$$

which is the formula given in ref. [5].

The constants B_{js} in (21) can be evaluated by matching Ψ_s^{TN} and Φ_s near $r = 0$ and using the orthogonality properties of the Wigner functions. This is done in appendix A. We find

$$B_{js} = i^j(2j + 1)\, \mathrm{e}^{i\sigma_{js}}.$$

Substituting the asymptotic expression for F_{js} into eq. (21) and comparing with eq.

(23) we find the partial-wave expansion of the scattering amplitude

$$
\begin{aligned}
f_s^{\mathrm{TN}}(\phi, \theta, \psi) &= \sum_{j=s}^{\infty} (2j+1)\psi_{jss}^{+}(\phi, \theta, \psi)\frac{e^{2i\sigma_{js}}-1}{2ik} \\
&= e^{is(\phi-\psi)}\sum_{j=s}^{\infty} (2j+1) d_{-ss}^{j}(\theta)\frac{e^{2i\sigma_{js}}-1}{2ik} \\
&\quad + e^{is(\phi+\psi)}\sum_{j=s}^{\infty} (2j+1)(-1)^{j} d_{ss}^{j}(\theta)\frac{e^{2i\sigma_{js}}-1}{2ik} \\
&= e^{is(\phi-\psi)}\sum_{j=s}^{\infty} (2j+1) d_{-ss}^{j}(\theta)\frac{e^{2i\sigma_{js}}-1}{2ik} \\
&\quad +(-1)^{s} e^{is(\phi+\psi)}\sum_{j=s}^{\infty} (2j+1) d_{-ss}^{j}(\pi-\theta)\frac{e^{2i\sigma_{js}}-1}{2ik} \\
&= (-1)^{s} f_s^{-}(\phi, \theta, \psi) + f_s^{+}(\phi, \theta, \psi).
\end{aligned} \tag{28}
$$

In principle we now have all the necessary ingredients to carry out a partial-wave analysis of quantum scattering in the Atiyah–Hitchin metric. In practice we still have to evaluate F_{js} and G_{js} for large kr more accurately than (18) and (20). This can be done numerically using the asymptotic expansion given in ref. [11]. Thus we obtain asymptotic states

$$
\frac{F_{js}(kr)}{r} \quad \text{and} \quad \frac{G_{js}(kr)}{r},
$$

with respect to which the solutions of eq. (10) can be analysed. To do this we have to study the eqs. (15) numerically. The details of this will be the topic of the subsequent sections, but in all cases we will generate the values for a, b and c simultaneously by integrating eq. (3) numerically. This can only be done reliably by integrating ourtwards from the bolt. We therefore have to pay particular attention to the behaviour of eq. (15) near the bolt in order to impose the right initial conditions there.

Our partial-wave calculations will provide us with certain elements of the S-matrix. In order to extract information about the differential cross sections of various processes we then have to carefully take into account the underlying "free" dynamics in Taub-NUT space. Our treatment here is analogous to the discussion of modified Coulomb potentials in atomic and nuclear physics and we simply state the formulae for the scattering amplitudes derived, for example, in ref. [12]. We introduce the notation S^j for the restriction of the S-matrix to the subspace characterised by the total angular momentum j, and denote its matrix elements by $S^j_{s\bar{s}}$.

The scattering amplitude $f_{ss}(\theta)$ for elastic scattering of pure monopoles ($s = 0$) or dyons ($s \geqslant 1$) can be expressed in terms of the diagonal elements of S^j:

$$f_{ss} = f_s^{\mathrm{TN}} + \hat{f}_{ss}, \tag{29}$$

where $\hat{f}_{ss}(\theta)$ is given by the partial-wave expansion

$$\hat{f}_{ss}(\phi, \theta, \psi) = \frac{1}{k} \sum_{j=2}^{\infty} (2j+1)\, \mathrm{e}^{2i\sigma_{js}} \left(\frac{S^j_{ss} - 1}{2i} \right) \psi^+_{jss}(\phi, \theta, \psi).$$

Because of the symmetry of the $\psi^+_{jss}(\phi, \theta, \psi)$ the whole scattering amplitude is automatically invariant under (12). The decomposition

$$\psi^+_{jss} = D^j_{-ss} + (-1)^j D^j_{ss}$$

allows us to split $\hat{f}_{ss}$ into two parts,

$$\hat{f}_{ss} = (-1)^s \hat{f}^-_{ss} + \hat{f}^+_{ss},$$

such that $(-1)^s \hat{f}^-_{ss}$ is an infinite sum involving only the D^j_{-ss} and $\hat{f}^+_{ss}$ is an infinite sum involving only the D^j_{ss}. We then have

$$f_{ss} = (-1)^s \left(f^-_s + \hat{f}^-_{ss} \right) + \left(f^+_s + \hat{f}^+_{ss} \right).$$

For $s = 0$ we retain both parts of the scattering amplitude. But, by the argument given in the discussion of the Taub-NUT scattering amplitude, we keep only $(f^-_s + \hat{f}^-_{ss})(\phi, \theta, \psi)$ if we want to calculate the differential cross section for scattering of dyons of a definite electric charge into an interval $(\theta, \theta + \mathrm{d}\theta)$. As in the Taub-NUT approximation we obtain the elastic differential cross section by squaring (29) and averaging over ψ. For pure monopoles, this average is trivial, since f_{00} is independent of ψ. Thus

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{00} = |f_{00}|^2,$$

while for dyons we have

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right)_{ss} = \frac{1}{2\pi} \int \mathrm{d}\psi \left| f^-_s + \hat{f}^-_{ss} \right|^2.$$

This expression is complicated by the interference between the long-range and the short-range part of the scattering amplitude. The inelastic processes involve no such interference. The scattering amplitude for a process in which dyons of relative electric charge s turn into dyons of relative electic charge $\tilde{s} \neq s$ is given

$$f_{s\tilde{s}}(\phi, \theta, \psi) = \hat{f}_{s\tilde{s}}(\phi, \theta, \psi) = \frac{1}{2i\sqrt{\tilde{k}k}} \sum_{j=s}^{\infty} (2j+1)\, e^{i\sigma_{js}} S^j_{s\tilde{s}}\, e^{i\sigma_{j\tilde{s}}} \psi^+_{j\tilde{s}s}(\phi, \theta, \psi) \quad (30)$$

and the corresponding inelastic differential cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{s0} = \frac{\tilde{k}}{k} \left| \hat{f}_{s0} \right|^2, \qquad \left(\frac{d\sigma}{d\Omega}\right)_{s\tilde{s}} = \frac{\tilde{k}}{k} \left| \hat{f}_{s\tilde{s}} \right|^2,$$

if $\tilde{s} \neq 0$. The total cross sections are obtained from the differential cross sections by integrating over $\cos\theta$ and ϕ as usual.

3. Single-channel scattering

We investigate the four single-channel problems characterized by

$$(j, s) = (0, 0), (1, 1), (2, 1), (3, 2).$$

Physically, the first of these describes pure monopole scattering with no (relative) angular momentum. There are no bound states in this channel so scattering takes place at all positive energies. $(j, s) = (1, 1)$ and $(2, 1)$ corresponds to scattering of dyons with relative electric charge 1 and total angular momentum 1 and 2 respectively. There are bound states with energies $0 \leqslant \varepsilon \leqslant 0.25$ in these channels [7] and the continuum begins at $\varepsilon = 0.25$. Finally $(j, s) = (3, 2)$ refers to dyons having total angular momentum 3 whose electric charge differs by 2 units. Because of the selection rule mentioned in sect. 2 such dyons cannot turn into pure monopoles and consequently they can form bound states at energies $\varepsilon \leqslant 1$. These were overlooked in refs. [5,7]. Scattering takes place above that energy. The equations for the radial part of the wave function obtained from (15) read

$$\frac{1}{abcf} \frac{d}{dr} \left(\frac{abc}{f} \frac{du_{js}}{dr} \right) + \left(\varepsilon - \frac{l}{a^2} - \frac{m}{b^2} - \frac{n}{c^2} \right) u_{js} = 0, \quad (31)$$

where

$$(j, s) = \begin{Bmatrix} (0, 0) \\ (1, 1) \\ (2, 1) \\ (3, 2) \end{Bmatrix} \Rightarrow (l, m, n) = \begin{Bmatrix} (0, 0, 0) \\ (0, 1, 1) \\ (4, 1, 1) \\ (4, 4, 4) \end{Bmatrix}. \quad (32)$$

In all cases the behaviour near the bolt can be found by approximating

$$a \simeq 2h, \qquad b \simeq \pi, \qquad c \simeq -\pi,$$

where $h = r - \pi$. The radial equation then simplifies to Bessel's equation near the bolt and the unique solution regular at the bolt is the usual Bessel function

$$J_{p/2}\left(\sqrt{\varepsilon - \frac{j(j+1) - p^2}{\pi^2}}\, h\right).$$

$p \equiv \sqrt{l}$, whose dependence on (j, s) can be read off from eq. (32), is always an even integer. Using the well-known values of Bessel functions of integer order and their derivatives for small h, numerical integration from slightly outside the bolt $h = 10^{-10}$ is straightforward. Asymptotically the solution may be expressed as a linear combination of the regular and the irregular solution of (17), with the appropriate values for (j, s). We define and calculate the relative phase shift δ_{js} by requiring

$$u_{js}(r) \propto \frac{F_{js}(kr)}{r} \cos \delta_{js} - \frac{G_{js}(kr)}{r} \sin \delta_{js}. \tag{33}$$

Surprisingly, δ_{00} is found to be zero at *all* energies (to five decimal places, which is the accuracy that we can numerically achieve with our method). The phase shifts for the other channels are not trivial, and we plot them against ε in fig. 2. There are two plots of δ_{11} with different energy ranges. In the first plot the range is chosen so that the interesting behaviour near $\varepsilon = 0.5$ is clearly visible. Note that this is precisely the energy for which the parameter η describing the strength of the Coulomb potential is zero.

A qualitative analysis suggests that the Taub-NUT approximation should be good for elastic scattering of pure monopoles below the dyon production threshold in all partial waves except the s-wave. Although all partial waves contribute to this process, the expectation value of ξ_1^2 cannot vanish for $j \neq 0$ and $s = 0$. Hence there will always be a repulsive "centrifugal" term $1/a^2$ which prevents the monopoles from reaching the region near the bolt where the Atiyah–Hitchin metric differs substantially from the Taub-NUT metric. For $j = 0$ there is no such term and one would expect the phase of the Atiyah–Hitchin radial wave function to be shifted substantially relative to the phase of the Taub-NUT wave function. This is why it is remarkable that this phase shift turns out to vanish at all energies.

At high energy, i.e. short wavelengths, the WKB approximation gives a geometrical interpretation of this fact. While the high energy behaviour of phase shifts is not directly relevant for a partial-wave analysis of low-energy scattering, the application of the WKB method for scattering on a riemannian manifold is interesting, because it exhibits the interplay between geometric and quantum

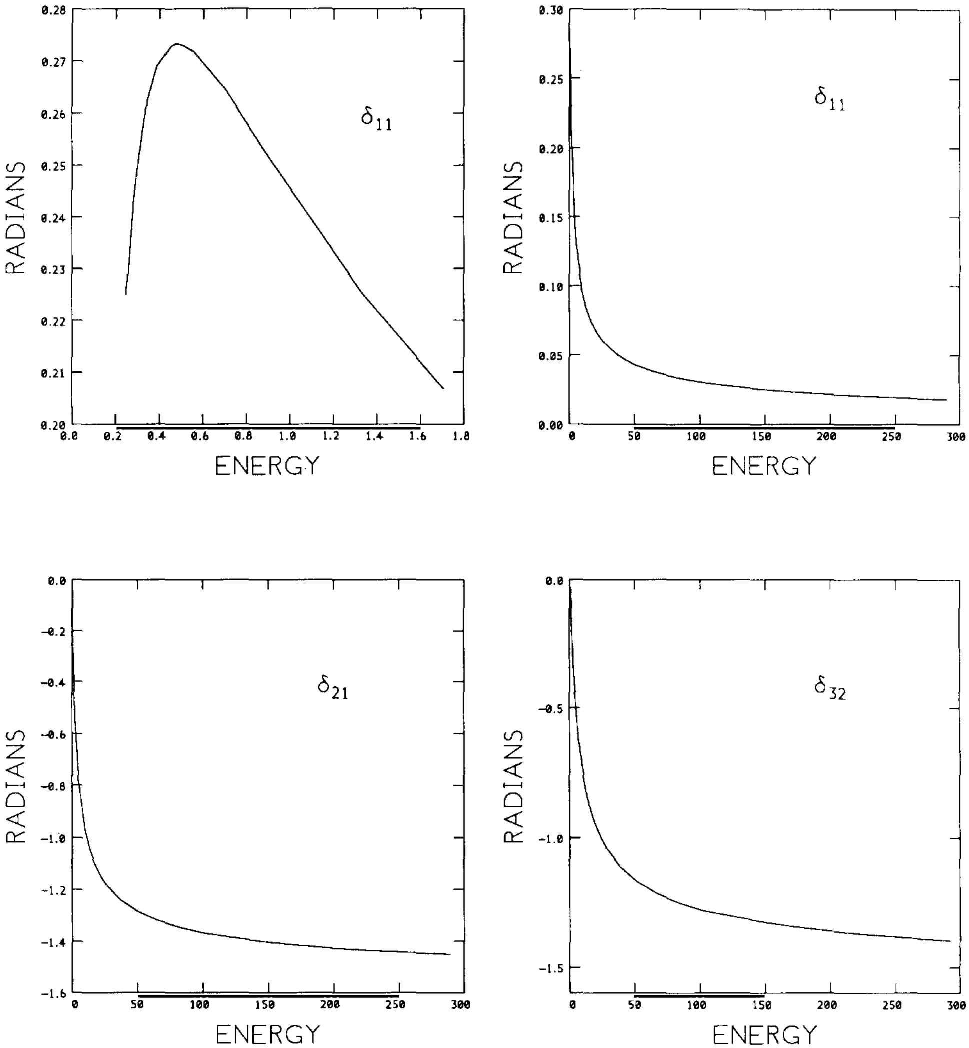

Fig. 2. Phase shifts for single-channels as a function of the energy ε. δ_{11} is shown on two different energy scales.

phenomena. Consider first the Taub-NUT radial equation for the $j=0$ channel. Eq. (31) with $(l, m, n)=(0, 0, 0)$ and the Taub-NUT expressions for a, b and c can be brought into standard form by introducing a new coordinate τ via

$$abc\, \mathrm{d}\tau = f\, \mathrm{d}r,$$

or

$$\tau(r) = \int_{r_0}^{r} \frac{f}{abc} \mathrm{d}\tilde{r}. \tag{34}$$

We fix the constant of integration by setting $r_0 = 2$ and find that

$$\tau(r) = \frac{1}{4}\left(1 - \frac{2}{r}\right),$$

so that τ ranges from $-\infty$ to 0 as r varies from 0 to 2 and τ ranges from 0 to $\frac{1}{4}$ as r varies from 2 to ∞. The radial equation now reads

$$\left(\frac{d^2}{d\tau^2} + \varepsilon(abc)^2\right)u_{00} = 0. \tag{35}$$

This looks like a one-dimensional Schrödinger equation with "potential" $-\varepsilon(abc)^2$ at zero energy. For $\varepsilon \neq 0$ this "potential" is zero only at $r = 2$ ($\tau = 0$). Thus classical trajectories in the potential at zero energy which start outside $r = 2$ will reach $r = 2$, stop and escape to $r = \infty$ ($\tau = \frac{1}{4}$). This is the classical motion we should keep in mind when applying the WKB method, although it differs from the geodesic motion in Taub-NUT space, which has a singularity at $r = 2$. The quasi-classicality condition requires

$$\left|\frac{d(1/p)}{d\tau}\right| \ll 1,$$

where

$$p \equiv \sqrt{\varepsilon(abc)^2} = \sqrt{\varepsilon}\,\frac{\sqrt{\tau}}{\left(\frac{1}{4} - \tau\right)^2}$$

so that for large ε and τ not too close to 0 we have indeed

$$\left|\frac{d(1/p)}{d\tau}\right| \propto \frac{1}{\sqrt{\varepsilon}} \ll 1.$$

Applying the standard WKB formulae to the radial wave function one finds

$$u_{00}^{\text{WKB}}(r) = \frac{C}{\sqrt{p(\tau(r))}}\sin\left(\int_{\tau_0}^{\tau(r)} p(\tilde{\tau})\, d\tilde{\tau} + \frac{\pi}{4}\right).$$

Here C is some normalisation constant and τ_0 is the classical turning point, i.e. $\tau_0 = 0$ for the Taub-NUT metric. Using the definition of $\tau(r)$ and $k = \sqrt{\varepsilon}$ we can write

$$u_{00}^{\text{WKB}}(r) = \frac{C}{\sqrt{p(\tau(r))}}\sin\left(\int_2^r kf(\tilde{r})\, d\tilde{r} + \frac{\pi}{4}\right). \tag{36}$$

We see that, up to the $\pi/4$ stemming from the standard connection formulae in semiclassical analysis, the WKB phase is given by the proper radial distance from any point with radial coordinate r to the singularity ($r = 2$) times the wavenumber. For the Taub-NUT metric the integral can be worked out explicitly

$$\int_2^r f(\tilde{r})\, \mathrm{d}\tilde{r} = \sqrt{r(r-2)} - \ln\left(r - 1 + \sqrt{r(r-2)}\right)$$
$$\approx r - 1 - \ln 2r \quad \text{for large } r. \tag{37}$$

So the WKB phase is

$$\phi_{\mathrm{TN}}^{\mathrm{WKB}}(r) = k\int_2^r f(\tilde{r})\, \mathrm{d}\tilde{r} + \tfrac{1}{4}\pi$$
$$\approx k(r - 1 - \ln 2r) + \tfrac{1}{4}\pi \text{ for large } r. \tag{38}$$

The exact partial-wave analysis of the Taub-NUT scattering problem yields the Coulomb phase for $j = 0$ and $\eta = k$:

$$\phi_{\mathrm{C}}(r) = kr - k \ln 2kr + \arg \Gamma(1 + ik)$$

which is obviously not equal to (38). However for large k one can use Stirling's formula for the Γ-function,

$$\arg \Gamma(1 + ik) \sim -k + k \ln k + \tfrac{1}{4}\pi + \mathrm{O}\left(\frac{1}{k}\right).$$

Then

$$\phi_{\mathrm{C}}(r) \sim k(r - 1 - \ln 2r) + \tfrac{1}{4}\pi.$$

Thus the WKB approximation (38) agrees with the exact phase for large k, as one expects.

Turning to the radial equation in the Atiyah–Hitchin manifold, we follow the same steps, but now the integral defining $\tau(r)$ diverges logarithmically as $r \downarrow \pi$. However, we need not bother with an exact definition of τ since it is only used in intermediate steps. We read off from eq. (35) (now with the Atiyah–Hitchin coeffients a, b, c) that the turning point is the value of τ corresponding to $r_0 = \pi$. But r_0 is all we need to evaluate the WKB wave function, which is now

$$u_{00}^{\mathrm{WKB}}(r) = \frac{C}{\sqrt{p(\tau(r))}} \sin\left(\int_\pi^r kf(\tilde{r})\, \mathrm{d}\tilde{r} + \frac{\pi}{4}\right). \tag{39}$$

The integral is again the proper radial distance from any point with the radial coordinate r to the bolt, where $r = r_0 = \pi$. In the Atiyah–Hitchin case the integration has to be done numerically, but the phase is found to be identical to the Taub-NUT result (38) for sufficiently large r: the values agree to seven decimal places at $r = 200$. In the WKB approximation the absence of a relative phase shift is thus seen to be a consequence of the fact that a point in Taub-NUT space with large radial coordinate r has the same proper radial distance from the singularity ($r = 2$) as a point with the same radial coordinate r in the Atiyah–Hitchin space has from the bolt ($r = \pi$).

Such a geometrical interpretation of the scattering data is a common feature of scattering by metrics. The (classical) relative motion of two vortices can also be modelled by a curved riemannian manifold, namely a rounded cone. In ref. [13] it was shown that the WKB approximation to the quantum scattering of two vortices gives an s-wave phase shift which is proportional to the length deficit of the rounded cone relative to the standard cone with the same opening angle, i.e. the difference in geodesic distance to the apex in each case.

We have only managed to explain the absence of a relative phase shift for high energies. Comparison of ϕ_{C} with $\phi_{\mathrm{TN}}^{\mathrm{WKB}}$ shows that the WKB approximation is not exact at lower energies. It remains an open question why the relative phase shift should vanish at *all* energies.

For the other three single-channel problems we can again understand the behaviour for large ε in terms of the WKB approximation, but now we have to keep track more carefully of the classical turning points. The equation generalising eq. (35) reads

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} + (abc)^2\left(\varepsilon - \frac{l}{a^2} - \frac{m}{b^2} - \frac{n}{c^2}\right)\right)u_{js} = 0, \tag{40}$$

where a, b and c may stand for either the Atiyah–Hitchin or the Taub-NUT expression and $\tau(r)$ is defined as before. For large but fixed r we can certainly choose ε so large that the "potential"

$$-(abc)^2\left(\varepsilon - \frac{l}{a^3} - \frac{m}{b^2} - \frac{n}{c^2}\right)$$

is negative, but now it may have zeros other than r_0, giving rise to "classical turning points" r_{t} whose precise location will depend on ε. The expression for the WKB phase generalising (38) is

$$\phi_{js}^{\mathrm{WKB}}(r) = \int_{r_{\mathrm{t}}}^{r} \sqrt{\varepsilon - \frac{l}{a^2} - \frac{m}{b^2} - \frac{n}{c^2}}\, f\, \mathrm{d}\tilde{r} + \frac{\pi}{4}. \tag{41}$$

By carefully estimating the energy dependence of r_t it is possible to derive the high-energy limit of this expression analytically and to find an asymptotic expansion of the phaseshift in powers of $\varepsilon^{-1/2}$. For $(j, s) = (1, 1)$ we find the following asymptotic formula for both the Atiyah–Hitchin ($r_0 = \pi$) and the Taub-NUT ($r_0 = 2$) metric

$$\phi_{js}^{\mathrm{WKB}}(r) \approx \sqrt{\varepsilon}\int_{r_0}^{r} f\,\mathrm{d}\tilde{r} + \frac{\pi}{4} - \frac{1}{2\sqrt{\varepsilon}}\int_{r_0}^{r}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) f\,\mathrm{d}\tilde{r}.$$

Hence, for $\varepsilon \to \infty$, the leading term in $\varepsilon^{-1/2}$ for the phase shift δ_{11} comes from

$$-\frac{1}{2\sqrt{\varepsilon}}\int_{r_0}^{r}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) f\,\mathrm{d}\tilde{r}.$$

We calculate this integral numerically for the Taub-NUT and the Atiyah–Hitchin space and find that the results differ by 0.612. Thus we get an asymptotic formula for the phase shift δ_{11},

$$\delta_{11} \approx \frac{0.612}{2\sqrt{\varepsilon}}.$$

For $(j, s) = (2, 1)$ and $(3, 2)$ a similar analysis shows that the asymptotic form of δ_{21} and δ_{32} for high energy is

$$-\frac{\pi}{2} + \mathrm{O}\left(\frac{1}{\sqrt{\varepsilon}}\right).$$

These asymptotic expressions approximate the high-energy part of the graphs shown in fig. 2 very well.

We can now use formula (29) to estimate some elastic cross sections. Consider the case of pure monopoles. Because $\delta_{00} = 0$ at all energies the first correction to the cross section calculated in the Taub-NUT approximation stems from phase shift of the partial wave with $j = 2$. In sect. 4 we will show that this phase shift is indeed small at low energies ($\varepsilon \leqslant 0.7$) so that in this energy regime the cross section is to a good approximation given by the Taub-NUT expression (25). Note that for small k the interference in this cross section term is slowly varying and would be important in an experimental check. This is the opposite situation from the standard Coulomb problem where the low-energy limit corresponds to the semiclassical limit in which the interference term becomes rapidly oscillating.

At low energy the elastic cross section for dyons of relative electric charge 1 can similarly be estimated by neglecting higher partial waves. δ_{11} is non-trivial so one should keep at least the lowest partial wave. Numerically we find that for $\varepsilon \to 0.25$, δ_{11} tends to some finite value ≈ 0.22 but δ_{21} tends to zero. Thus we keep only the first term in the expression for $\hat{f}_{11}^-$ to estimate the scattering amplitude at low energy:

$$\left(f_1^- + \hat{f}_{11}^-\right)(\phi, \theta, \psi) = f_1^-(\phi, \theta, \psi) - \frac{3}{k} e^{2i\sigma_{11}} e^{i\delta_{11}} \sin \delta_{11} e^{i(\phi-\psi)} \frac{(1-\cos\theta)}{2}$$

The resulting cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{11} = \frac{1}{4}\left(1 + \frac{1}{4k^2}\right)^2 \frac{1}{\sin^4\theta/2}$$
$$- \frac{3}{k}\left(1 + \frac{1}{4k^2}\right) \sin \delta_{11} \cos\left(\delta_{11} + \eta \ln \sin^2\frac{\theta}{2} - \arctan\frac{1}{\eta}\right)$$
$$+ \frac{9}{k^2} \sin^2\delta_{12} \sin^4\frac{\theta}{2}. \tag{42}$$

The interference term is interesting because it depends on the sign on δ_{11} and the last term shows that the Atiyah–Hitchin metric predicts a cross section for scattering in the backward directions ($\theta = \pi$) that is significantly enhanced relative to Taub-NUT cross section.

The phase shift δ_{32} does not give the main correction to the Taub-NUT cross section for scattering of dyons with relative electric charge 2. This stems from a two-channel problem to which we now turn.

4. Multi-channel scattering

A typical two-channel problem arises in the $j = 2$ sector. We will use it to explain the general formalism for multi-channel scattering and to illustrate the qualitatively new features that occur when channels which were uncoupled in the Taub-NUT approximation become coupled. In accordance with the remarks in sect. 2 we seek solutions to the Schrödinger equation (6) of the form

$$\Psi_0(r, \phi, \theta, \psi) = u_{20}(r)\psi_{200}^+(\phi, \theta, \psi) + u_{22}(r)\psi_{220}^+(\phi, \theta, \psi), \tag{43}$$

if we are interested in processes where the incoming particles are pure monopoles, and

$$\Psi_2(r, \phi, \theta, \psi) = u_{20}(r)\psi_{202}^+(\phi, \theta, \psi) + u_{22}(r)\psi_{222}^+(\phi, \theta, \psi) \tag{44}$$

if the colliding particles are initially dyons of relative electric charge 2. For definiteness let us consider the first of these possibilities. The system of second order coupled differential equations for the radial wave functions corresponding to (15) does not depend on this choice, but the interpretation of the S-matrix elements in terms of scattering amplitudes does, as eq. (30) shows. Using the formulae (13) we find that the system of differential equations (15) takes the form

$$\frac{r}{ab^2c}\frac{\mathrm{d}}{\mathrm{d}r}\left(acr\frac{\mathrm{d}u_{20}}{\mathrm{d}r}\right)+\left(\varepsilon-\frac{3}{a^2}-\frac{3}{b^2}\right)u_{20}+\sqrt{3}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)u_{22}=0,$$

$$\frac{r}{ab^2c}\frac{\mathrm{d}}{\mathrm{d}r}\left(acr\frac{\mathrm{d}u_{22}}{\mathrm{d}r}\right)+\left(\varepsilon-\frac{1}{a^2}-\frac{1}{b^2}-\frac{4}{c^2}\right)u_{22}+\sqrt{3}\left(\frac{1}{b^2}-\frac{1}{a^2}\right)u_{20}=0. \quad (45)$$

Since the incoming particles are pure monopoles, we have to distinguish between two energy regimes, below and above the threshold for dyon production $\varepsilon = 1$. We first look at the scattering problem for $\varepsilon \geqslant 1$. Then both u_{20} and u_{22} will be oscillatory for large r. Asymptotically we may neglect exponentially small terms, the equations decouple and we get two equations of the type (16) for $h_{js}(r) = ru_{js}(r)$ with the appropriate values of j and s. We set

$$k_0 = \sqrt{\varepsilon}, \quad k_2 = \sqrt{\varepsilon - 1}$$

as well as

$$\eta_0 = k_0, \quad \eta_2 = k_2 - \frac{1}{k_2}.$$

It is also convenient to introduce the notation

$$U(r) \equiv \begin{pmatrix} u_{20}(r) \\ u_{22}(r) \end{pmatrix}.$$

We then try to find solutions that behave asymptotically like

$$U(r) \approx \frac{1}{r}\begin{pmatrix} \dfrac{\alpha_0}{\sqrt{k_0}}F_{20}(k_0 r) + \dfrac{\beta_0}{\sqrt{k_0}}G_{20}(k_0 r) \\ \dfrac{\alpha_2}{\sqrt{k_2}}F_{22}(k_2 r) + \dfrac{\beta_2}{\sqrt{k_2}}G_{22}(k_2 r) \end{pmatrix} \quad (46)$$

for constant vectors

$$A = \begin{pmatrix} \alpha_0 \\ \alpha_2 \end{pmatrix}, \quad B = \begin{pmatrix} \beta_0 \\ \beta_2 \end{pmatrix}.$$

We will see explicitly that there are two linearly independent solutions which are regular at the origin. This will give us two sets of vectors,

$$A^0,\ B^0 \quad \text{and} \quad A^1,\ B^1$$

and we can ask for the matrix R that satisfies

$$RA^0 = B^0, \qquad RA^1 = B^1.$$

This is the restriction of the so-called reactance matrix **R** to the subspace of asymptotic states defined by $j = 2,\ s = 0,\ 2$. **R** is related to the **S**-matrix via a Cayley transformation

$$\mathbf{S} = (\mathbf{1} - i\mathbf{R})(\mathbf{1} + i\mathbf{R})^{-1}.$$

It follows from general arguments of formal scattering theory [14] that **S** is unitary and symmetric. Hence **R** is real and symmetric. We will explicitly check the symmetry of R later. In order to find the two solutions numerically we have to analyse (45) near the bolt. Setting again $h = r - \pi$ and using the approximations

$$a = 2h, \quad b = \pi, \quad c = -\pi,$$

we find that near the bolt the term $(b^{-2} - c^{-2})$ vanishes and we can decouple the equations using the transformation

$$T = \frac{1}{2}\begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}.$$

The differential equation for the pair

$$V(r) \equiv \begin{pmatrix} v_{20}(r) \\ v_{22}(r) \end{pmatrix} = TU(r) \tag{47}$$

is then

$$\frac{r}{ab^2c}\frac{\mathrm{d}}{\mathrm{d}r}\left(acr\frac{\mathrm{d}v_{20}}{\mathrm{d}r}\right) + \left(\varepsilon - \frac{3}{b^2} - \frac{3}{c^2}\right)v_{20} + \sqrt{3}\left(\frac{1}{b^2} - \frac{1}{c^2}\right)v_{22} = 0,$$

$$\frac{r}{ab^2c}\frac{d}{dr}\left(acr\frac{\mathrm{d}v_{22}}{\mathrm{d}r}\right) + \left(\varepsilon - \frac{4}{a^2} - \frac{1}{b^2} - \frac{1}{c^2}\right)v_{22} + \sqrt{3}\left(\frac{1}{b^2} - \frac{1}{c^2}\right)v_{20} = 0. \tag{48}$$

Near the bolt this simplifies to

$$\frac{\mathrm{d}^2v_{20}}{\mathrm{d}h^2} + \frac{1}{h}\frac{\mathrm{d}v_{20}}{\mathrm{d}h} + \left(\varepsilon - \frac{6}{\pi^2}\right)v_{20} = 0,$$

$$\frac{\mathrm{d}^2v_{22}}{\mathrm{d}h^2} + \frac{1}{h}\frac{\mathrm{d}v_{22}}{\mathrm{d}h} + \left(\varepsilon - \frac{2}{\pi^2} - \frac{1}{h^2}\right)v_{22} = 0. \tag{49}$$

We can easily write down two linearly independent solutions of this system, regular at $h=0$, in terms of Bessel functions J_n of integer order n. Labelling the solutions of eq. (48) by the order of the Bessel function to which they reduce at the bolt, we have, for small h.

$$V^0 \approx \begin{pmatrix} J_0\left(\sqrt{\varepsilon - \frac{6}{\pi^2}h}\right) \\ 0 \end{pmatrix}, \qquad V^1 \approx \begin{pmatrix} 0 \\ J_1\left(\sqrt{\varepsilon - \frac{2}{\pi^2}h}\right) \end{pmatrix}. \tag{50}$$

Using the well-known values of the Bessel functions and their derivatives near $h=0$ we can impose these as initial conditions slightly outside the bolt (in practice at $h=10^{-10}$) and integrate (48) outwards. Having obtained the two independent solutions V^0 and V^1, we translate their asymptotic behaviour into that of the U^i by inverting the linear transformation (47).

As in the case of potential scattering one shows the symmetry of R by considering the generalized wronskian

$$W(r) \equiv u_{20}^0 \frac{\mathrm{d}u_{20}^1}{\mathrm{d}r} - u_{20}^1 \frac{\mathrm{d}u_{20}^0}{\mathrm{d}r} + u_{22}^0 \frac{\mathrm{d}u_{22}^1}{\mathrm{d}r} - u_{22}^1 \frac{\mathrm{d}u_{22}^0}{\mathrm{d}r}.$$

We find, using eq. (45),

$$\frac{\mathrm{d}W}{\mathrm{d}r}(r) = -\frac{1}{r}\left(\frac{b}{a} + \frac{b}{c} + 1 - \frac{b^2}{ac}\right) W(r).$$

As a result of the boundary conditions at the bolt we have

$$W(\pi) = 0,$$

and hence

$$W(\infty) = 0.$$

Expressing W in terms of the asymptotic form of the wave functions

$$\frac{1}{r}\begin{pmatrix} \frac{1}{\sqrt{k_0}}\sin(k_0 r - \eta_0 \ln 2k_0 r + \sigma_{20}) + \frac{R_{00}}{\sqrt{k_0}}\cos(k_0 r - \eta_0 \ln 2k_0 r + \sigma_{20}) \\ \frac{R_{02}}{\sqrt{k_2}}\cos(k_2 r - \eta_2 \ln 2k_2 r + \sigma_{22}) \end{pmatrix},$$

$$\frac{1}{r}\begin{pmatrix} \frac{R_{20}}{\sqrt{k_0}}\cos(k_0 r - \eta_0 \ln 2k_0 r + \sigma_{20}) \\ \frac{1}{\sqrt{k_2}}\sin(k_2 r - \eta_2 \ln 2k_2 r + \sigma_{22}) + \frac{R_{22}}{\sqrt{k_2}}\cos(k_2 r - \eta_2 \ln 2k_2 r + \sigma_{22}) \end{pmatrix}$$

we find that, in the normalisation we have chosen,

$$R_{02} = R_{20}.$$

We can therefore parametrize R by two eigenphases δ^+ and δ^- and one mixing angle ϵ as follows [12]. We write

$$R = O(\epsilon)\begin{pmatrix} \tan\delta^+ & 0 \\ 0 & \tan\delta^- \end{pmatrix} O(\epsilon)^{-1}$$

where O is the orthogonal matrix

$$O(\epsilon) = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix}.$$

The parameters δ^+, δ^- and ϵ are useful for translating the R-matrix into the S-matrix:

$$S = O(\epsilon)\begin{pmatrix} e^{2i\delta^+} & 0 \\ 0 & e^{2i\delta^-} \end{pmatrix} O(\epsilon)^{-1}.$$

They are also sometimes used in the literature to display the result of a partial-wave analysis, and we will follow that practice. Another expedient set of parameters are the so-called "bar" parameters:

$$\delta^+ + \delta^- = \bar{\delta}^+ + \bar{\delta}^-$$

$$\sin(\bar{\delta}^+ - \bar{\delta}^-) = \frac{\tan 2\bar{\epsilon}}{\tan 2\epsilon}, \qquad \sin(\delta^+ - \delta^-) = \frac{\sin 2\bar{\epsilon}}{\sin 2\epsilon}. \tag{51}$$

One finds $S_{02} = i\, e^{i(\bar{\delta}^+ + \bar{\delta}^-)} \sin 2\bar{\epsilon}$. Together with eq. (30) this shows that $\sin 2\bar{\epsilon}$ is a direct measure of the probability for dyon production in a collision of pure monopoles. We therefore calculate and plot $\bar{\epsilon}$ as well.

The parameters δ^+, δ^-, ϵ, $\bar{\epsilon}$ are displayed in fig. 3 as a function of the energy ε, which is dimensionless. The differential elastic cross section depends on the parameters δ^+, δ^- and ϵ in a fairly complicated way and moreover involves the interference with the Taub-NUT scattering amplitude. The total elastic cross section is always infinite due to the long-range Coulomb forces between monopoles. Thus we will not attempt to discuss elastic processes quantitatively but simply point out some noteworthy qualitative features of our plots. Firstly we notice that the energy at which the Coulomb potential vanishes plays a special role, as it did for the phase shift δ_{11}. Near $\varepsilon = 2.0$, δ^+, δ^- and ϵ have maxima. The second observation concerns the behaviour of $\bar{\epsilon}$ as one approaches the threshold energy $\varepsilon = 1.0$ for dyon production from above. $\bar{\epsilon}$ decreases rapidly but tends to a

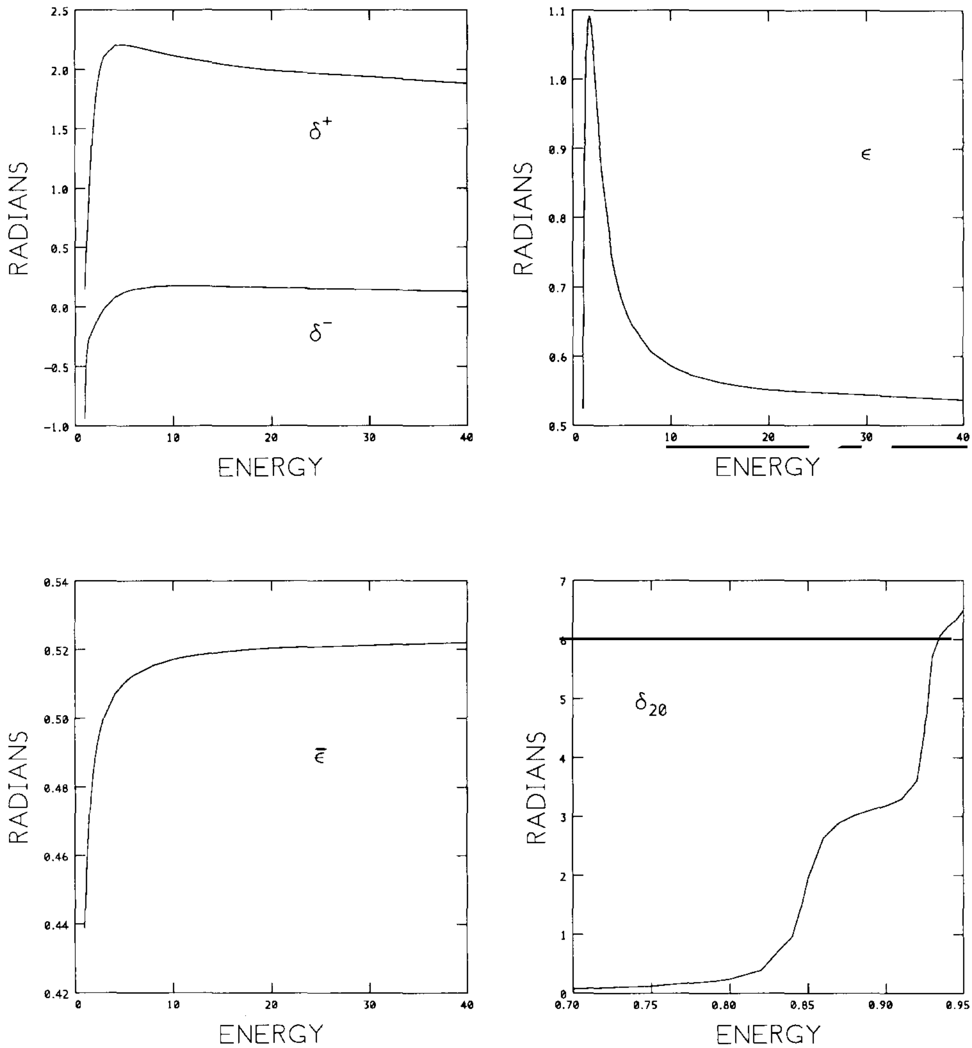

Fig. 3. $j=2$ coupled problem. Parameters for S-matrix above threshold and phase shift for scattering of pure monopoles below threshold.

non-zero value at the threshold. This is the threshold behaviour one expects in coupled channel problems if both channels have Coulomb-like potentials [15].

For energies below the dyon production threshold u_{22} will be exponentially increasing or decreasing for sufficiently large r. Only the latter solution makes sense physically, but one cannot easily characterize it by its behaviour at the bolt. We find it numerically by an iterative method. The u_{20} part of that solution can then be compared with the corresponding solution of the Taub-NUT radial equation (16) and we obtain the relative phase shift δ_{20} as in sect. 3. δ_{20} is also

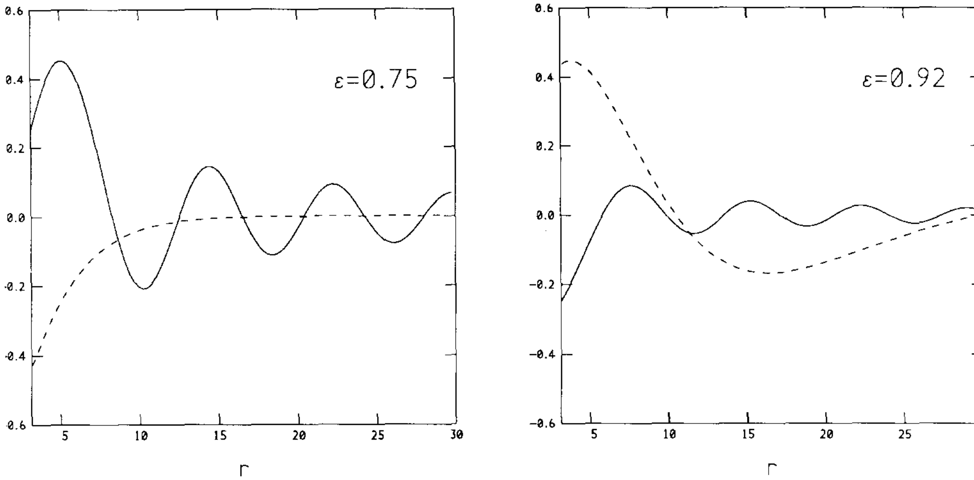

Fig. 4. Radial wave function for $j = 2$ coupled problem below threshold. u_{20} is shown with a full line, u_{22} is shown with a dashed line.

plotted in fig. 3 as a function of the energy. It tends to zero monotonically from above as $\varepsilon \to 0$ and is small ($\leqslant 0.1$) for $\varepsilon \leqslant 0.7$. For $\varepsilon > 0.8$, however, it shows a very interesting behaviour. As the energy increases from 0.8 to 0.9 the phase shift increases very rapidly by π, passing through $\pi/2$ at $\varepsilon \approx 0.85$. It then rises even more quickly, taking on the value $3\pi/2$ at about $\epsilon = 0.925$ and reaching 2π at $\varepsilon = 0.94$. This is accompanied by changes in the wave function displayed in fig. 4. Below $\varepsilon = 0.75$, u_{22} is a monotonically increasing function having some finite negative value at the bolt and tending to 0 for large r. At an energy of about 0.8, u_{22} developes a local minimum just outside the bolt which moves outwards as the energy increases further. Also, the amplitude of u_{22} relative to u_{20} increases, reaching a maximum at $\varepsilon \approx 0.85$. At $\varepsilon \approx 0.9$, u_{22} shows another local extremum, whose relative amplitude peaks at about $\varepsilon = 0.92$. The special values $\varepsilon = 0.85$ and $\varepsilon = 0.925$ are close to the energies of the two lowest bound states of the second equation in (45) with the coupling term removed. In fact, that Sturm–Liouville problem has infinitely many discrete eigenvalues which accumulate at $\varepsilon = 1.0$. The numerical results shown in fig. 3 suggest that the phase shift increases by π whenever the energy crosses one of these discrete eigenvalues and hence tends to ∞ as $\varepsilon \to 1.0$. This behaviour is typical of threshold resonances with Coulomb potentials in both channels. The phenomenon has been investigated in some detail for the case of potential scattering [15] and this analysis can presumably be adapted to our situation without much difficulty. We will not do this here, but make a qualitative remark instead. It is known that resonances are related to the time delay observed in the scattering of wavepackets [16]. On the other hand, such a time delay occurs also in the classical scattering of pure monopoles under certain conditions, described in refs. [2,5]. These scattering processes are modelled by

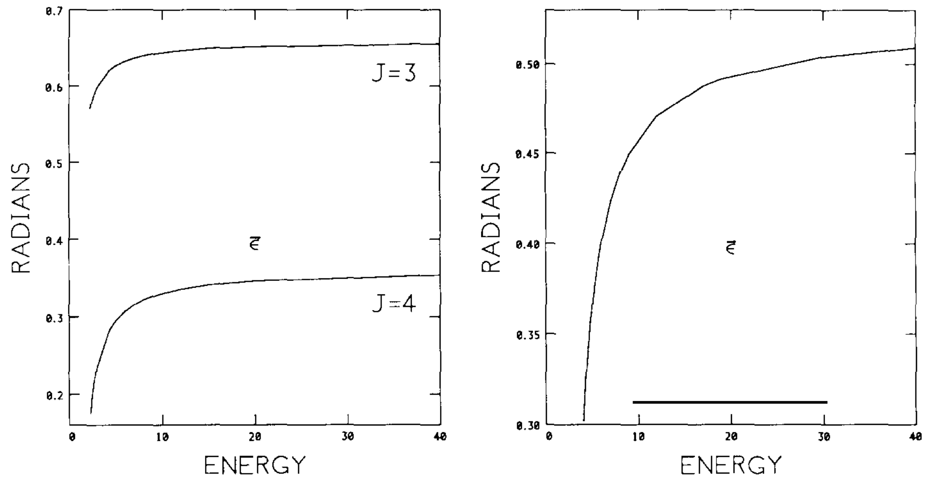

Fig. 5. $\bar{\epsilon}$-parameter for $j = 3$ and $j = 4$ (left) and $j = 5$ (right).

geodesics on the so-called Atiyah–Hitchin trumpet, a two-dimensional geodesic submanifold of M_2^0. It is asymptotic to a cone at one end and to a cylinder at the other. Geodesics crossing the "neck" of the trumpet and passing into the cylindrical region correspond to pure monopoles turning into dyons. But only for an impact parameter $b \leqslant 2$ will such geodesics continue infinitely far into the cylinder or, in the language of particle scattering, will the dyons escape to infinity. For $2 < b < \pi$ pure monopoles turn into dyons which move back-to-back in the line perpendicular to plane of the initial motion, but then return and turn into pure monopoles again. The time delay in such a process increases indefinitely as $b \downarrow 2$. It would be fascinating if one could relate this classical phenomenon to the threshold behaviour of the phase shift described above.

We have similarly analysed the coupled channel problems labelled by $3 \leqslant j \leqslant 5$ (cf. table 1). As we do not intend to give a detailed quantitative description of elastic cross sections we only report our findings concerning the inelastic cross sections and the threshold behaviour. All coupled channels display the threshold behaviour found in the $j = 2$ case. Below the threshold energy there is only one phaseshift and in all cases it shows qualitatively the same resonance behaviour as δ_{20}. In fig. 5 we show the variation of $\bar{\epsilon}$ with energy. The partial waves with $j = 3$ and $j = 4$ both contribute to scattering processes involving dyons of relative electric charge 1 and 3 so we have plotted the parameters $\bar{\epsilon}$ for these angular momenta together. The $j = 5$ partial waves give the leading contribution to the scattering of dyons of relative electric charges 2 and 4, and the corresponding parameter $\bar{\epsilon}$ is shown separately. Again we find that $\bar{\epsilon}$ tends to a non-zero value at the threshold. We can use that limit to estimate some inelastic total cross sections at the threshold energy.

Assuming $\tilde{s} > s$ we can use (30) and the orthogonality of the $\psi^{+}_{j\tilde{s}s}$ to write down the partial inelastic cross section for a process in which the relative electric charge changes from s to $\tilde{s}$,

$$Q^{j}_{s0} = 2\frac{\pi(2j+1)}{k^2}\left|S^{j}_{s0}\right|^2 \quad \text{if } \tilde{s} = 0$$

and, for $\tilde{s} \neq 0$,

$$Q^{j}_{s\tilde{s}} = \frac{\pi(2j+1)}{k^2}\left|S^{j}_{s\tilde{s}}\right|^2.$$

The total cross section for such a process is

$$Q_{s\tilde{s}} = \sum_{j \geqslant \tilde{s}} Q^{j}_{s\tilde{s}}.$$

Near the threshold, we can estimate the cross section for the production of dyons of relative electric charge 2 in a collision of pure monopoles by considering only

$$Q^{2}_{02} = \frac{5\pi}{\varepsilon}(\sin 2\bar{\epsilon})^2.$$

Taking $\bar{\epsilon} \approx 0.4$ at the threshold, we estimate $Q^{2}_{02} \approx 2\pi$. Recalling that our unit of length is the size of a monopole, we find that the inelastic cross section for production of dyons with relative electric charge 2 is roughly three times the cross sectional area of a monopole. It is instructive to compare this with the classical cross section of the special scattering processes modelled by geodesics on the Atiyah–Hitchin trumpet described above. There we saw that for an impact parameter $b \leqslant 2$ pure monopoles turn into dyons. This corresponds to a classical cross section of 4π, which is close to our quantum mechanical estimate. We can also estimate the angular distribution of the emitted dyons from the angular wave function multiplying S^{2}_{02} in the partial-wave expansion of the cross section (30). This is D^{j}_{20}, which is proportional to $\sin^2\theta$. Thus dyons emerge preferentially at right angles to the beam axis, just as in classical scattering of monopoles. We have not checked that the next partial cross section Q^{4}_{02} is much smaller than Q^{2}_{02}, because we did not calculate coupled problems with more than two channels. For the process in which the relative electric charge changes from 1 to 3, however, we can estimate both Q^{3}_{13} and Q^{4}_{13}. We find, near threshold, that

$$Q^{3}_{13} \approx 3\pi, \qquad Q^{4}_{13} \approx 0.4\pi$$

so that only a modest increase of j from 3 to 4 reduces the partial cross section by a factor of 7. Finally we calculate, again at the threshold energy

$$Q^{5}_{24} \approx 0.9\pi.$$

5. Conclusion

In this paper we extended the work of ref. [5] on the moduli space approximation for the quantum scattering of BPS monopoles by considering the true moduli space of two monopoles (the Atiyah–Hitchin manifold) and not just the manifold modelling the asymptotic dynamics of two monopoles (the Taub-NUT space). Even before one does any detailed calculations it is clear from the different geometries of the two manifolds that a description of the scattering in terms of the Atiyah–Hitchin manifold will differ from one using the Taub-NUT approximation in that it predicts inelastic scattering. Using a partial-wave analysis we could give quantitative estimates of various elastic and inelastic scattering processes by considering only partial waves of the lowest contributing angular momentum. The most surprising result of our calculations is the vanishing of the s-wave phase shift in the elastic scattering of pure monopoles. This implies that the Taub-NUT cross section (25) for this process is an unexpectedly good approximation at energies below the dyon production threshold. But how accurate are our estimates at higher energy? One can give an estimate of the largest total angular momentum that contributes significantly to the scattering of pure monopoles at a given energy by requiring that the classical motion in the effective potential of the Taub-NUT radial equation (16) remains outside the region $r \leqslant R$ where the Atiyah–Hitchin metric differs significantly from the Taub-NUT metric. In sect. 2 we estimated $R = 5$ for the range of this core region. We then find for scattering of pure monopoles at the threshold energy $\varepsilon = 1.0$ that $j_{\max} = 3$. If the colliding particles are dyons of relative electric charge 1, a similar analysis at the threshold of inelasticity $\varepsilon = 2.25$ gives $j_{\max} = 5$. Indeed we saw that the $j = 4$ contribution to the inelastic cross section is already much smaller than the $j = 3$ contribution. For the scattering of dyons with relative electric charge 2 we find, at the threshold $\varepsilon = 4.0$, $j_{\max} = 7$. We conclude that our calculations provide the numerical information necessary to give a good quantitative description of the scattering of pure monopoles and of dyons with relative electric charge 1 at energies up to and slightly above the thresholds of inelasticity. For dyons of relative charge 2 we also calculated the dominant contributions to the scattering up to energies slightly above the threshold of inelasticity, but here one should expect significant corrections from the partial waves with $j = 6$.

More generally we believe that our discussion together with the work done in ref. [7] on bound states captures the essential qualitative features of quantum dynamics of monopoles in the moduli space approximation. These include the existence of infinitely many bound states of dyons, all of which are embedded in the continuum, the occurrence of elastic and inelastic scattering, and the Coulomb-like resonance behaviour of the elastic cross sections near the thresholds of inelasticity.

These phenomena are of interest beyond the specific question of monopole scattering. Scattering in the Atiyah–Hitchin manifold is interesting because it provides an example of scattering by a metric defined on a space that includes both spatial and internal parameters. This is different from standard problems in non-relativistic mechanics, where interactions are described by a potential defined on flat $\mathbb{R}^3$ (or several copies thereof for many body problems). While the standard partial wave formalism had to adapted carefully to take into account the non-trivial geometry of the configuration space (recall that the symmetrised, bosonic cross section for pure monopoles resulted from purely geometric requirements) the final results – bound state energies in ref. [7] and cross sections in this paper – show the type of phenomena that are familiar from quantum mechanics defined by standard hamiltonians. In this respect our calculations are relevant to the more general question of low energy dynamics of solitons as discussed in ref. [17]. More specifically they give hints for the discussion of the nuclear two body problem in the Skyrme model. This model treats nucleons as solitons in a classical field theory. Again one can truncate the field theory to a finite-dimensional lagrangian dynamical system defined on a manifold of collective coordinates for the two-skyrmion system [17]. This manifold is 12-dimensional and has a potential V as well as a riemannian metric defined on it. While the potential has been studied extensively, little attention has been paid to the metric. It has long been known that the kinetic energy of the two-skyrmion system is not just the sum of the kinetic energies of two free skyrmions. The extra terms have been studied for example in ref. [18] and were interpreted as velocity-dependent potentials. These were then projected into inter-nucleon potentials via projection techniques that involved some rather ad-hoc operator ordering. From the point of view adopted in this paper one should rather think of the kinetic energy in terms of the non-trivial metric on the space of collective coordinates. A natural and coordinate independent quantum hamiltonian is then given by $-\Delta + V$ where Δ is the covariant laplacian associated to the metric.

It is known that skyrmion and monopole dynamics even share certain qualitative features – such as 90 degree scattering in a head-on collison [19] – and it has been conjectured [7] that the manifold of collective coordinates for the two-skyrmion space contains a four-dimensional submanifold modelling the relative motion of two skyrmions in a fixed relative orientation ("attractive channel"), which is topically and metrically similar to the Atiyah–Hitchin manifold. While these possibilities are intriguing, the more immediate relevance of the discussion of BPS monopoles for skyrmion dynamics is to emphasize the general remarks made in ref. [17]. One cannot fully appreciate the the predictions of the Skyrme model for nucleon–nucleon interactions without a better understanding of the topological and geometrical structure of the space of collective coordinates for two-skyrmions.

I would like to warmly thank my supervisor Dr. N.S. Manton for suggesting the problem addressed here, for much useful advice and for a critical reading of the manuscript. I also thank T.M. Samols for many helpful discussions about monopole and vortex scattering.

An SERC research grant and a research studentship from Emmanuel College, Cambridge, are gratefully acknowledged.

Appendix A

We want to determine B_{js} so that

$$\Psi_{\mathrm{TN},s}(r, \phi, ,\theta, \psi) = \frac{1}{kr} \sum_{j \geqslant s} B_{js} \psi^+_{jss}(\phi, \theta, \psi) F_{js}(kr)$$

equals

$$\Phi_s = C_s\left(\Phi_s^+ + (-1)^s \Phi_s^-\right).$$

The definition of F_{js} and Φ_s involve constants C_{js} and C_s which are chosen so that we get the asymptotic behaviour given in eqs. (18) and (23). Using the asymptotic form of the hypergeometric function

$$F(a, b, u) \approx \frac{\Gamma(b)}{\Gamma(b-a)(-u)^a}\left(1 - \frac{a(a-b+1)}{u} + \frac{(-1)^a \, \mathrm{e}^u \Gamma(b-a)}{\Gamma(a) u^{b-2a}}\right)$$

we find

$$C_{js} = \frac{2^j \, \mathrm{e}^{-\pi\eta/2} |\Gamma(1+j+i\eta)|}{(2j+1)!}, \qquad C_s = \frac{\mathrm{e}^{\pi\eta/2} \Gamma(1+s+i\eta)}{\Gamma(2s+1) \, \mathrm{e}^{i\pi s/2}}. \tag{A.1}$$

Using the orthogonality of the ψ^+_{jss} we have

$$\frac{16\pi^2}{(2j+1)} B_{js} \frac{F_{js}(kr)}{kr} = \int \left(\psi^+_{jss}\right)^* \Phi_s \, \mathrm{d}\phi \, \mathrm{d}\psi \, \mathrm{d}\cos\theta. \tag{A.2}$$

This is an identity of two analytic functions. To find B_{js} we expand both sides into a power series in r around 0 and compute the coefficient of the lowest power of r. For the l.h.s. this is easy. One finds

$$\begin{aligned}
\text{l.h.s.} &= \frac{16\pi^2}{(2j+1)} B_{js} C_{js} (kr)^j \, \mathrm{e}^{ikr} \sum_{n=0}^{\infty} \frac{\Gamma(j+1+n+i\eta)\Gamma(2j+2)}{\Gamma(j+1+i\eta)\Gamma(2j+n+2)} \frac{(-2ikr)^n}{n!} \\
&= \frac{16\pi^2}{(2j+1)} B_{js} C_{js} k^j r^j + \text{higher powers of } r.
\end{aligned} \tag{A.3}$$

On the r.h.s. only the ϕ and ψ integration are trivial:

$$\text{r.h.s.} = 4\pi^2 C_s\Big((-1)^s I_{js}^- + (-1)^j I_{js}^+ \Big), \tag{A.4}$$

where

$$I_{js}^+ = \int_{-1}^{1} d_{ss}^j(\theta) k^s (r+z)^s \, e^{-ikz} F(s - i\eta, 2s+1, ik(r+z)) \, d\cos\theta,$$

$$I_{js}^- = \int_{-1}^{1} d_{-ss}^j(\theta) k^s (r-z)^s \, e^{ikz} F(s - i\eta, 2s+1, ik(r-z)) \, d\cos\theta. \tag{A.5}$$

To evaluate these integrals we write the Wigner functions in terms of the Jacobi polynomials and use Rodrigues' formula. Setting $x = \cos\theta$ we have

$$\begin{aligned} d_{ss}^j(\theta) &= \frac{1}{2^s}(1+x)^s P_{j-s}^{0,2s}(x) \\ &= \frac{(-1)^{j-s}}{2^j (j-s)!}(1+x)^{-s}\left(\frac{d}{dx}\right)^{j-s}\left[(1-x)^{j-s}(1+x)^{j+s}\right], \end{aligned}$$

$$\begin{aligned} d_{-ss}^j(\theta) = d_{s-s}^j(\theta) &= \frac{1}{2^s}(1-x)^s P_{j-s}^{2s,0}(x) \\ &= \frac{(-1)^{j-s}}{2^j (j-s)!}(1-x)^{-s}\left(\frac{d}{dx}\right)^{j-s}\left[(1+x)^{j-s}(1-x)^{j+s}\right]. \end{aligned} \tag{A.6}$$

Clearly

$$I_{js}^+ = (-1)^{j-s} I_{js}^-,$$

so that

$$\text{r.h.s.} = 8\pi^2 I_{js}^+ (-1)^j C_s,$$

Now

$$\begin{aligned} I_{j,s}^+ = &\frac{(-1)^{j-s}(kr)^s}{s^j (j-s)!} e^{ikr} \int_{-1}^{1} dx \, e^{-ikr(1+x)} \\ &\times F(s - i\eta, 2s+1, ikr(1+x)) \left(\frac{d}{dx}\right)^{j-s} (1-x)^{j-s}(1+x)^{j+s}. \end{aligned} \tag{A.7}$$

Integrating by parts $(j-s)$ times, changing variables to $y=1+x$, we get

$$I_{js}^{+}=\frac{(kr)^{s}}{s^{j}(j-s)!}e^{ikr}\int_{0}^{2}dy$$

$$\times\left(\frac{d}{dy}\right)^{j-s}\left[e^{-ikry}F(s-i\eta,2s+1,ikry)\right](2-y)^{j-s}y^{j+s}. \qquad (A.8)$$

Noting that the expression in square brackets depends on k, r, y only in the combination $ikry$ we write

$$\left[e^{-ikry}F(s-i\eta,2s+1,ikry)\right]=\sum_{n=0}^{\infty}a_{n}(ikry)^{n},$$

and find that the expansion of I_{js}^{+} in the powers of r begins as follows:

$$I_{js}^{+}=\left[\frac{k^{s}}{2^{j}(j-s)!}\int_{-1}^{1}(1+x)^{j-s}(1-x)^{j+s}\,dx(j-s)!(ik)^{j-s}a_{j-s}\right]r^{j}$$

$$+\text{ higher powers of } r$$

$$=\left[\frac{(i)^{j-s}2^{j+1}(j-s)!(j+s)!k^{j}}{(2j+1)!}a_{j-s}\right]r^{j}+\text{higher powers of } r. \qquad (A.9)$$

Using the power series for e^{-ikry} and the hypergeometric function we get an expression for the a_{j-s},

$$a_{j-s}=\sum_{n+m=j-s}\frac{(-1)^{m}}{m!}\frac{1}{n!}\frac{\Gamma(s+n-i\eta)\Gamma(2s+1)}{\Gamma(s-i\eta)\Gamma(2s+n+1)}. \qquad (A.10)$$

To evaluate this we need an amusing identity involving Γ-functions.

Lemma. For all $\eta\in\mathbb{C}$ and $j,s\in\mathbb{N}$, $j\geqslant s$

$$\sum_{n+m=j-s}\frac{(-1)^{m}}{m!}\frac{1}{n!}\frac{\Gamma(s+n-i\eta)\Gamma(2s+1)}{\Gamma(s-i\eta)\Gamma(2s+n+1)}=\frac{(-1)^{j-s}(2s)!}{(j+s)!(j-s)!}\frac{\Gamma(j+1+i\eta)}{\Gamma(s+1+i\eta)}.$$

Proof. Eliminating the summation index $m=j-s-n$ on the left and using the factorial properties of the Γ-function the above identity is seen to be equivalent to

$$(j+s)!\sum_{n=0}^{j-s}\frac{(-1)^{n}}{(2s+n)!}\binom{j-s}{n}(s+n-1-i\eta)\dots(s-i\eta)$$

$$=(j+i\eta)\dots(s+1+i\eta).$$

Both sides are clearly polynomials of degree $(j-s)$ in η. We show their equality by evaluating them at $j-s+1$ distinct points. For $-i\eta = m+s+1$, $0 \leqslant m < j-s$ the expression on the right-hand side vanishes and on the left we have

$$\frac{(j+s)!}{(2s+m)!}\sum_{n=0}^{j-s}(-1)^n\frac{(2s+m+n)!}{(2s+n)!}\binom{j-s}{n},$$

which is zero because $(2s+m+n)!/(2s+n)!$ is a polynomial of degree m in n and it is well known that

$$\sum_{n=0}^{N}(-1)^n\binom{N}{n}n^m = 0$$

for $0 \leqslant m < N$. Finally we consider $-i\eta = s$. Then we get $(j-s)!$ on the right. Using the identity

$$\sum_{n=0}^{N}(-1)^n\binom{N}{n}\frac{a}{a+n} = \frac{N!a!}{(N+a)!},$$

which can be proved for all $a \in \mathbb{N}$ by induction over N we find for the sum on the left

$$\frac{(j+s)!}{(2s)!}\sum_{n=0}^{j-s}(-1)^n\binom{j-s}{n}\frac{2s}{2s+n} = (j-s)!.$$

This completes the proof.

Putting this lemma together with eqs. (A.3), (A.4) and (A.9) and inserting the expressions for C_s and C_{js} we finally get

$$B_{js} = i^j(2j+1)\frac{2^j(2s)!}{(2j+1)!}\frac{\Gamma(j+1+i\eta)}{\Gamma(s+1+i\eta)}\frac{C_s}{C_{js}}$$

$$= i^j(2j+1)\,\mathrm{e}^{i\sigma_{js}}. \qquad \text{(A.11)}$$

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