Module F12MS3: Oscillations and Waves

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This course begins with the mathematical description of simple oscillating systems such as a mass on a spring or a simple pendulum. We study the behaviour of such systems with and without damping, and when subjected to a periodic external force. We then study coupled oscillators: several masses connected by springs, or beads on an elastic string. The equations of motion for these systems have some particularly simple solutions, called normal modes, which we describe explicitly and in detail. By smearing out the masses of the beads on a string we construct a mathematical model for a continuous elastic medium: the elastic string. We show that the transverse displacement of such a string obeys a partial differential equations, called the wave equation, and find solutions which describe standing waves. The standing waves are the continuous analogue of the normal modes of the coupled oscillators. The idea that a general configuration of the string is a sum of standing waves leads into the mathematical theory of Fourier series: the decomposition of arbitrary functions in terms of cosine and sine functions. Finally we study travelling wave solutions of the wave equation.

A note on units: We will be using the SI system. Length is measured in metres (m), time in seconds (s or sec), mass in kilogrammes (kg) and force in Newtons (N). The Newton is a derived unit and can be expressed in terms of m, s and kg: 1 N=kg $\cdot$ m/s^2. Note that this is consistent with Newton's second law Force=mass $\times$ acceleration. In these units the gravitational acceleration on earth is $g = 9.8$ m/s^2=9.8N/kg.

Table of Contents

1 Simple harmonic motion

1.1 Hooke's law

Consider a spring attached to a wall and lying horizontally on a smooth surface. It is found empirically that the force required to stretch the spring elastically is proportional to the stretching. This result is named after the English physicist Robert Hooke (1635-1703):

Physical Law 1.1.1 *(***Hooke's law***) The force F_H required to stretch the spring by an amount x is given by*

$$F_H = kx, \tag{1.1}$$

where k is a constant of proportionality which is characteristic of the spring and called the spring constant.

Suppose we now attach a trolley of mass m to the free end of the spring. When the trolley is displaced from the equilibrium position by an amount x the spring exerts a force on the trolley which, by Newton's third law, is equal and opposite to the force F_H exerted by the trolley on the spring. It is therefore given by

$$F = -kx. \tag{1.2}$$

and sometimes called the *elastic restoring force*. According to Newton's second law the motion of the trolley will be such that its acceleration $\ddot{x}$ satisfies

$$m\ddot{x} = -kx. \tag{1.3}$$

This is the equation of simple harmonic motion (SHM). With the definition

$$\omega^2 = \frac{k}{m} \tag{1.4}$$

the equation becomes

$$\ddot{x} = -\omega^2 x \tag{1.5}$$

It is easy to check that $x(t) = \cos(\omega t)$ and $x(t) = \sin(\omega t)$ solve (1.6). Further solutions are $A\cos(\omega t)$ for any constant A, but also the sum $x(t) = \cos(\omega t) + \sin(\omega t)$.

More generally, we have the following

Theorem 1.1.2 *(***Principle of superposition***) If $x_1(t)$ and $x_2(t)$ are solutions of* (1.5)*, then so is $x(t) = Ax_1(t) + Bx_2(t)$, where A and B are constants.*

To prove this theorem, compute $\ddot{x} = A\ddot{x}_1 + B\ddot{x}_2$. But since $\ddot{x}_1 = -\omega^2 x_1$ and $\ddot{x}_2 = -\omega^2 x_2$, we have $\ddot{x} = -A\omega^2 x_1 - B\omega^2 x_2 = -\omega^2 x$ as required. □

The principle of superposition allows one to generate new solutions from two given solutions. If certain conditions on the differential equation and the two given solutions are satisfied one can show that all solutions are obtained from the superposition principle. This is the case for the solutions $x_1(t) = \cos(\omega t)$ and $x_2(t) = \sin(\omega t)$ of (1.5). The general solution is therefore

$$x(t) = A\cos(\omega t) + B\sin(\omega t). \tag{1.6}$$

General solutions of second order differential equations depend on two constants.

1.2 Properties of simple harmonic motion

We can write the solution (1.6) also in the form

$$x(t) = R\cos(\omega t - \phi), \tag{1.7}$$

where the angle ϕ lies in the interval $(-\pi, \pi]$. Using the trigonometric identity $\cos(\alpha-\beta) = \cos\alpha\cos\beta+\sin\alpha\sin\beta$, and comparing with (1.6) we deduce the following relation between the constants A, B on one hand and R, ϕ on the other:

$$R\cos\phi = A, \qquad \text{and} \qquad R\sin\phi = B. \tag{1.8}$$

These can be inverted to give $R = \sqrt{A^2 + B^2}$. We also deduce $\tan\phi = B/A$, but note that, since $\tan(\phi + \pi) = \tan(\phi)$, this only determines ϕ up to multiples of π. To get the correct value of ϕ you need to refer back to (1.8). R is the furthest distance the trolley travels from the equilibrium position during the motion and is called the amplitude of the oscillation. Since cos is a periodic function with period 2π, the motion repeats itself after a time T which is such that $\omega T = 2\pi$, i.e.

$$T = \frac{2\pi}{\omega}. \tag{1.9}$$

This is called the period of the motion. The inverse $\nu = 1/T$ is the frequency and $\omega = 2\pi\nu$ is called the angular frequency. The angular frequency $\omega = \sqrt{k/m}$ is sometimes called the **characteristic frequency**.

Example 1.2.1 *A trolley of mass $m = 1$ kg is attached to a spring with spring constant $k = 64N/m$. The trolley is pulled $1/4$ m to the right of the equilibrium position and released from rest. Find its subsequent motion. What is its amplitude and period?*

$x(t) = \frac{1}{4}\cos 8t$. Amplitude $R = 1/4$ m, period $T = \pi/4$ seconds. □

1.3 Energy in simple harmonic motion

Kinetic energy is a measure of the energy in the motion of a particle and was discussed in the module "Mathematics of motion". We recall the definition:

Definition 1.3.1 *(***Kinetic energy***) The kinetic energy K of a particle of mass m moving with velocity $\dot{x}$ is*

$$K = \frac{1}{2} m \dot{x}^2. \tag{1.10}$$

The potential energy of a particle attached to a spring is a measure of the energy stored in the spring-particle system when the spring is stretched or compressed. The precise definition is motivated by the requirement that the sum of kinetic energy and potential energy should be conserved.

Definition 1.3.2 *(***Potential energy***) The potential energy of simple harmonic motion is*

$$V = \frac{1}{2} k x^2. \tag{1.11}$$

With these definitions we have

Theorem 1.3.3 Energy conservation *The total energy $E = K + V$ is conserved during simple harmonic motion.*

Proof: Differentiating, using the product rule and chain rule, we find

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx) = 0, \tag{1.12}$$

by virtue of the equations of motion (1.3). □

Both the kinetic and potential energy change during the motion, but a good measure of how the total energy is divided into kinetic and potential energy is given by the average kinetic and potential energy.

Definition 1.3.4 *The average kinetic energy is*

$$K_{av} = \frac{1}{T} \int_0^T \frac{1}{2} m \dot{x}^2 dt \tag{1.13}$$

and the average potential energy is

$$V_{av} = \frac{1}{T} \int_0^T \frac{1}{2} k x^2 dt. \tag{1.14}$$

Example 1.3.5 *Compute the kinetic and potential energy, the total energy and the average kinetic and potential energy for the motion of the trolley in example 1.2.1*

For the potential energy, measured in Joule, we find, with $x(t) = \frac{1}{4}\cos 8t$,

$$V = \frac{1}{2} \cdot 64 \cdot (\frac{1}{4}\cos 8t)^2 = 2\cos^2 8t \tag{1.15}$$

and for the kinetic energy, also measured in Joule, we use $\dot{x}(t) = 2\sin 8t$ and find

$$K = \frac{1}{2} \cdot (2\sin 8t)^2 = 2\sin^2 8t \tag{1.16}$$

Hence the total energy in Joule is

$$E = 2(\cos^2 8t + \sin^2 8t) = 2. \tag{1.17}$$

In order to compute the average kinetic and potential energy we use the trigonometric identities

$$\sin^2 z = \frac{1}{2}(1 - \cos(2z)), \quad \cos^2 z = \frac{1}{2}(1 + \cos(2z)). \tag{1.18}$$

Combine them with the fact that the definite integrals of both $\cos(nz)$ and $\sin(nz)$ from 0 to 2π vanish for any integer n

$$\int_0^{2\pi} \sin(nz)\, dz = \int_0^{2\pi} \cos(nz)\, dz = 0. \tag{1.19}$$

to conclude

$$\int_0^{2\pi} \sin^2 z\, dz = \int_0^{2\pi} \frac{1}{2} dz - \frac{1}{2}\int_0^{2\pi} \cos(2z)\, dz = \pi \tag{1.20}$$

and similarly

$$\int_0^{2\pi} \cos^2 z\, dz = \pi. \tag{1.21}$$

Then, with $T = \pi/4$ and $z = 8t$

$$K_{av} = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \sin^2(8t)\, dt = \frac{1}{\pi}\int_0^{2\pi} \sin^2 z\, dz = 1 \tag{1.22}$$

By a very similar calculation

$$V_{av} = 1. \tag{1.23}$$

□

In the example we saw that both the kinetic and the potential energy oscillate in such a way that one is zero when the other is maximal: the total energy changes back and forth between kinetic and potential energy. The sum of kinetic and potential energy is constant and *on average* there is an equal amount of both in every cycle of the oscillation. The conclusions generalise from the example to general simple harmonic motion.

It's worth recording one general result here, which we used in the example and which we will need again.

Lemma 1.3.6 *Consider an arbitrary angular frequency ω and the associated period $T = \frac{2\pi}{\omega}$. Then*

$$\frac{1}{T}\int_0^T \sin^2(\omega t)\,dt = \frac{1}{T}\int_0^T \sin^2(\omega t)\,dt = \frac{1}{2} \tag{1.24}$$

To prove this, let $z = \omega t$, so that

$$\frac{1}{T}\int_0^T \sin^2(\omega t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} \sin^2 z\,dz \tag{1.25}$$

and

$$\frac{1}{T}\int_0^T \cos^2(\omega t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} \cos^2 z\,dz \tag{1.26}$$

Now use the results (1.21) and (1.19) to obtain the claim. □

1.4 The vertical spring

Suppose a spring his hanging vertically as shown in in Fig. 1.1 and an object of mass m is attached to the free end of the spring, thus exerting a gravitational force gm. According to Hooke's law (1.1) the spring will stretch by an amount l satisfying

$$mg = kl. \tag{1.27}$$

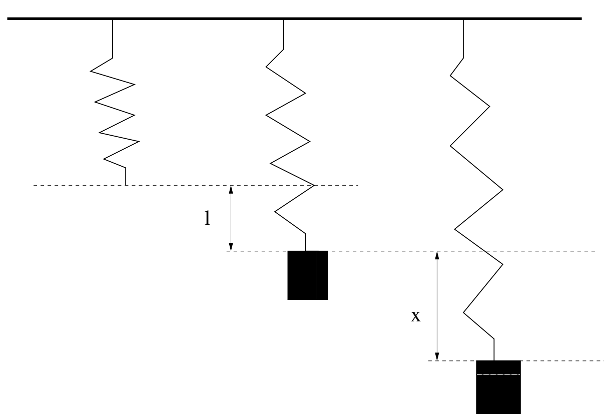

Figure 1.1: The vertical spring

We are interested in the motion of the object when we stretch the spring by an additional amount x_0 and release it, possibly with some initial velocity v_0. Ignoring friction, the forced acting on the spring are the gravitational force mg (downwards) and the elastic restoring force $-k(l + x)$. Hence, by Newton's second law, the equation of motion is

$$m\frac{d^2x}{dt^2} = mg - k(l + x) = -kx \tag{1.28}$$

where we used (1.27). Thus we find that the effect of gravity disappears from the equation, and the displacement x from the equilibrium obeys the same equation as the horizontal spring!

Example 1.4.1 *Suppose an object of* 1 *kg mass stretches a spring* $\frac{49}{20}$*m. If the object is given an initial speed* 5 *m s*$^{-1}$ *downwards at the equilibrium position, find the subsequent motion. Neglect air resistance.*

From the data given and equation (1.27) we compute the spring constant

$$k = \frac{mg}{l} = 9.8 \cdot \frac{20}{49}\mathrm{N/m} = 4\mathrm{N/m}.$$

Since $m = 1kg$ we find $\omega = 2\mathrm{s}^{-1}$. The general solution is thus

$$x(t) = A\cos(2t) + B\sin(2t), \tag{1.29}$$

where it is understood that t gives the time measured in seconds and x the distance from equilibrium measured in metres. From the initial condition we deduce $A = 0$ and $B = 5/2$ and thus

$$x(t) = \frac{5}{2}\sin(2t). \tag{1.30}$$

□

1.5 The simple pendulum

The simple pendulum is a bob of mass m suspended from a fixed point O by a light, inextensible rod (or string) of length l. The rod is hinged and allowed to move in one plane only. The equation of motion for the simple pendulum can be derived in two ways, either using the vector description of two-dimensional motion developed in the module "Mathematics of motion" or using rotational dynamics of rigid bodies. Since we have not covered the latter I give a derivation based on the first method. This is a little intricate and you are welcome to skip to the result (1.43) if you are willing to take it on trust.

First we set up some notation to describe the position of bob, see also Fig. (1.2). In terms of the coordinate axis $\vec{i}$ and $\vec{j}$ and the angle θ shown in the figure the position vector of the bob relative to O is

$$\vec{r}(t) = l(\sin\theta\,\vec{i} - \cos\theta\,\vec{j}). \tag{1.31}$$

When the bob moves, the angle θ changes with time but l is constant since the rod is supposed to be inextensible. Thus the velocity is

$$\dot{\vec{r}} = l\dot{\theta}(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) \tag{1.32}$$

and the acceleration is

$$\ddot{\vec{r}} = l\ddot{\theta}(\cos\theta\,\vec{i} + \sin\theta\,\vec{j}) - l\dot{\theta}^2(\sin\theta\,\vec{i} - \cos\theta\,\vec{j}). \tag{1.33}$$

With the abbreviation

$$\vec{n} = \sin\theta\,\vec{i} - \cos\theta\,\vec{j} \tag{1.34}$$

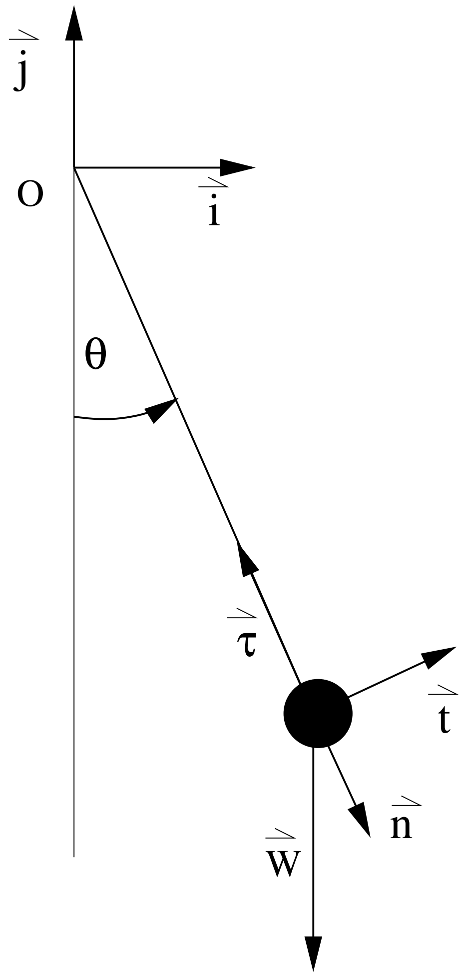

Figure 1.2: The simple pendulum

for the normalised vector in the direction of the rod and

$$\vec{t} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \tag{1.35}$$

for the normalised tangential vector we can write the acceleration as

$$\ddot{\vec{r}} = l\ddot{\theta}\vec{t} - l\dot{\theta}^2\vec{n} \tag{1.36}$$

Neglecting air resistance, there are two forces on the bob, namely its weight and the tension in the rod. The weight $\vec{W}$ has magnitude mg and points in the $-\vec{j}$ direction; the tension always points in opposite direction to $\vec{r}$ and we denote its magnitude by τ. The total force on the bob is therefore

$$\vec{F} = -\tau(\sin\theta\,\vec{i} - \cos\theta\,\vec{j}) - mg\,\vec{j}. \tag{1.37}$$

Expressing $\vec{j}$ in terms of $\vec{n}$ and $\vec{t}$ as $\vec{j} = \sin\theta\,\vec{t} - \cos\theta\,\vec{n}$ we write the force as

$$\vec{F} = -\tau\vec{n} - mg\sin\theta\,\vec{t} + mg\cos\theta\,\vec{n}. \tag{1.38}$$

Thus decomposing Newton's second law

$$m\ddot{\vec{r}} = \vec{F} \tag{1.39}$$

into its $\vec{t}$ and $\vec{n}$ component we find

$$ml\ddot{\theta} = -mg\sin\theta \tag{1.40}$$

and

$$-m\dot{\theta}^2 = -\tau + mg\cos\theta. \tag{1.41}$$

Of the two equations we found, the first (1.40) is the equation of motion for the bob; the equation (1.41) merely determines the tension in the rod resulting from the bob's motion. The equation (1.40) is difficult to solve in general, but if we restrict ourselves to small oscillations we can approximate

$$\sin\theta \approx \theta \tag{1.42}$$

and the equation becomes

$$\ddot{\theta} = -\frac{g}{l}\theta. \tag{1.43}$$

This is the equation for simple harmonic motion with angular frequency

$$\omega = \sqrt{\frac{g}{l}} \tag{1.44}$$

so that the period of the oscillation is

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{1.45}$$

Thus a long pendulum swings more slowly than a short one, and a pendulum on the moon more slowly than the same pendulum on earth. Moreover, the period is independent of the mass of the bob! This should be contrasted with the situation for the mass on a spring.

2 Revision interlude

2.1 Complex numbers

Complex numbers are pairs of real numbers $x, y \in \mathbb{R}$ and written as

$$z = x + iy \tag{2.1}$$

where i is called the **imaginary unit**. The real numbers x and y are called the real and imaginary part of z, written as

$$x = \mathrm{Re}(z), \quad y = \mathrm{Im}(z). \tag{2.2}$$

The set of all complex numbers is denoted $\mathbb{C}$. Addition of complex numbers is defined component-wise. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2). \tag{2.3}$$

The multiplication rule is determined once we fix

$$i^2 = -1. \tag{2.4}$$

Then

$$z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1). \tag{2.5}$$

The complex conjugate $\bar{z}$ of the complex number z is defined as

$$\bar{z} = x - iy \tag{2.6}$$

and satisfies

$$z\bar{z} = x^2 + y^2. \tag{2.7}$$

It follows that the inverse of z is

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x - iy}{x^2 + y^2}. \tag{2.8}$$

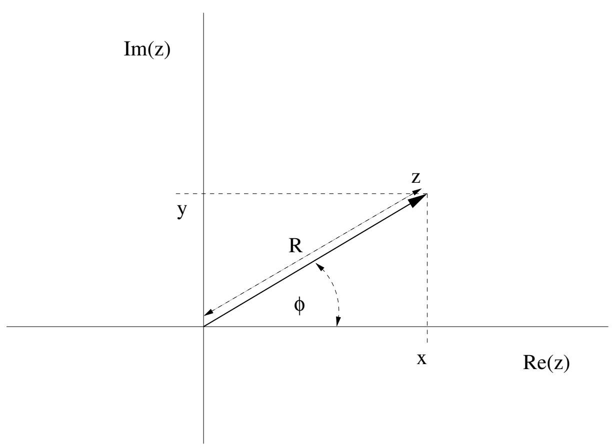

Figure 2.1: The Argand diagram

Complex numbers can be depicted as vectors in the two-dimensional plane, called the **Argand diagram**, where the real part is plotted along the x-axis and the imaginary part along the y-axis. The length R of the vector representing the complex number z is the called **modulus** and the angle ϕ (measured in radians) it makes with the x-axis is called the **argument** of z. The angle ϕ is only defined up to multiples of 2π, but we adopt the convention that ϕ lies in the interval $(-\pi, \pi]$ (often called the principal value). Thus the modulus is given by

$$R = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}} \tag{2.9}$$

and the argument ϕ satisfies

$$\tan\phi = \frac{y}{x}. \tag{2.10}$$

From the figure we find

$$x = R\cos\phi, \quad y = R\sin\phi \tag{2.11}$$

or

$$z = R(\cos\phi + i\sin\phi). \tag{2.12}$$

Using the important **Euler relation**

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{2.13}$$

we have the modulus-argument representation of z:

$$z = Re^{i\phi}. \tag{2.14}$$

The modulus of z is also sometimes written as $|z|$, so $|z| = R$ when z is given by (2.14). The multiplication of complex numbers is particularly simple in the modulus-argument form. If $z_1 = R_1 e^{i\phi_1}$ and $z_2 = R_2 e^{i\phi_2}$ then

$$z_1 z_2 = R_1 R_2 e^{i(\phi_1+\phi_2)}. \tag{2.15}$$

The inverse of z in (2.14) is

$$\frac{1}{z} = \frac{1}{R}e^{-i\phi} \tag{2.16}$$

and its complex conjugate is

$$\bar{z} = Re^{-i\phi}. \tag{2.17}$$

To end, we note an important consequence of the relation (2.13) and its complex conjugate

$$e^{-i\phi} = \cos\phi - i\sin\phi. \tag{2.18}$$

Adding and subtracting (2.13) and (2.18) we find

$$\cos\phi = \frac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right) \quad \text{and} \quad \sin\phi = \frac{1}{2i}\left(e^{i\phi} - e^{-i\phi}\right). \tag{2.19}$$

2.2 Second order differential equations with constant coefficients

2.2.1 Homogeneous equations with constant coefficients

Consider

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = 0, \tag{2.20}$$

where a_1 and a_0 are real constants. We try solutions of the form

$$x(t) = e^{\lambda t}. \tag{2.21}$$

Inserting (2.21) into (2.20) leads to

$$(\lambda^2 + a_1\lambda + a_0)e^{\lambda t} = 0. \tag{2.22}$$

Since $e^{\lambda t}$ is never zero, we deduce that (2.21) is a solution if λ satisfies the equation

$$\lambda^2 + a_1\lambda + a_0 = 0. \tag{2.23}$$

This is called the **characteristic equation** of the differential equation (2.20). Its roots are

$$\lambda_1 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0}}{2} \quad \text{and} \quad \lambda_2 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0}}{2} \tag{2.24}$$

Inserting the roots into (2.21) we thus obtain solutions to the differential equation (2.20. The nature of the solution depends on the roots.

(i) $\lambda_1 \neq \lambda_2$ real. This is the easiest case: $x_1(t) = e^{\lambda_1 t}$ and $x_2(t) = e^{\lambda_2 t}$ are independent solutions: they form a **fundamental set** of solutions. The general solution is a linear combination of x_1 and x_2:

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}, \tag{2.25}$$

with A and B constants.

(ii) $\lambda_1 = \lambda_2$ real. In that case we obtain only one solution from (2.21), namely $x_1(t) = e^{\lambda_1 t}$. A second independent solution is given by $x_2(t) = te^{\lambda_1 t}$, and the general solution is of the form

$$x(t) = (A + Bt)e^{\lambda_1 t} \tag{2.26}$$

(iii) λ_1, λ_2 complex. In that case $\lambda_2 = \bar{\lambda}_1$, so if $\lambda_1 = p + iq$ then $\lambda_2 = p - iq$. The functions $e^{\lambda_1 t}$ and $e^{\lambda_2 t}$ are independent solutions of (2.20) but are complex. To obtain real solutions we take the linear combinations

$$\begin{aligned} x_1(t) &= \frac{1}{2}(e^{(p+iq)t} + e^{(p-iq)t}) = e^{pt}\cos(qt) \\ x_2(t) &= \frac{1}{2i}(e^{(p+iq)t} - e^{(p-iq)t}) = e^{pt}\sin(qt) \end{aligned} \tag{2.27}$$

and obtain a real fundamental set. The general solution is of the form

$$x(t) = e^{pt}(A\cos(qt) + B\sin(qt)). \tag{2.28}$$

roots	fundamental set of solutions
$\lambda_1 \neq \lambda_2$ real	$u_1(t) = e^{\lambda_1 t}, \quad u_2(t) = e^{\lambda_2 t}$
$\lambda_1 = \lambda_2$ real	$u_1(t) = e^{\lambda_1 t}, \quad u_2(t) = te^{\lambda_1 t}$
$\lambda_1 = p + iq, \lambda_2 = p - iq$	$u_1(t) = e^{pt}\cos(qt), \quad u_2(t) = e^{pt}\sin(qt)$

Table 1: fundamental sets for constant coefficient equations

2.2.2 Inhomogeneous equations

Consider now

$$\frac{d^2x}{dt^2} + a_1\frac{dx}{dt} + a_0 x = f, \tag{2.29}$$

where f is some function of t.

In order to find all solutions of (2.29) we require

1. A fundamental set $\{x_1, x_2\}$ of solutions of $\dfrac{d^2x}{dt^2} + a_1\dfrac{dx}{dt} + a_0 x = 0$.
2. A particular solution x_p of (2.29).

The general solution is then given by

$$x = Ax_1 + Bx_2 + x_p, \tag{2.30}$$

where A and B are real constants.

If the function f in (2.29) is a polynomial, exp, sin or cos, then one can find a particular solution of a similar form via the **method undetermined coefficients**. The following table gives recipes involving unknown coefficients which one can determine by substituting into the equation. There is no deep reason for these recipes other than that they work. In the table I have abbreviated homogeneous equation by HE.

$f(t)$	particular solution
$b_0 + b_1 t + ... b_n t^n$	$c_0 + c_1 t + ... c_n t^n$
$e^{\lambda t}$	$e^{\lambda t}$ is not a solution of HE $\Rightarrow$ try $ce^{\lambda t}$ $e^{\lambda t}$ is a solution of HE $\Rightarrow$ try $cte^{\lambda t}$ $e^{\lambda t}$ and $te^{\lambda t}$ solutions of HE $\Rightarrow$ try $ct^2 e^{\lambda t}$
$e^{i\omega t}$	$e^{i\omega t}$ not solution of the HE $\Rightarrow$ try $Ce^{i\omega t}$, $C \in \mathbb{C}$, and take real part $e^{i\omega t}$ solution of the HE $\Rightarrow$ try $Cte^{i\omega t}$, $C \in \mathbb{C}$, and take real part
$\cos(\omega t)$	Find particular solution for equation with $f(t) = e^{i\omega t}$ and take real part

Table 2

To illustrate the recipes given in the table, we consider some examples, beginning with the **polynomial case**.

$$\frac{d^2x}{dt^2} + x = t^2. \tag{2.31}$$

We try

$$x_p(t) = b_0 + b_1 t + b_2 t^2, \tag{2.32}$$

and find by inserting into (2.31)

$$(2b_2 + b_0) + b_1 t + b_2 t^2 = t^2 \tag{2.33}$$

Comparing coefficients yields

$$b_2 = 1, \quad b_1 = 0, \quad b_0 = -2 \tag{2.34}$$

so that

$$x_p(t) = t^2 - 2. \tag{2.35}$$

Continuing with the **exponential case**, consider

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{3t} \tag{2.36}$$

The roots of the characteristic polynomial are -1 and -2. Thus the right hand side is not a solution of the homogeneous equation, and we try

$$x_p(t) = ce^{3t} \tag{2.37}$$

and deduce from inserting into (2.36) that x_p is a solution provided we choose $c = 1/20$. Now change the right hand side of (2.36) to a solution of the homogeneous equation, say

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{-t}. \tag{2.38}$$

Now we try

$$x_p(t) = cte^{-t}. \tag{2.39}$$

After slightly tedious differentiations, you should find that this is indeed a solution provided we pick $c = 1$.

Finally we turn to the **oscillatory case** $f(t) = \cos(\omega t)$. It is particularly important for applications and we will study it in detail in Sect. 3. According to method summarised given in the last row of table 2 we should solve the equation with $\cos(\omega t)$ replaced by $e^{i\omega t}$, and take the real part at the end. The method can be justfied as follows. Suppose $x(t)$ is a solution of

$$\ddot{x} + a_1\dot{x} + a_0x = e^{i\omega t}, \tag{2.40}$$

with a_1, a_0 and ω all real. Then, taking real parts,

$$\text{Re}\,(\ddot{x} + a_1\dot{x} + a_0x) = \text{Re}\left(e^{i\omega t}\right) = \cos(\omega t) \tag{2.41}$$

so, with $x_p = \text{Re}(x)$, we have $\text{Re}(\ddot{x}) = \ddot{x_p}$, $\text{Re}(a_1\dot{x}) = a_1\dot{x_p}$ and $\text{Re}(a_0x) = a_0x_p$. Hence

$$\ddot{x_p} + a_1\dot{x_p} + a_0x_p = \cos(\omega t) \tag{2.42}$$

To illustrated the method, we study one example here:

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = \cos(2t). \tag{2.43}$$

We consider the associated complex equation

$$\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = e^{2it}. \tag{2.44}$$

Our strategy is to solve this equation, and take the real part of the (complex) solution as a particular solution for (2.43). Thus we try $x(t) = Ce^{2it}$ with $C \in \mathbb{C}$. Differentiating and inserting into (2.44) we find

$$(-4C + 6iC + 2C)e^{2it} = e^{2it} \tag{2.45}$$

so that

$$C = \frac{1}{-2+6i} = -\frac{1}{20} - \frac{3}{20}i. \tag{2.46}$$

Thus the particular solution is

$$\begin{aligned} x_p(t) &= \mathrm{Re}\left(\left(-\frac{1}{20} - \frac{3}{20}i\right)(\cos(2t) + i\sin(2t))\right) \\ &= -\frac{1}{20}\cos(2t) + \frac{3}{20}\sin(2t). \end{aligned} \tag{2.47}$$

3 Damped and forced oscillations

3.1 The oscillating spring revisited

Consider a spring hanging vertically as in 1.4, with an object of mass m attached to it. We are now going to consider the more complicated situation where the object is acted on by an additional force $f(t)$, and we are going to take friction into account.

The forces acting on the object are

1. The downward gravitational force: mg.
2. The elastic restoring force $-k(l+x)$.
3. Air resistance. This is an example of a damping force which is proportional to the velocity but acts in the opposite direction: $-r\frac{dx}{dt}$.
4. Any other force exerted on the object, denoted $f(t)$.

Note that the damping coefficient r has units $\mathrm{N} \cdot \mathrm{s/m}$ and the spring constant k has units $\mathrm{N/m}$.

According to Newton's second law, the motion of the object is thus governed by the equation

$$m\frac{d^2x}{dt^2} = mg - k(l+x) - r\frac{dx}{dt} + f(t). \tag{3.1}$$

Re-arranging the terms, and using (1.27) we thus arrive at the linear second-order inhomogeneous ODE

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = f(t). \tag{3.2}$$

This is the sort of equation which we have learnt to solve in the previous subsection. Here we will see how to interpret our solutions physically. I have summarised the physical meaning of the various parameters and functions in the following table.

$x(t)$	downward displacement from equilibrium at time t
$\dot{x}(t)$	velocity at time t
$\ddot{x}(t)$	acceleration at time t
$f(t)$	external or driving force at time t
m	mass
r	damping coefficient
k	spring constant

Table 3

We are now going to study the equation (3.2), starting with the simplest situation and building up towards the general form (3.2). First consider the case where the damping constant r is zero, and no additional force f acts on the mass. The equation (3.2) thus becomes

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{3.3}$$

This is the equation of **simple harmonic motion** we studied at the beginning of the course. Let's derive the solution using the general technique developed in subsection 2.2. The characteristic equation is

$$\lambda^2 + \frac{k}{m} = 0. \tag{3.4}$$

With the abbreviation

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{3.5}$$

a fundamental set of solutions is given by

$$x_1(t) = \cos(\omega_0 t), \quad x_2(t) = \sin(\omega_0 t). \tag{3.6}$$

The general solution is therefore

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) \tag{3.7}$$

in agreement with (1.6).

3.2 Unforced oscillations with damping

There is no external force, but the damping coefficient r is not zero. The equation is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = 0 \tag{3.8}$$

or

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = 0 \tag{3.9}$$

with the abbreviations

$$\gamma = \frac{r}{m} \tag{3.10}$$

and ω_0 as in (3.5). The characteristic equation is

$$\lambda^2 + \gamma\lambda + \omega_0^2 = 0 \tag{3.11}$$

with roots

$$\lambda_1 = -\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \quad \text{and} \quad \lambda_2 = -\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{3.12}$$

Provided the roots are different, a fundamental set is therefore given by

$$u_1(t) = e^{\lambda_1 t} \quad , u_2(t) = e^{\lambda_2 t}. \tag{3.13}$$

Both physically and mathematically the discussion of the general solution is best organised according to the sign of $\gamma^2 - 4\omega_0^2$.

(i) **Underdamped case**: $\gamma^2 < 4\omega_0^2$

We introduce the abbreviation

$$\beta = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{3.14}$$

so that $\lambda_1 = -\gamma/2 + i\beta$ and $\lambda_2 = -\gamma/2 - i\beta$. Comparing with table 1 in section 2.5 we deduce that a real fundamental set is given by

$$x_1(t) = e^{-\frac{\gamma}{2}t}\cos(\beta t) \qquad x_2(t) = e^{-\frac{\gamma}{2}t}\sin(\beta t). \tag{3.15}$$

The general solution is therefore given by

$$x(t) = e^{-\frac{\gamma}{2}t}(A\cos(\beta t) + B\sin(\beta t)) = R\,e^{-\frac{\gamma}{2}t}\cos(\beta t - \phi) \tag{3.16}$$

with R and ϕ defined as after Eq. (1.6).

The solution still oscillates and has infinitely many zeroes, but the amplitude of the oscillation decreases exponentially with time. For $t \to \infty$ all solutions tend to zero: the oscillation dies down.

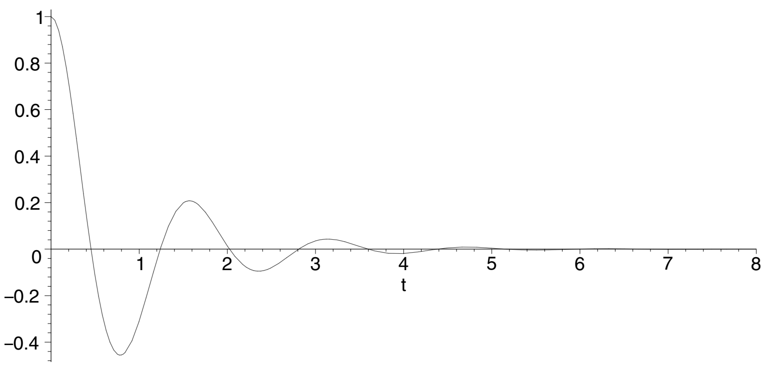

Figure 3.1: Plot of the underdamped solution (3.16) for $A = 1, B = 0.25, \beta = 4, \gamma = 2$

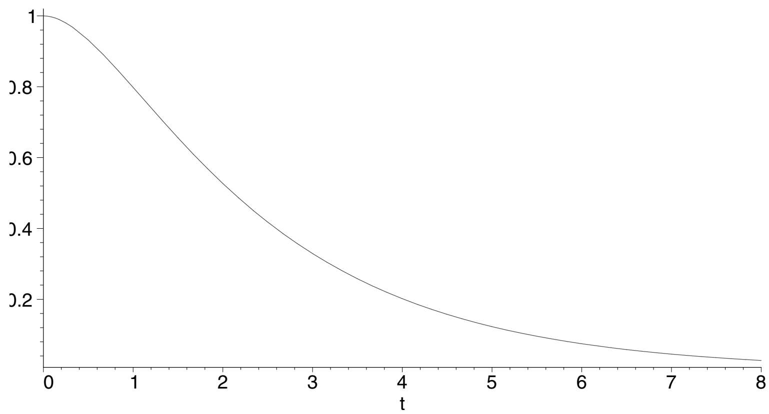

Figure 3.2: The overdamped solution (3.17) for $A = 1.5, B = -0.5, \lambda_1 = -0.5, \lambda_2 = -1.5$

(ii) **Overdamped case**: $\gamma^2 > 4\omega_0^2$

In this case both roots λ_1 and λ_2 (3.12) are real and negative. The general solution is

$$x(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}. \tag{3.17}$$

Solutions are zero for at most one value of t (provided A and B do not both vanish) and tend to 0 for $t \to \infty$.

(iii) **Critically damped case**: $\gamma^2 = 4\omega_0^2$

In this case $\lambda_1 = \lambda_2 = -\gamma/2$ and (compare sect. 2.4.3) the general solution is

$$x(t) = (A + Bt)e^{-\frac{\gamma}{2}t} \tag{3.18}$$

Again solutions are zero for at most one value of t (provided A and B do not both vanish) and tend to 0 for $t \to \infty$.

The terminology "underdamped", "overdamped" and "critically damped" has its origin in engineering applications. The theory developed here applies, for example, to the springs that provide the damping in cars. When perturbed from equilibrium, underdamped springs return to the equilibrium position quickly but overshoot. Overdamped springs take a long time to return to equilibrium. In the critically damped case the

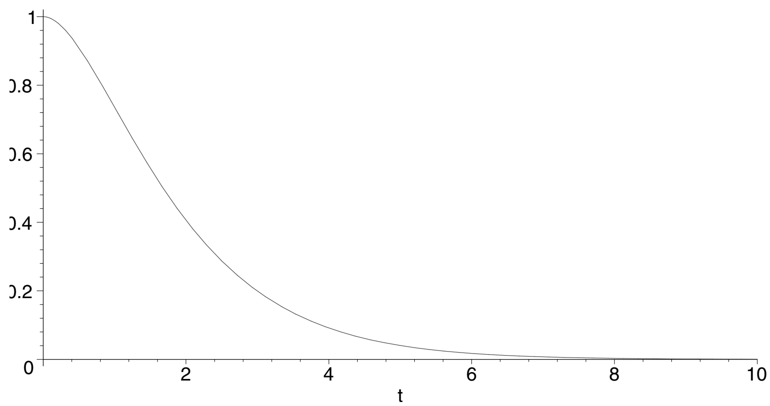

Figure 3.3: Plot of the critically damped solution (3.18) for $A = B = 1, \gamma = 2$

spring returns to the equilibrium position very quickly but avoids overshooting. Thus the critically damped case provides the most efficient damping.

Example 3.2.1 *An object of mass $m = 2$ kg is attached to a spring with spring constant $k = 4$ N m^{-1} and immersed in a viscous liquid with damping constant $r = 6$ N m^{-1} s. At time $t =$ the object is raised 1 m and given an initial downward velocity of 3 m s^{-1}. Find the subsequent motion of the object.*

The equation of motion is

$$\ddot{x} + 3\dot{x} + 2x = 0. \tag{3.19}$$

The characteristic equation has the two real roots $\lambda_1 = -2$ and $\lambda_2 = -1$, so the general solution is

$$x(t) = Ae^{-t} + Be^{-2t} \tag{3.20}$$

Since $x(0) = A + B = -1$ and $\dot{x}(0) = -A - 2B = 3$ we deduce $A = 1$ and $B = -2$. The solution

$$x(t) = e^{-t} - 2e^{-2t} \tag{3.21}$$

passes through the equilibrium $x = 0$ once at $t = \ln 2$ s.

3.3 Forced oscillations with damping

This is the most general case, with all terms in eq. (3.2) playing a role. If the external force f grows indefinitely, it is clear that the spring will eventually break. Remarkably this can also happen when f is a periodic function which averages to zero. Since this case is particularly important in applications, we focus on it here. Suppose therefore that

$$f(t) = f_0 \cos(\omega t), \tag{3.22}$$

where f_0 is some constant (in units of Newtons). Then the equation of motion is

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = f_0\cos(\omega t). \tag{3.23}$$

After dividing by m the equation takes the form

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{f_0}{m}\cos(\omega t). \tag{3.24}$$

The quickest way to solve this is to use complex numbers. Since $\cos\omega t$ is the real part of $\exp(i\omega t)$ we solve

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \frac{f_0}{m}e^{i\omega t}. \tag{3.25}$$

first and then take the real part of the solution we obtain. This turns out to be an efficient method. Suppose first that $i\omega$ is not a solution of the characteristic equation. Then try $x(t) = C\exp(i\omega t)$. Inserting into (3.25) yields

$$Ce^{i\omega t}(-\omega^2 + i\gamma\omega + \omega_0^2) = \frac{f_0}{m}e^{i\omega t}. \tag{3.26}$$

Dividing by $\exp(i\omega t)$ and solving for C we find

$$C = \frac{(f_0/m)}{-\omega^2 + i\gamma\omega + \omega_0^2} = \frac{(f_0/m)e^{-i\phi}}{\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}} \tag{3.27}$$

where

$$\tan\phi = \frac{\gamma\omega}{\omega_0^2 - \omega^2} \tag{3.28}$$

Taking the real part of $x(t) = C\exp(i\omega t)$ we therefore find the solution

$$x_p(t) = \frac{(f_0/m)}{\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}}\cos(\omega t - \phi). \tag{3.29}$$

The function

$$R(\omega) = \frac{(f_0/m)}{\sqrt{(\omega_0^2-\omega^2)^2 + \gamma^2\omega^2}} \tag{3.30}$$

gives the ω-dependent amplitude of the forced motion. For later use we write it as

$$R(\omega) = \frac{f_0}{m\omega_0\omega}\frac{1}{\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{\gamma^2}{\omega_0^2}}} \tag{3.31}$$

The general solution of (3.24) is a linear combination of the particular solution (3.29) and the fundamental solutions of the free damped system discussed in sect. 2.6.2. Let us for definiteness consider the underdamped case $\gamma^2 < 4\omega_0^2$. Then the relevant fundamental set is (3.15) and the general solution of the inhomogeneous equation (3.24) is

$$x(t) = \underbrace{e^{-\frac{\gamma}{2}t}(A\cos(\beta t) + B\sin(\beta t))}_{\text{transient solution}} + \underbrace{x_p(t)}_{\text{steady state solution}}, \tag{3.32}$$

with α and β as defined in (3.14). The first part is called the **transient solution** because it tends to zero for $t \to \infty$. After a long time the solution is dominated by the **steady state solution**. Note that this has the same frequency as the driving term (3.22). Also note that the amplitude of the steady state solution depends on the frequency ω and is very large when $\omega = \omega_0$. This phenomenon is called **resonance**.

Resonance occurs when the driving frequency is equal to the characteristic frequency of the spring.

Depending on the value of r, m and ω the amplitude of the steady state solution at resonance could be much bigger than the amplitude of the driving force. Finally note that at resonance $\tan\phi = \infty$ so that $\phi = \pi/2$. Thus the steady solution lags behind the driving force by $\pi/2$ at the resonance frequency. Resonance manifests itself more dramatically when the damping is small or zero, as we shall see in the next section.

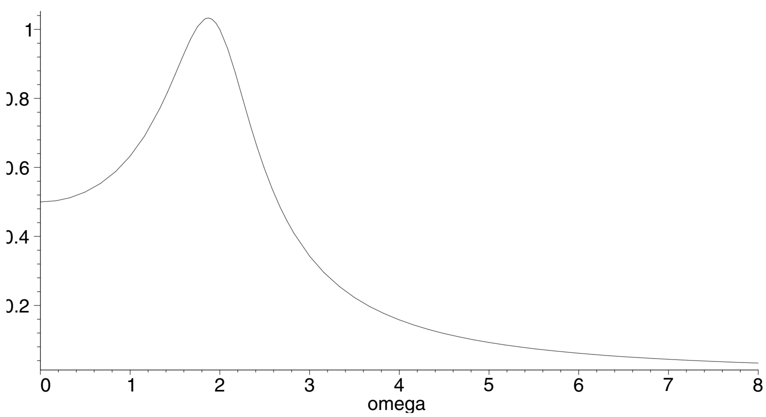

Figure 3.4: The amplitude (3.30) for $f_0 = 2, \omega_0 = 2, \gamma = 1$

The amplitude of displacement is not maximal at resonance. We now show that the amplitude of the velocity is maximal there. Differentiating (3.29) we have

$$\dot{x}_p(r) = -\omega R(\omega)\sin(\omega t - \phi). \tag{3.33}$$

This is an oscillation with amplitude

$$V(\omega) = \omega R(\omega) = \frac{f_0}{m\omega_0}\frac{1}{\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{\gamma^2}{\omega_0^2}}} \tag{3.34}$$

where we used the expression (3.31). Written in this way it is obvious the denominator is minimal as a function of ω when $\omega = \omega_0$. Hence the velocity amplitude V is maximal at resonance.

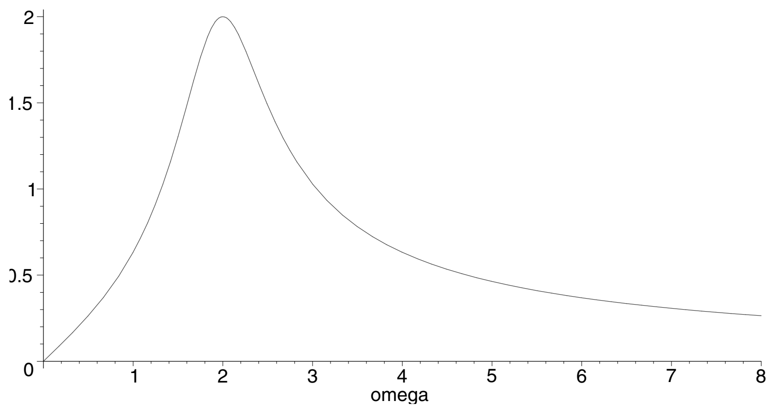

Figure 3.5: The velocity amplitude (3.34) for $f_0 = 2, \omega_0 = 2, \gamma = 1$

3.4 Forced oscillations without damping

In the absence of damping, the equation (3.23) simplifies to

$$m\frac{d^2x}{dt^2} + kx = f_0 \cos(\omega t). \tag{3.35}$$

This is a special case of the discussion of oscillatory driving terms in sect. 2.5.1. First consider the case $\omega \neq \omega_0$, i.e. $i\omega$ is not a root of the characteristic equation. Then we have the particular solution

$$x_p(t) = \frac{(f_0/m)}{(\omega_0^2 - \omega^2)} \cos(\omega t) \tag{3.36}$$

and therefore the general solution is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t) + \frac{f_0/m}{(\omega_0^2 - \omega^2)} \cos(\omega t) \tag{3.37}$$

which is a fairly complicated superposition of oscillations of different frequencies. It remains bounded as $t \to \infty$.

Next consider the case of resonance. The driving frequency equals the spring's characteristic frequency, i.e.

$$\omega = \omega_0. \tag{3.38}$$

Again we think of $\cos(\omega t)$ as the real part of $\exp(i\omega t)$ and study

$$\frac{d^2x}{dt^2} + \omega^2 x = e^{i\omega t} \tag{3.39}$$

Since $i\omega$ is a solution of the characteristic equation, we try $x(t) = Ct\exp(i\omega t)$ in accordance with Table 2. We find

$$2i\omega C e^{i\omega t} = e^{i\omega t} \tag{3.40}$$

or

$$C = \frac{1}{2i\omega}. \tag{3.41}$$

Taking the real part of $x(t) = Ct\exp(i\omega t) = \frac{t}{2\omega}\left(-i\cos(\omega t) + \sin(\omega t)\right)$ we obtain the particular solution

$$x_p(t) = \frac{t}{2\omega}\sin(\omega t) \tag{3.42}$$

The general solution is therefore

$$x(t) = A\cos(\omega t) + B\sin(\omega t) + \frac{f_0 t}{2m\omega}\sin(\omega t). \tag{3.43}$$

The remarkable property of the particular solution is that its amplitude grows linearly with time and becomes infinite as $t \to \infty$. In real life ever increasing oscillations mean that the oscillating system (be it a spring or a more complicated object - such as a building) will break. This can lead to very dramatic manifestations of resonance, as for example in the collapse of the Tacoma Narrows bridge in the USA in 1940.

3.5 Energy in damped and driven oscillations

In the presence of damping and an external force, the energy of an oscillating spring is no longer conserved. Instead we find, with $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ defined as in Sect. 1.3,

$$\frac{dE}{dt} = (m\ddot{x} + kx)\dot{x} = -r\dot{x}^2 + \dot{x}f, \tag{3.44}$$

where we used the equation of motion (3.23) and $f(t) = f_0\cos(\omega t)$. The terms on the right hand describe how the energy of the spring changes. The first term is always negative and describes the energy lost per unit time due to friction. The second term describes the rate at which energy is put into or taken out of (depending on the sign) the spring by the external force. Rate of change of energy is called power in physics, so we define the dissipated power as

$$P_{dis} = -r\dot{x}^2 \tag{3.45}$$

and the power input as

$$P = f\dot{x} \tag{3.46}$$

Let's compute the power input for the forced and damped spring. For the steady state solution (3.29) we have

$$\dot{x}_b = -R(\omega)\omega\sin(\omega t - \phi) \tag{3.47}$$

and hence

$$\begin{aligned} P(t) &= -f_0 R(\omega)\omega\cos(\omega t)\sin(\omega t - \phi) \\ &= -f_0 R(\omega)\omega(\cos(\omega t)\sin(\omega t)\cos\phi - \cos(\omega t)\cos(\omega t)\sin\phi) \end{aligned} \tag{3.48}$$

Using $\cos(\omega t)\sin(\omega t) = \frac{1}{2}\sin(2\omega t)$ and

$$\frac{1}{T}\int_0^T \sin(2\omega t)dt = 0 \tag{3.49}$$

as well as

$$\frac{1}{T}\int_0^T \cos^2(\omega t)dt = \frac{1}{2} \tag{3.50}$$

(see (1.24)) we find that the average power input over one cycle is

$$\bar{P} = \frac{1}{T}\int_0^T P(t)dt = \frac{1}{2}f_0\omega R(\omega)\sin\phi \tag{3.51}$$

With the expression

$$R(\omega) = \frac{f_0}{m\omega_0\omega}\frac{1}{\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{\gamma^2}{\omega_0^2}}} \tag{3.52}$$

(see (3.31)) for R and

$$\begin{aligned} \sin\phi &= \frac{\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \\ &= \frac{\gamma}{\omega_0}\frac{1}{\sqrt{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{\gamma^2}{\omega_0^2}}} \end{aligned} \tag{3.53}$$

we have

$$\bar{P} = \frac{\gamma f_0^2}{2m\omega_0^2}\frac{1}{\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right)^2 + \frac{\gamma^2}{\omega_0^2}} \tag{3.54}$$

Comparing with (3.34) we see that $\bar{P}$, like the velocity amplitude $V(\omega)$, is maximal when $\omega = \omega_0$.

4 Coupled oscillators

4.1 Coupled springs

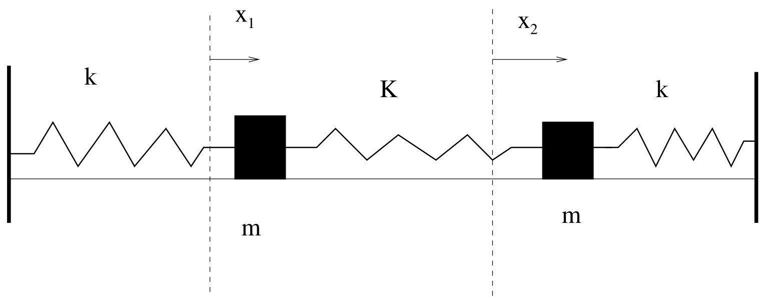

Figure 4.1: Coupled springs

Consider two particles, both of mass m, connected by a spring of spring constant K as shown in the figure. The particles lie on a horizontal plane and are connected to walls by springs with spring constant k. Their motion is restricted to a line containing all three springs. The force on each particle is obtained by applying Hooke's law to the two springs connected to the particle. Counting displacement of the particles to the right as positive, the force exerted by the spring to the left of particle 1 on particle 1 is $-kx_1$. The middle spring is stretched by an amount $(x_2 - x_1)$ when the the particles are in the position shown in the figure. It will therefore exert a force $-K(x_1 - x_2)$ on particle 1 and a force $-K(x_2 - x_1)$ on particle 2. Finally the spring on the right exerts a force $-kx_2$ on particle 2. Hence, by Newton's second law we have the equations of motion

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 - K(x_1 - x_2) \\ m\ddot{x}_2 &= -kx_2 - K(x_2 - x_1). \end{aligned} \tag{4.1}$$

This is a system of coupled, linear second order differential equations which looks difficult to solve. However, if we add the equations we obtain

$$m(\ddot{x}_1 + \ddot{x}_2) = -k(x_1 + x_2) \tag{4.2}$$

and when we subtract them we get

$$m(\ddot{x}_1 - \ddot{x}_2) = -k(x_1 - x_2) - 2K(x_1 - x_2). \tag{4.3}$$

Hence, with abbreviations

$$y_1 = x_1 + x_2, \qquad y_2 = x_1 - x_2 \tag{4.4}$$

and

$$\omega_1^2 = \frac{k}{m}, \qquad \omega_2^2 = \frac{k}{m} + 2\frac{K}{m} \tag{4.5}$$

we have two decoupled equations

$$\begin{aligned}\ddot{y}_1 &= -\omega_1^2 y_1 \\ \ddot{y}_2 &= -\omega_2^2 y_2.\end{aligned} \tag{4.6}$$

Each of them is just the equation for simple harmonic motion discussed in Sect. 1. Hence the general solution is

$$\begin{aligned}y_1(t) &= A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) \\ y_2(t) &= A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)\end{aligned} \tag{4.7}$$

where A_1, B_1, A_2, B_2 are arbitrary real constants. Having solved the equation (4.6) in terms of the coordinates y_1 and y_2 we recover the general solution x_1 and x_2 of (4.1) via

$$\begin{aligned}x_1(t) &= \frac{1}{2}(y_1(t) + y_2(t)) \\ x_2(t) &= \frac{1}{2}(y_1(t) - y_2(t)).\end{aligned} \tag{4.8}$$

The coordinates y_1 and y_2 are called normal mode coordinates, or simply normal modes of the coupled springs. To understand their physical interpretation, consider the motion when only one of them is non-vanishing.

1. Assume that $y_2 = 0$ (i.e. $A_2 = B_2 = 0$ in (4.7)), and consider the motion of the particles according to (4.8) when $y_1 \neq 0$. Since $x_1 = x_2$ in this case, the two particles oscillate "in tandem". The spring in the middle remains unstretched since $x_1 - x_2 = 0$.

2. Assume that $y_1 = 0$ (i.e. $A_1 = B_1 = 0$ in (4.7)), and consider the motion of the particles according to (4.8) when $y_2 \neq 0$. Since $x_1 = -x_2$ in this case, the two particles oscillate "in opposition". The spring in the middle gets stretched and compressed in every cycle of the motion.

Since $\omega_1 < \omega_2$, the period $T_1 = 2\pi/\omega_1$ of the mode y_1 is larger than the period $T_2 = 2\pi/\omega_2$ of y_2. The normal mode y_1 is therefore sometimes called the slow mode, and y_2 the fast mode.

The following example illustrates how normal modes can be used to solve initial value problems.

Example 4.1.1 *Two particles of mass $m = 1$ kg are connected to springs as shown in Fig. (4.1). The spring constant of the outer springs is $k = 25$ N/m and the spring constant of the middle spring is $K = 11/2$ N/m . Particle 1 is initially displaced 1 m to the right and released from rest; particle 2 is initially at rest and at the equilibrium position. Find the subsequent motion of both particles.*

We solve this problem in three steps.

Step 1: *Translate the initial values into intial values for the normal modes y_1 and y_2.* From the question we know that $x_1(0) = 1$, $\dot{x}_1(0) = 0$ and $x_2(0) = 0$, $\dot{x}_2(0) = 0$. Hence the initial values for the normal coordinates are

$$y_1(0) = x_1(0) + x_2(0) = 1, \quad \dot{y}_1(0) = \dot{x}_1(0) + \dot{x}_2(0) = 0$$

and

$$y_2(0) = x_1(0) - x_2(0) = 1, \quad \dot{y}_2(0) = \dot{x}_1(0) - \dot{x}_2(0) = 0$$

Step 2: *Solve the initial value problem for the normal modes.* With k and K given in the example, we find from (4.5) that $\omega_1 = 5$ s^{-1} and $\omega_2 = 6$ s^{-1}. Hence the general solution of the normal mode equations (4.6) are $y_1(t) = A_1 \cos(5t) + B_1 \sin(5t)$ and $y_2(t) = A_2 \cos(6t) + B_2 \sin(6t)$. Imposing the initial conditions found in step 1 gives $y_1(t) = \cos(5t)$ and $y_2(t) = \cos(6t)$.

Step3 : *Translate the results of step 2 into the particle coordinates x_1 and x_2.* From the formula (4.8) we find $x_1(t) = \frac{1}{2}(\cos(5t) + \cos(6t))$ and $x_2(t) = \frac{1}{2}(\cos(5t) - \cos(6t))$. □

Remarks on beats

It is instructive to re-write the superposition of two oscillations of different frequencies which appear in the final answer for the displacements x_1 and x_2 in the previous question. Returning to the general situation where the normal mode frequencies are denoted ω_1 and ω_2, consider the superposition

$$x(t) = \cos(\omega_1 t) + \cos(\omega_2 t) \tag{4.9}$$

and suppose that $\omega_2 - \omega_1$ is small compared to $\omega_1 + \omega_2$ (like in the example). Then, using the trigonometric identity

$$2 \cos A \cos B = \cos(A + B) + \cos(A - B), \tag{4.10}$$

we have

$$x(t) = 2 \cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \cos\left(\frac{\omega_1 + \omega_2}{2} t\right). \tag{4.11}$$

We interpret this formula as follows. The function

$$A(t) = 2 \cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \tag{4.12}$$

is the slowly varying amplitude of an oscillation with the rapid angular frequency $(\omega_1 + \omega_2)/2$. This is illustrated in Fig 4.2 where we plot (4.12) for the values $\omega_1 = 5$ s^{-1} and $\omega_2 = 6$ s^{-1} from the example.

In physics, the periodic fluctuations of the amplitude of an oscillation are called beats. The time interval between two successive maxima of the modulus of the amplitude is called the period T_{beats} of the beats, and the angular frequency of the beats is defined as

$$\omega_{\text{beats}} = \frac{2\pi}{T_{\text{beats}}}. \tag{4.13}$$

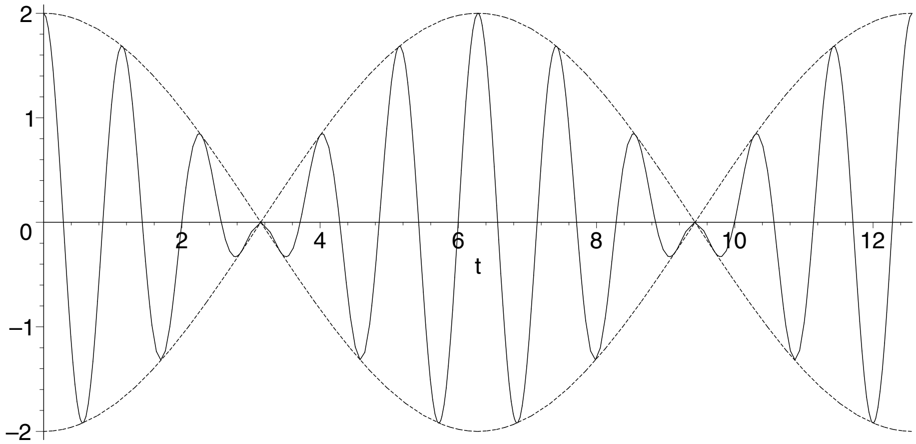

Figure 4.2: Beats: plot of $\cos(5t)+\cos(6t)$ against t. The dashed curve shows the slowly-varying amplitude $\pm 2\cos(t/2)$

The angular frequency of beats which occur in the superposition (4.12) of is twice the angular frequency of the amplitude oscillation $A(t)$ in (4.12) since the modulus of $A(t)$ has half the period of $A(t)$. Thus

$$\omega_{\text{beats}} = \omega_2 - \omega_1. \tag{4.14}$$

Note that Beats are used in tuning instruments. When two notes are slightly out of tune the difference of their frequencies is small, leading to slow and clearly audible fluctuations in the amplitude of their combined sound. The notes can be tuned by reducing the frequency of these beats, ideally to zero.

4.2 N coupled oscillators: beads on a string

Consider N identical beads on a flexible string whose endpoints are connected to walls. Each bead has a mass m, and we neglect the mass of the string itself. In the equilibrium position, depicted at the top in Fig. 4.3, the tension in the string is τ and the beads are separated from each other and the walls by a distance l. The beads are allowed to move in one transverse direction, and we denote the transverse displacement of the p-th bead from the equilibrium position by z_p, $p = 1, \ldots, N$, see Fig. (4.3). It turns out to be convenient to introduce the notation z_0 and z_{N+1} for the "displacement" of the points where the string is connected to the walls. By assumption

$$z_0 = z_{N+1} = 0. \tag{4.15}$$

When the beads are displaced from equilibrium the segments of string connecting them are no longer horizontal. We call the angle between the horizontal and the segment connecting the p-th and $p+1$-st string α_p and assume in the following that all these angles are small

$$\alpha_p \ll 1, \quad p = 1, \ldots, N. \tag{4.16}$$

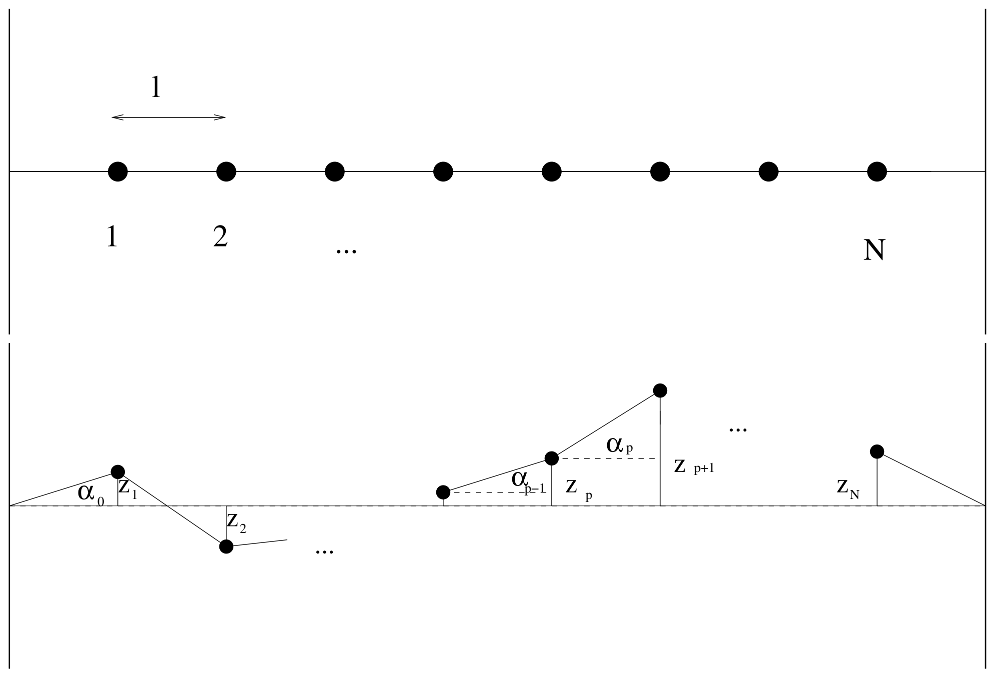

Figure 4.3: N beads on a string

We can therefore use the small angle approximation for trigonometric functions

$$\sin\alpha_p \approx \tan\alpha_p \approx \alpha_p, \quad p = 1, \ldots, N \tag{4.17}$$

and

$$\cos\alpha_p \approx 1, \quad p = 1, \ldots, N. \tag{4.18}$$

The total force F_p acting on the p-th bead is the difference between the tensions in the string segments meeting at that bead. Decomposing into horizontal and vertical components we have from Fig. (4.3)

$$F_p^{\text{horizontal}} = -\tau\cos\alpha_{p-1} + \tau\cos\alpha_p. \tag{4.19}$$

For small angles, it follows from (4.18) that

$$F_p^{\text{horizontal}} \approx 0. \tag{4.20}$$

For the vertical component of the force we find, by looking at Fig. (4.3),

$$F_p^{\text{vertical}} = -\tau\sin\alpha_{p-1} + \tau\sin\alpha_p. \tag{4.21}$$

Now we express the angles α_p in terms of the vertical displacements via

$$\tan\alpha_{p-1} = \frac{z_p - z_{p-1}}{l}, \tag{4.22}$$

noting that, because of our convention (4.15), this formula holds for all $p = 1, \ldots, N$. Using the approximation (4.17) we obtain

$$F_p^{\text{vertical}} = -\frac{\tau}{l}(z_p - z_{p-1}) + \frac{\tau}{l}(z_{p+1} - z_p). \tag{4.23}$$

Thus, Newton's second law yields the equations of motion

$$m\ddot{z}_p = -\frac{\tau}{l}(z_p - z_{p-1}) - \frac{\tau}{l}(z_p - z_{p+1}), \tag{4.24}$$

where p runs from 1 to N. Because of the condition (4.15) the equation for the first and N-th beads are

$$\begin{aligned} m\ddot{z}_1 &= -\frac{\tau}{l}z_1 - \frac{\tau}{l}(z_1 - z_2) \\ \ddot{z}_N &= -\frac{\tau}{l}(z_N - z_{N-1}) - \frac{\tau}{l}z_N. \end{aligned} \tag{4.25}$$

In the special case $N = 2$ of two coupled beads we obtain the coupled equations

$$\begin{aligned} \ddot{z}_1 &= -\frac{\tau}{ml}z_1 - \frac{\tau}{ml}(z_1 - z_2) \\ \ddot{z}_2 &= -\frac{\tau}{ml}(z_2 - z_1) - \frac{\tau}{ml}z_2. \end{aligned} \tag{4.26}$$

With the replacements $z_1 \mapsto x_1, z_2 \mapsto x_2$ and $\tau/(lm) \mapsto K/m = k/m$, these are the same equations (4.1) that we found for the coupled springs in the previous section when, in the notation of Fig 4.1 $k = K$. Applying our experience gained there we can decouple the equations by introducing the normal mode coordinates

$$y_1 = z_1 + z_2, \quad y_2 = z_1 - z_2 \tag{4.27}$$

and the normal mode angular frequencies ω_1 and ω_2

$$\omega_1^2 = \frac{\tau}{ml}, \quad \omega_2^2 = \frac{3\tau}{ml}. \tag{4.28}$$

The equations (4.26) are then equivalent to

$$\begin{aligned} \ddot{y}_1 &= -\omega_1^2 y_1 \\ \ddot{y}_2 &= -\omega_2^2 y_2 \end{aligned} \tag{4.29}$$

and solved by

$$\begin{aligned} y_1(t) &= A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t) \\ y_2(t) &= A_2\cos(\omega_2 t) + B_2\sin(\omega_2 t) \end{aligned} \tag{4.30}$$

where A_1, B_1, A_2, B_2 are arbitrary real constants.

We can visualise the normal modes as follows. When the beads move in the normal mode y_1 (i.e. $y_2 = 0$) the beads oscillate together. When $y_1 = 0$ and the beads move in the normal mode y_2, they move in opposition: when one is "up" the other is "down" and vice-versa.

4.3 The eigenvector method for finding normal modes

In this subsection we introduce a systematic way of finding normal modes which also works for $N > 2$ beads. To motivate it, we re-write the results of the previous section in matrix form. The inverse of the relation (4.27) is

$$z_1 = \frac{1}{2}(y_1 + y_2), \quad z_2 = \frac{1}{2}(y_1 - y_2). \tag{4.31}$$

Hence, when the beads move in the normal mode y_1 given in (4.30) with $y_2 = 0$, the displacements z_1 and z_2 at time t can be combined into the two component vector

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \frac{1}{2}(A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.32}$$

When the beads move in the normal mode y_2, with $y_1 = 0$, their displacements z_1 and z_2 at time t are given by

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \frac{1}{2}(A_2 \cos(\omega_2 t) + B_2 \sin(\omega_2 t)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4.33}$$

The key property of the vectors (4.32) and (4.33) is that in both cases all components oscillate with the same frequency. The following method, called the eigenvector method, takes this as the starting point for finding normal modes. We illustrate how it works using the example for two beads on a string.

Step 1: Write the equations of motion (4.26) in matrix form:

$$\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \tag{4.34}$$

where we introduced the abbreviation $\omega_0^2 = \tau/(ml)$ (*Check that this really is the same equation!*)

Step 2: Try to find a non-zero solution of the form

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = (A \cos(\omega t) + B \sin(\omega t)) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{4.35}$$

where ω and A, B, v_1, v_2 are unknown real constants. Inserting into (4.34) leads to

$$-\omega^2((A \cos(\omega t) + B \sin(\omega t)) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\omega_0^2((A \cos(\omega t) + B \sin(\omega t)) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{4.36}$$

Assuming that A and B are not both zero, this can only be true for all t if

$$-\omega^2 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \tag{4.37}$$

i.e. if

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{\omega^2}{\omega_0^2} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{4.38}$$

Step 3: Solve the matrix equation (4.38). This is an eigenvalue equation for the matrix

$$M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \tag{4.39}$$

We need to find its eigenvectors and eigenvalues, i.e. the real numbers ϵ and vectors $\boldsymbol{v}$ such that

$$M\boldsymbol{v} = \epsilon \boldsymbol{v}. \tag{4.40}$$

In this case it is straightforward to solve the eigenvalue problem - see example sheets. The matrix M has the eigenvalue $\epsilon_1 = 1$ with eigenvector

$$\boldsymbol{v}^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{4.41}$$

and eigenvalue $\epsilon_2 = 3$ with eigenvector

$$\boldsymbol{v}^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4.42}$$

Step 4: Obtain a solution of the form (4.35) for every eigenvalue and eigenvector found in Step 3. Comparing (4.38) and (4.40), we have

$$\epsilon = \frac{\omega^2}{\omega_0^2}. \tag{4.43}$$

Hence $\epsilon_1 = 1$ is equivalent to

$$\omega_1 = \omega_0, \tag{4.44}$$

and inserting the eigenvector (4.41) into (4.35) we obtain the solution

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = (A\cos(\omega_1 t) + B\sin(\omega_1 t)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.45}$$

Renaming the arbitrary constants A and B as $A_1/2$ and $B_1/2$ we recover the solution (4.32). Similarly, from $\epsilon_2 = 3$ we deduce

$$\omega_2 = \sqrt{3}\omega_0. \tag{4.46}$$

Inserting the eigenvector (4.42) into (4.35) we obtain the solution

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = (A\cos(\omega_2 t) + B\sin(\omega_2 t)) \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{4.47}$$

Renaming the arbitrary constants A and B as $A_2/2$ and $B_2/2$ we recover the solution (4.33).

We end this section by considering the case of an arbitrary number of beads. According to the eigenvector method, **step 1** is to write the equations of motion (4.24) as a matrix equation:

$$\begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \vdots \\ \\ \ddot{z}_N \end{pmatrix} = -\omega_0^2 \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ & \dots & & & \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ \\ z_N \end{pmatrix}, \tag{4.48}$$

where we introduced the abbreviation $\omega_0^2 = \tau/(ml)$. Next, we implement **step 2** by looking for a solution of the form

$$\begin{pmatrix} \ddot{z}_1(t) \\ \ddot{z}_2(t) \\ \vdots \\ \\ \ddot{z}_N(t) \end{pmatrix} = (A\cos(\omega t) + B\sin(\omega t)) \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ \\ v_N. \end{pmatrix} \tag{4.49}$$

Inserting this into (4.48) leads to the eigenvalue problem

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ & \dots & & & \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ \\ v_N \end{pmatrix} = \frac{\omega^2}{\omega_0^2} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ \\ v_N \end{pmatrix}. \tag{4.50}$$

Solving this is **step 3** in our method. We give the solution in the following lemma. The proof is left as an exercise on Sheet 9.

Lemma 4.3.1 *The matrix*

$$M = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ & \dots & & & \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix} \tag{4.51}$$

has N eigenvalues and eigenvectors given by

$$\epsilon_n = 2 - 2\cos\left(\frac{n\pi}{N+1}\right), \quad \boldsymbol{v}^n = \begin{pmatrix} \sin\left(\frac{n\pi}{N+1}\right) \\ \sin\left(\frac{2n\pi}{N+1}\right) \\ \vdots \\ \\ \sin\left(\frac{Nn\pi}{N+1}\right) \end{pmatrix} \tag{4.52}$$

where $n = 1, \dots, N$.

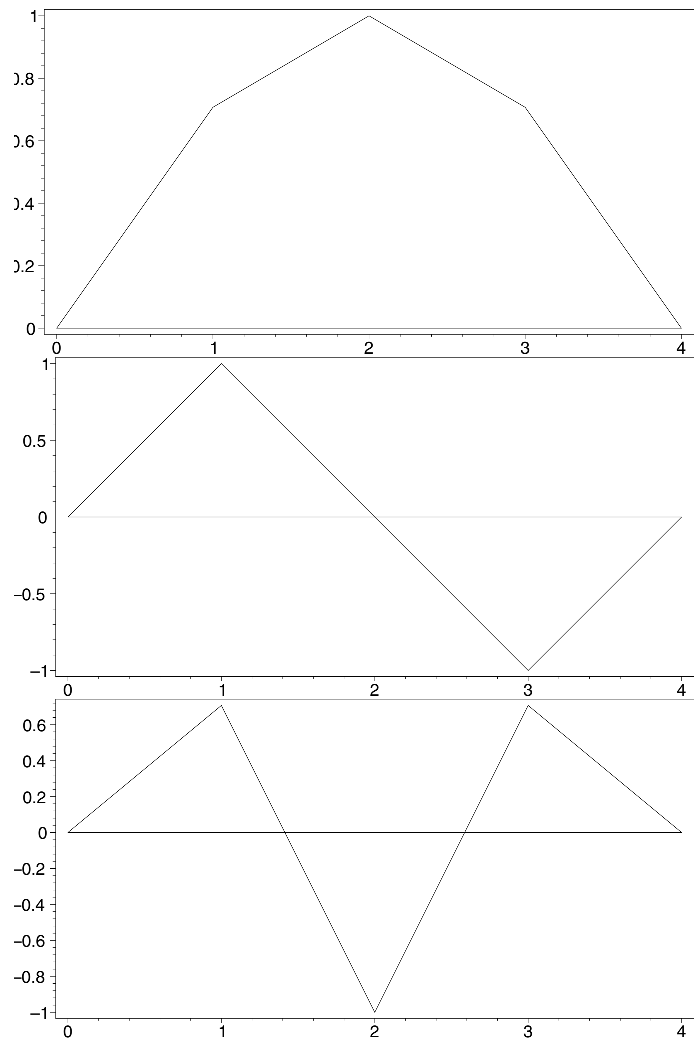

Figure 4.4: Normal modes of three beads on a string

Finally, we carry out **step 4** by inserting our results for the eigenvalue problem into the solution (4.49). Renaming the constants A and B to A_n and B_n for the solution corresponding to the n-the eigenvalue ϵ_n we obtain N independent solutions. With the abbreviation

$$\boldsymbol{z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ \\ z_N \end{pmatrix} \tag{4.53}$$

we have the N solutions

$$\boldsymbol{z}^n(t) = (A_n \cos(\omega_n t) + B_n \sin(\omega_n t))\, \boldsymbol{v}^n, \tag{4.54}$$

where

$$\omega_n^2 = \epsilon_n \omega_0^2. \tag{4.55}$$

The eigenvector method produces N solutions, containing two arbitrary real constants A_n and B_n each. The general solution of (4.34) is a linear combination of these. It contains $2N$ arbitrary constants as we expect for a general solution of a system of second order differential equation for N functions. The eigenmodes for $N = 3$ are shown in Fig. 4.4.

Example 4.3.2 *Consider the case of $N = 3$ beads on a string. Each bead has mass $m = 10$ gram. If the tension in the string is $\tau = 1$ N and the beads are separated from each other and the wall by $l = 1$ m, find the normal modes of the system.*

.

With the data given we find $\omega_0^2 = \tau/(ml) = 0.01 \text{ s}^{-2}$, so that $\omega_0 = 0.1 \text{ s}^{-1}$. The angular frequencies of the three normal modes are given by (4.55) with ε_n given by (4.52). Inserting $N = 3$ and $n = 1, 2, 3$ we find

$$\omega_1^2 = (2 - \sqrt{2})\omega_0^2, \quad \omega_2^2 = 2\omega_0^2, \quad \omega_3^2 = (2 + \sqrt{2})\omega_0^2. \tag{4.56}$$

The normal modes are given by (4.54) with

$$\boldsymbol{v}^1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad \boldsymbol{v}^2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \boldsymbol{v}^3 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -1 \\ \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{4.57}$$

he displacements of the beads at time $t = 0$ for each of these normal modes are shown in Fig. (4.4). $\square$.

5 Waves

In this section we derive the wave equation for transverse waves on string. We consider a string of length L and with a constant mass density μ per unit length stretched between two walls. We can think of this system as the $N \to \infty$ limit of the N beads on a massless string considered in the previous section. This may sound complicated, but we shall see in this section that a string with constant mass density is actually easier to describe mathematically than a finite but large number of beads on a (massless) string.

5.1 The wave equation

Consider the general displacement of the string from equilibrium shown in Fig. (5.1). The string is allowed to move in one transverse direction (the vertical direction in the figure). We introduce a coordinate $x \in [0, L]$ along the string and denote and the transverse displacement by z. Since the transverses displacement depends on the time t and on x, z is a function of t and x. The fact that the string is fixed at its endpoints is mathematically expressed via

$$z(t, 0) = z(t, L) = 0, \quad \forall t \in \mathbb{R} \tag{5.1}$$

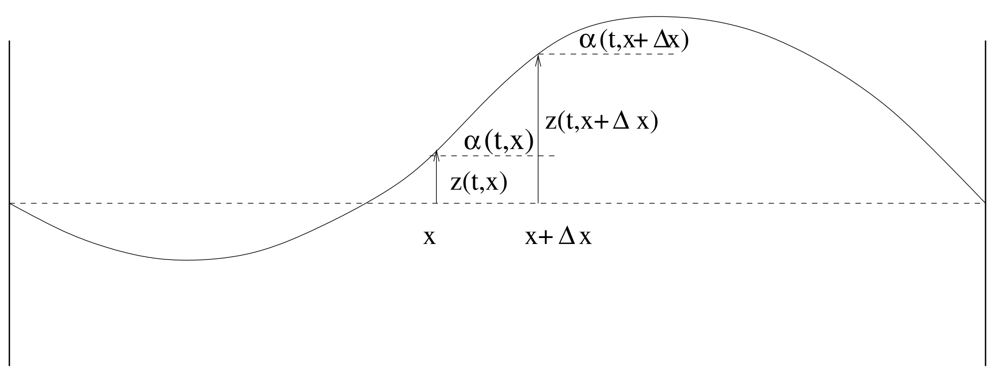

Figure 5.1: The stretched string

In order to derive an equation of motion for z, consider a small segment $(x, x + \Delta x)$ of the string. The angle between the string and the horizontal is denoted by α; since the angle varies along the string and also with time, α is a function of t and x. As in the derivation of the equation of motion for beads on a string we assume that α is small, i.e.

$$\alpha(t, x) \ll 1, \quad \forall t \in \mathbb{R},\, x \in [0, L], \tag{5.2}$$

so that

$$\sin\alpha(t, x) \approx \tan\alpha(t, x) \approx \alpha(t, x), \quad \forall t \in \mathbb{R}, x \in [0, L], \tag{5.3}$$

and

$$\cos\alpha(t, x) \approx 1, \quad \forall t \in \mathbb{R}, x \in [0, L]. \tag{5.4}$$

Next we consider the forces acting on the segment $(x, x + \Delta x)$. Denoting the tension in the string by τ, the horizontal force on the segment is

$$F^{\text{horizontal}}(t, x) = -\tau \cos \alpha(t, x) + \tau \cos \alpha(t, x + \Delta x). \tag{5.5}$$

However, with the approximation (5.4) we find

$$F^{\text{horizontal}}(t, x) \approx 0. \tag{5.6}$$

For the vertical component of the force we find

$$F^{\text{vertical}}(t, x) = -\tau \sin \alpha(t, x) + \tau \sin \alpha(t, x + \Delta x). \tag{5.7}$$

Using the approximation (5.3) we simplify the right hand side

$$\sin \alpha(t, x + \Delta x) - \sin \alpha(t, x) \approx \alpha(t, x + \Delta x) - \alpha(t, x). \tag{5.8}$$

Furthermore, the Taylor expansion

$$\alpha(t, x + \Delta x) = \alpha(t, x) + \Delta x \frac{\partial \alpha}{\partial x}(t, x) \quad + \text{terms of order}(\Delta x)^2 \tag{5.9}$$

implies, for sufficiently small Δx,

$$\alpha(t, x + \Delta x) - \alpha(t, x) \approx \frac{\partial \alpha}{\partial x}(t, x) \Delta x. \tag{5.10}$$

Hence the vertical component of the force is

$$F^{\text{vertical}}(t, x) \approx \tau \frac{\partial \alpha}{\partial x}(t, x) \Delta x. \tag{5.11}$$

Now we express the angles $\alpha(t, x)$ in terms of the vertical displacements via

$$\tan \alpha(t, x) = \frac{z(t, x + \Delta x) - z(t, x)}{\Delta x} \approx \frac{\partial z}{\partial x}, \tag{5.12}$$

where we have used the Taylor expansion again, and assumed that Δx is small. For small angles we therefore have

$$\alpha(t, x) \approx \frac{\partial z}{\partial x}, \tag{5.13}$$

so that

$$F^{\text{vertical}}(t, x) \approx \tau \frac{\partial^2 z}{\partial x^2}(t, x) \Delta x. \tag{5.14}$$

In order to derive the equations of motion for the string we apply Newton's second law to the string segment $[x, x + \Delta x]$. Recalling that the string has constant mass per unit length μ, we note that segment $[x, x+\Delta x]$ has mass $\mu \Delta x$. We approximate the transverse displacement of that the segment by $z(t,x)$, which becomes exact in the limit $\Delta x \to 0$. Hence the acceleration of the segement is $\frac{\partial^2 z}{\partial t^2}$ and we obtain the equation of motion

$$\mu \Delta x \frac{\partial^2 z}{\partial t^2} = \tau \frac{\partial^2 z}{\partial x^2} \Delta x. \tag{5.15}$$

Dividing by Δx we have the partial differential equation

$$\frac{1}{v^2} \frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}, \tag{5.16}$$

where the parameter v is defined in terms of the string tension τ and the mass density μ

$$v = \sqrt{\frac{\tau}{\mu}}. \tag{5.17}$$

The equation (5.16) is called the wave equation. It is one of the most important equations of mathematical physics, arising in numerous applications of mathematics to physics. Before we describe a systematic approach to solving the wave equation in the next section, we consider a simple solution.

It is easy to check that the function

$$z(t,x) = \cos(\omega t) \sin(kx) \tag{5.18}$$

solves the wave equation (5.16) for any pair of real parameters ω and k satisfying

$$\omega = vk. \tag{5.19}$$

The solution (5.18) is periodic in both space and time. We have already discussed the periodicity of $\cos(\omega t)$ in our treatment of simple harmonic motion. Since

$$\cos(\omega(t+T)) = \cos(\omega t) \tag{5.20}$$

we have the familiar formula for the period

$$T = \frac{2\pi}{\omega} \tag{5.21}$$

However, the solution (5.18) is also periodic in space. The equation

$$\sin(k(x+\lambda)) = \sin(kx) \tag{5.22}$$

holds for all x if we take

$$\lambda = \frac{2\pi}{k}. \tag{5.23}$$

The parameter λ characterises the spatial periodicity of the wave; it is called the wavelength.

5.2 Standing waves

The solution (5.18) does not, in general, satisfy the boundary condition (5.1) at $x = L$. When we require

$$z(t, L) = 0 \quad \forall t \in \mathbb{R} \tag{5.24}$$

we get the extra condition

$$\sin(kL) = 0. \tag{5.25}$$

This imposes a condition on the allowed values for k. The equation (5.25) holds if

$$k = \frac{n\pi}{L}, \quad n = 1, 2, 3 \ldots. \tag{5.26}$$

We thus obtain n solutions of the wave equation (5.16) satisfying the boundary condition (5.24):

$$z_n(t, x) = \cos(\omega_n t)\sin(\frac{n\pi}{L}x), \tag{5.27}$$

where

$$\omega_n = v\frac{n\pi}{L}. \tag{5.28}$$

In this section we explain a systematic way of obtaining solutions like (5.27). The method is called **separation of variables** and is useful in solving many partial differential equations. The idea is to solve the partial differential equation (5.16) by assuming that the unkown function $z(t, x)$ is a product of an unknown function $g(t)$ which depends only on t and an unknown function $f(x)$ which depends only on x. Thus we insert

$$z(t, x) = g(t)f(x) \tag{5.29}$$

into (5.16) and hope that we can solve the resulting equations for g and f. It follows from (5.29) that

$$\frac{\partial^2 z}{\partial t^2}(t, x) = \frac{d^2 g}{dt^2}(t)f(x) \tag{5.30}$$

and

$$\frac{\partial^2 z}{\partial x^2}(t, x) = g(t)\frac{d^2 f}{dx^2}(x). \tag{5.31}$$

The insertion of (5.29) into the wave equation leads to

$$\frac{1}{v^2}\frac{d^2 g}{dt^2}(t)f(x) = g(t)\frac{d^2 f}{dx^2}(x). \tag{5.32}$$

Assuming that neither f nor g vanish this is equivalent to

$$\frac{1}{v^2 g(t)}\frac{d^2 g}{dt^2}(t) = \frac{1}{f(x)}\frac{d^2 f}{dx^2}(x). \tag{5.33}$$

The left hand side of this equation depends only on t, and the right hand side depends only on x. The only way two functions of different variables can be identical is if they are both constants. Assuming for now that the constant is negative, we denote it by $-k^2$. Thus we obtain two ordinary differential equations, one for the function f which depends on x

$$\frac{d^2 f}{dx^2} = -k^2 f \tag{5.34}$$

and one for the function g which depends on t only:

$$\frac{d^2 g}{dt^2} = -v^2 k^2\, g. \tag{5.35}$$

Our assumption (5.29) has thus allowed us to trade one partial differential equation for two ordinary differential equations with an arbitrary parameter k. Moreover, both equations are the familiar differential equation for simple harmonic motion. Thus the general solution of (5.34) is

$$f(x) = \alpha \cos(kx) + \beta \sin(kx) \tag{5.36}$$

with α, β arbitrary real constants. The condition (5.1) implies

$$f(0) = 0, \quad f(L) = 0. \tag{5.37}$$

In order to satisfy $f(0) = 0$ we must have $\alpha = 0$. The requirement $f(L) = 0$ imposes a condition on k, which we already discussed after (5.25). Thus we conclude that k must take one of the values

$$k = \frac{n\pi}{L}, \quad n = 1, 2, 3 \ldots. \tag{5.38}$$

Inserting these values into (5.35) we obtain

$$\frac{d^2 g}{dt^2} = -v^2 \left(\frac{n\pi}{L}\right)^2 g. \tag{5.39}$$

For each value of n, this is again the equation for simple harmonic motion. So, for each value of n, we get the general solution

$$g_n(t) = A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \tag{5.40}$$

where

$$\omega_n = v\frac{n\pi}{L}. \tag{5.41}$$

Finally assembling f and g_n into a function of t and x according to (5.29) we obtain the following solutions of the wave equation and the boundary condition (5.1):

$$z_n(t, x) = (A_n \cos(\omega_n t) + B_n \sin(\omega_n t)) \sin\left(\frac{n\pi}{L} x\right). \tag{5.42}$$

Here we have set the constant $\beta = 1$ in (5.36). This can be done without loss of generality because β multiplies the the constants A_n and B_n which are themselves arbitrary.

In analogy to the solutions (4.54) the solutions (5.42) are called the the normal modes of the string. The mode $n = 1$

$$z_1(t, x) = (A_n \cos(\omega_1 t) + B_n \sin(\omega_1 t)) \sin\left(\frac{\pi}{L} x\right). \tag{5.43}$$

is called the fundamental mode of the string. The solutions for $n > 1$ are called higher modes or, in accoustics, overtones. Each mode has a characteristic angular frequency, period and wavelength. With the angular frequency of the n-th mode given by (5.41), the period of the n-th mode is

$$T_n = \frac{2\pi}{\omega_n} = \frac{2L}{vn}. \tag{5.44}$$

The wavelength can be computed from (5.38) via (5.23). We find

$$\lambda_n = \frac{2\pi}{\frac{n\pi}{L}} = \frac{2L}{n}. \tag{5.45}$$

The following example illustrates the importance of normal modes.

Example 5.2.1 *Consider a string of length $L = 1m$, with uniform mass density and total mass $M = 0.1$ kg. The string is stretched so that the tension in the string is $\tau = 10$ Newton, and fixed at its endpoints. It is allowed to oscillate freely in one transverse direction.*

(a) Find all normal modes of the string and give their angular frequencies, periods and wavelengths. Sketch the lowest three modes at time $t = 0$.

(b) If the string is given an initial displacement $z_0(x) = \sin(\pi x)$ and released from rest, find its subsequent motion.

(a) The mass density μ is related to the total mass and the length of the string via $\mu = M/L$. Hence, with the values given in the example, $\mu = 0.1$kg/m. Thus

$$v = \sqrt{\frac{\tau}{\mu}} = \sqrt{100}\text{m s}^{-1} = 10\text{m s}^{-1}.$$

Inserting the numerical values for v and L into (5.42) we find

$$z_n(t, x) = (A_n \cos(10n\pi t) + B_n \sin(10n\pi t)) \sin(n\pi x). \tag{5.46}$$

The angular frequency of the n-th mode is $\omega_n = 10\pi n \text{ s}^{-1}$, the period is $T_n = \frac{1}{5n}$ s and the wavelength is $\lambda_n = \frac{2}{n}$ m. The functions $z_n(0, x)$ are sketchted for $n = 1, 2, 3$ in Fig 5.2.

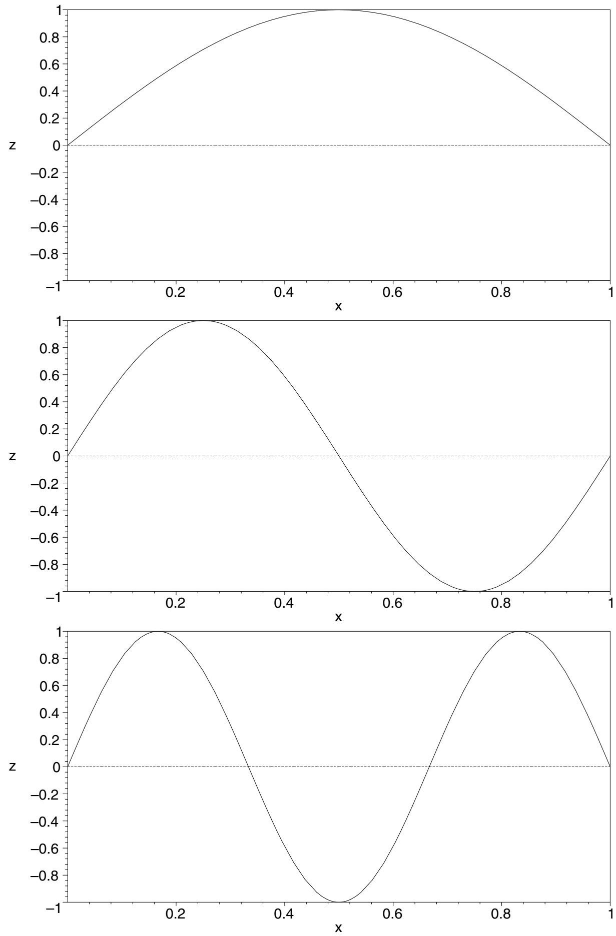

Figure 5.2: Lowest normal modes of a string of length $L = 1$ m ($n = 1, 2, 3$)

(b) The information "is released from rest" means that the required solution should have vanishing time derivative at time $t = 0$. Thus we seek a solution of the wave equation and the boundary condition (5.1) which also satisfies the intial conditions

$$z_n(0, x) = z_0(x), \quad \frac{\partial z}{\partial t}(0, x) = 0 \tag{5.47}$$

We check if one of the normal modes (5.46) satisfies these conditions. We have

$$z_n(0, x) = A_n \sin(n\pi x) \tag{5.48}$$

and

$$\frac{\partial z_n}{\partial t}(0, x) = 10\pi n B_n \sin(n\pi x) \tag{5.49}$$

Hence we can satisfy (5.47) by taking $n = 1$, $A_1 = 1$ and $B_1 = 0$, and the required solution is

$$z(t, x) = \cos(10\pi t) \sin(\pi x). \tag{5.50}$$

□

In the example we were lucky that we could satisfy the initial conditions with one of the normal modes of the string. For more general initial conditions this will not be possible. The next section introduces a tool for tackling general intial value problems for the string. The tool is called Fourier analysis, and widely used in applied mathematics.

5.3 Fourier series

We consider functions on the interval $[-L, L]$, where $L > 0$, and show that any reasonable function $f : [-L, L] \mapsto \mathbb{R}$ can be respresented as a series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{5.51}$$

with a_0, a_n and b_n, $n = 1, 2, 3 \ldots$. We are going to give an explicit formula for these coefficients further below. To motivate the formula, we recall the definition of an inner product space, and the expansion of a vector in such a space in an orthognal basis.

Example 5.3.1 *The vector space $\mathbb{R}^2$ has the inner product*

$$\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = x_1 y_1 + x_2 y_2. \tag{5.52}$$

(a) Check that the vectors $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ form an orthogonal basis of $\mathbb{R}^2$ with respect to the inner product $\langle , \rangle$.

(b) Expand the vector $\boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the basis $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$

(a) Using the formula (5.52) we find $\langle \boldsymbol{v}_1, \boldsymbol{v}_2 \rangle = 1 - 1 = 0$ so that $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are indeed orthogonal. They form a basis since any two orthogonal vectors form a basis of the two-dimensional vector space $\mathbb{R}^2$.
(b) We are looking for coefficients c_1 and c_2 such that

$$\boldsymbol{v} = c_1 \boldsymbol{v}_2 + c_2 \boldsymbol{v}_2. \tag{5.53}$$

In order to compute the coefficient c_1 , we take the inner product with $\boldsymbol{v}_1$ to find

$$\langle \boldsymbol{v}_1, \boldsymbol{v} \rangle = c_1 \langle \boldsymbol{v}_1, \boldsymbol{v}_1 \rangle \tag{5.54}$$

and hence

$$3 = 2c_1 \Rightarrow c_1 = \frac{3}{2}. \tag{5.55}$$

Similarly, to find the coefficient c_2 we take the inner product with $\boldsymbol{v}_2$ to find

$$\langle \boldsymbol{v}_2, \boldsymbol{v} \rangle = c_1 \langle \boldsymbol{v}_2, \boldsymbol{v}_2 \rangle \tag{5.56}$$

and hence

$$-1 = 2c_2 \Rightarrow c_2 = -\frac{1}{2}. \tag{5.57}$$

□

The formulae (5.54) and (5.56) can be summarised in one formla as

$$c_n = \frac{\langle \boldsymbol{v}_n, \boldsymbol{v} \rangle}{\langle \boldsymbol{v}_n, \boldsymbol{v}_n \rangle} \quad n = 1, 2. \tag{5.58}$$

In this formulation it holds for any vector space with inner product. In order to apply it to (5.51) we need to consider the space of all functions on $[-L, L]$ as a vector space, and equip it with an inner product. More precisely, let

$$H = \text{space of all piecewise continuous functions on}[-L, L] \tag{5.59}$$

and define the inner product of two functions $f, g \in H$ as

$$\langle f, g \rangle = \int_{-L}^{L} f(x)g(x)dx. \tag{5.60}$$

Having defined the vector space H and the inner product we want to expand an arbitray element of H in a fixed orthogonal set. The set we want to use is

$$S = \{1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots, \sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \ldots\}. \tag{5.61}$$

Thus we need to show

Lemma 5.3.2 *The set S given in* (5.61) *is an orthogonal set in H.*

Proof : The following general results will be useful in the proof. Recall that a function $F : [-L, L] \to \mathbb{R}$ is called odd if $F(-x) = -F(x)$ and even if $F(-x) = F(x)$.

1. The integral of any odd function $F : [-L, L] \to \mathbb{R}$ from $-L$ to L vanishes. i.e.

$$F(-x) = -F(x) \Rightarrow \int_{-L}^{L} F(x)dx = 0. \tag{5.62}$$

2. For a non-zero integer N we have

$$\int_{-L}^{L} \cos\frac{N\pi x}{L}\, dx = 0 \tag{5.63}$$

The proof of the first result is an exercise on the example sheets, and the proof of the second result is an elementary integration. The results allow us to deduce immediately that the function 1 is orthogonal to all other elements of S:

$$\langle 1, \sin\frac{n\pi x}{L} \rangle = \int_{-L}^{L} \sin\frac{n\pi x}{L} = 0 \tag{5.64}$$

since the sine function is odd and, by (5.63),

$$\langle 1, \cos\frac{n\pi x}{L}\rangle = \int_{-L}^{L} \cos\frac{n\pi x}{L} = 0. \tag{5.65}$$

Next consider the inner products

$$\langle \cos\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\rangle = \int_{-L}^{L} \cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx. \tag{5.66}$$

Now use the trigonometric identity $2\sin A\cos B = \sin(A+B) + \sin(A-B)$ and the fact that the sine function is odd to deduce

$$\langle \cos\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\rangle = \frac{1}{2}\int_{-L}^{L} \sin\frac{(m+n)\pi x}{L} + \sin\frac{(n-m)\pi x}{L}\,dx = 0. \tag{5.67}$$

Using $2\cos A\cos B = \cos(A+B) + \cos(A-B)$ we find

$$\langle \cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L}\rangle = \frac{1}{2}\int_{-L}^{L} \cos\frac{(m+n)\pi x}{L} + \cos\frac{(n-m)\pi x}{L}\,dx. \tag{5.68}$$

Integrating the first terms gives zero by (5.63), and the same holds for the second term except when $n = m$. Thus

$$\langle \cos\frac{m\pi x}{L}, \cos\frac{n\pi x}{L}\rangle = \begin{cases} 0 & \text{if} \quad n \neq m \\ L & \text{if} \quad n = m. \end{cases} \tag{5.69}$$

Finally, using $2\sin A\sin B = \cos(A-B) - \cos(A+B)$ we find

$$\langle \sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\rangle = \frac{1}{2}\int_{-L}^{L} \cos\frac{(m-n)\pi x}{L} - \cos\frac{(n+m)\pi x}{L}\,dx. \tag{5.70}$$

This time integrating the second terms gives zero by (5.63), but the first term only gives zero when $n \neq m$. Thus

$$\langle \sin\frac{m\pi x}{L}, \sin\frac{n\pi x}{L}\rangle = \begin{cases} 0 & \text{if} \quad n \neq m \\ L & \text{if} \quad n = m \end{cases} \tag{5.71}$$

Hence the set S is an orthogonal set. Moreover we note the inner products of the elements with themselves

$$\langle 1, 1\rangle = 2L, \quad \langle \sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\rangle = \langle \cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\rangle = L \tag{5.72}$$

$\square$.

The expansion

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{5.73}$$

of an element $f \in H$ is therefore an expansion in an orthogonal set. Hence we compute the coefficients a_0, a_n and b_n in analogy to (5.58) via

$$
\begin{aligned}
a_0 &= \frac{\langle 1, f\rangle}{\langle 1, 1\rangle} = \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
a_n &= \frac{\langle \cos\frac{n\pi x}{L}, f\rangle}{\langle \cos\frac{n\pi x}{L}, \cos\frac{n\pi x}{L}\rangle} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx \\
b_n &= \frac{\langle \sin\frac{n\pi x}{L}, f\rangle}{\langle \sin\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\rangle} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx
\end{aligned} \tag{5.74}
$$

The series (5.73) with the coefficients defined by (5.74) is called the **Fourier series of f**.

Example 5.3.3 *Find the Fourier series for $f(x) = x$, $x \in [-1, 1]$*

With $L = 1$ the formulae (5.74) give

$$
\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{1} x dx = 0 \quad \text{since the integrand is odd} \\
a_n &= \int_{-1}^{1} x\cos\frac{n\pi x}{L}dx = 0 \quad \text{since the integrand is odd} \\
b_n &= \int_{-1}^{1} f(x)\sin\frac{n\pi x}{L}dx = -\frac{1}{n\pi}\left[x\cos(n\pi x)\right]_{-1}^{1} + \frac{1}{n\pi}\int_{-1}^{1}\cos(n\pi x)dx = \\
&= (-1)^{n+1}\frac{2}{n\pi}
\end{aligned} \tag{5.75}
$$

Hence

$$
f(x) \approx \frac{2}{\pi}[\sin(\pi x) - \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x)\ldots \tag{5.76}
$$

□.

Even though the functions we consider are only defined on the interval $[-L, L]$, the Fourier series makes sense for all $x \in \mathbb{R}$. It converges almost everywhere to a function $\tilde{f}$ which is a periodic extension of f i.e.

$$
\tilde{f}(x) = f(x - 2nL) \quad \text{for} \quad (2n-1)L < x < (2n+1)L, \quad n \in \mathbb{Z} \tag{5.77}
$$

At points x where the function $\tilde{f}$ is not continuous, the Fourier series converges to the average of the right-limit and the left-limit of $\tilde{f}$ at x.[1] This phenomenon is called the **Gibbs phenomenon**.

[1] The right-limit of f at x is defined as $\lim_{\epsilon\to 0} f(x+\epsilon)$, with $\epsilon > 0$ Similarly, the left-limit is $\lim_{\epsilon\to 0} f(x-\epsilon)$, with $\epsilon > 0$.

5.4 Using Fourier series for solving the wave equation

In our study of standing waves on a string, the dependence of displacement z on x was given by a function f on the interval $[0, L]$. We now show how to obtain a series for such functions which involves only sine terms (or only cosine terms). The resulting series is called the **half range Fourier series**. We start with the Fourier sine series.

Given the function $f : [0, L] \to \mathbb{R}$ extend it to an odd function $f : [-L, L] \to \mathbb{R}$ by defining $f(-x) = -f(x)$. Then the new function $f : [-L, L] \to \mathbb{R}$ has a (full range) Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{5.78}$$

with coefficients defined as in (5.74). However, since f is odd we can use the rule (5.62) to deduce that $a_0 = a_n = 0$ in (5.78). Moreover, since both f and $\sin\frac{n\pi x}{L}$ are odd, their product is even and therefore

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx \tag{5.79}$$

Thus b_n is defined purely in terms of the original function $f : [0, L] \to \mathbb{R}$. The series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{5.80}$$

with b_n defined via (5.79) is called the **Fourier sine series** of $f : [0, L] \to \mathbb{R}$

The Fourier cosine series of a function $f : [0, L] \to \mathbb{R}$ is obtained by extending it to an even function $f : [-L, L] \to \mathbb{R}$ by defining $f(-x) = f(x)$. Then the new function $f : [-L, L] \to \mathbb{R}$ has a full range Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{5.81}$$

with coefficients defined as in (5.74). Since f is even we can now use the rule (5.62) to deduce that $b_n = 0$ in (5.81). Moreover, since both f and $\cos\frac{n\pi x}{L}$ are even, their product is even, too, and therefore

$$\begin{aligned} a_0 &= \;= \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{L}\int_{0}^{L} f(x)dx \\ a_n &= \;= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}dx \end{aligned} \tag{5.82}$$

Thus a_0, a_n are defined purely in terms of the original function $f : [0, L] \to \mathbb{R}$. The series

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \tag{5.83}$$

with a_0, a_n defined via (5.82) is called the **Fourier cosine series** of $f : [0, L] \to \mathbb{R}$

Example 5.4.1 *Find the Fourier sine series of the function $f(x) = x^2$, $x \in [0, \pi]$*

From (5.79) we find, with $L = \pi$,

$$b_n = \frac{2}{\pi} \int_0^L x^2 \sin(nx) dx \tag{5.84}$$

Integrating by parts twice and using $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$ we find

$$b_n = -\frac{2\pi}{n}(-1)^n + \frac{4}{\pi n^3}(-1)^n - \frac{4}{\pi n^3}. \tag{5.85}$$

□

Finally, we illustrate the application of Fourier series to initial values problems for the string.

Example 5.4.2 *(a) Find the Fourier sine series of the function $f(x) = x(\pi - x)$, with $0 < x < \pi$.*

(b) Consider a string of length $L = \pi$ meters, with tension $\tau = 10$ Newton and mass per unit length $\mu = 0.1$ kg /m. Using the result of (a) find the solution $z(t, x)$ of the wave equation for the transverse displacement of the string satisfying the initial conidition

$$z(0, x) = x(\pi - x), \ \frac{\partial z}{\partial t}(0, x) = 0.$$

(a) The Fourier sine series on the intervl $[0, \pi]$ takes the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx), \tag{5.86}$$

with

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin(nx) dx \\ &= \frac{2}{\pi} \int_0^\pi x(\pi - x) \sin(nx) dx \\ &= \frac{2}{\pi} \left(\left[-x(\pi - x) \ \frac{1}{n} \cos(nx) \right]_0^\pi + \frac{1}{n} \int_0^\pi (\pi - 2x) \cos(nx) dx \right) \\ &= \frac{2}{\pi} \left(\left[(\pi - 2x) \frac{1}{n^2} \sin(nx) \right]_0^\pi + \frac{2}{n^2} \int_0^\pi \sin(nx) dx \right) \\ &= \frac{4}{\pi n^2} \left[-\frac{1}{n} \cos(nx) \right]_0^\pi \\ &= \frac{4}{\pi n^3} (1 - (-1)^n) \end{aligned} \tag{5.87}$$

so

$$\begin{aligned} x(\pi - x) &= \sum_{n=1}^{\infty} \frac{4}{\pi n^3} (1 - (-1)^n) \sin(nx) \\ &\approx \frac{8}{\pi} \left(\sin(x) + \frac{1}{27} \sin(3x) + \frac{1}{125} \sin(5x) \dots \right). \end{aligned} \tag{5.88}$$

(b) With the parameters given in the question, we find $v = \sqrt{\tau/\mu} = 10\mathrm{m\ s}^{-1}$. Hence, using the general formula (5.42), the normal modes of the string are those found already found in (5.46):

$$z_n(t,x) = (A_n \cos(10nt) + B_n \sin(10nt)) \sin(nx) . \tag{5.89}$$

It is easy to check (we will do this in the next section) that arbitrary linear combination of the normal modes also satisfy the wave equation. In particular (ingoring convergence issues), the infinite sum

$$z(t,x) = \sum_{n=1}^{\infty} (A_n \cos(10nt) + B_n \sin(10nt)) \sin(nx) \tag{5.90}$$

solves the wave equation and satisfies the boundary condition $z(t,0) = z(t,\pi) = 0$. We now need to choose the constants A_n and B_n so that the given initial conditions are also satisfied. Since

$$\frac{\partial z}{\partial t}(t,x) = \sum_{n=1}^{\infty} (-10A_n \sin(10nt) + 10B_n \cos(10nt)) \sin(nx) \tag{5.91}$$

we have the initial velocity

$$\frac{\partial z}{\partial t}(0,x) = \sum_{n=1}^{\infty} 10B_n \sin(nx) . \tag{5.92}$$

To make this vanish, we simply choose $B_n = 0$ for all n. The initial displacement is

$$z(0,x) = \sum_{n=1}^{\infty} A_n \sin(nx) . \tag{5.93}$$

We would like this to be equal to $x(\pi - x)$. Comparing with the sine series (5.88), we see that we can achieve this by setting

$$A_n = \frac{4}{\pi n^3}(1 - (-1)^n). \tag{5.94}$$

Hence the required solution of the initial value problem for the vibrating string is

$$z(t,x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^3}(1 - (-1)^n) \cos(10nt) \sin(nx) . \tag{5.95}$$

To end this section, we collect the similarities between eigenvector method for studying the motion of beads on a string and the separation of variables method for solving the motion of a string in a table.

	N beads on a string	Vibrating string
equation of motion	$\frac{d^2\boldsymbol{z}}{dt^2} = -\frac{\tau}{m} M\boldsymbol{z}$	$\frac{\partial^2 z}{\partial t^2} = \frac{\tau}{\mu}\frac{\partial^2 z}{\partial x^2}$
normal mode equation	$M\boldsymbol{v}^n = \epsilon_n \boldsymbol{v}^n$	$\frac{d^2 f_n}{dx^2} = -k_n^2 f_n$
eigenvalues	$\epsilon_n = 2 - 2\cos\left(\frac{n\pi}{N+1}\right)$	$k_n = \frac{n\pi}{L}$
normal modes	$v_j^n = \sin\left(\frac{nj\pi}{N+1}\right)$	$f_n(x) = \sin\left(\frac{nx\pi}{L}\right)$
	$\langle \boldsymbol{v}^n, \boldsymbol{v}^m \rangle = \frac{N+1}{2}\,\delta_{nm}$	$\int_0^L f_n(x) f_m(x) dx = \frac{L}{2}\,\delta_{nm}$
normal frequencies	$\omega_n^2 = \frac{\tau}{m}\epsilon_n$	$\omega_n^2 = \frac{\tau}{\mu}k_n^2$
general solution	$\sum_{n=1}^{N} \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) \boldsymbol{v}^n$	$\sum_{n=1}^{\infty} \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) f_n(x)$

Table 5: The eigenvector method versus the separation of variables

5.5 Travelling waves

In this subsection we return to the wave equation

$$\frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} = \frac{\partial^2 z}{\partial x^2}, \tag{5.96}$$

and discuss a class of solutions called travelling waves. So far we have only considered standing waves of the form

$$z_k(t,x) = \cos(\omega t)\sin(kx). \tag{5.97}$$

However, using the trigonometric identity $2\cos A \sin B = \sin(A+B) + \sin(B-A)$ we can write this solution as

$$z_k(t,x) = \frac{1}{2}\left(\sin(kx+\omega t) + \sin(kx-\omega t)\right). \tag{5.98}$$

Now it is easy to check that both

$$z_{k,+}(t,x) = \sin(kx+\omega t) \tag{5.99}$$

and

$$z_{k,-}(t,x) = \sin(kx-\omega t) \tag{5.100}$$

satisfy the wave equation (5.96) provided that the relation

$$\omega = vk \tag{5.101}$$

is satisfied. Note that neither of these solutions satisfy the boundary condition $z(t,0) = z(t,L) = 0$. However, recall that our derivation of the wave equation in Sect. 5.1 was independent of the boundary condition. Hence we can think of the wave equation without boundary conditions as a mathematical description of the transverse displacement of an infinitely long string. This is the point of view we take in this section: the range of x is all of $\mathbb{R}$, and no boundary conditions are imposed. In particular, we therefore do not have any conditions on the parameter k: it is an arbitrary real number. We may assume without loss of generality that $k > 0$ since changing the sign of k in (5.99) and (5.100) does not lead to an essentially new solution. Similarly, we assume without loss of generality that $\omega > 0$.

Both the solution (5.99) and (5.100) are examples of travelling waves. To understand in what sense they "travel" consider the spatial dependence of, for example, z_+ at time time $t = 0$:

$$z_{k,+}(0,x) = \sin(kx) \tag{5.102}$$

A time interval $\Delta t > 0$ later the wave is

$$z_{k,+}(\Delta t, x) = \sin(kx + \omega\Delta t). \tag{5.103}$$

While (5.102) is a sine function with period $T = \frac{2\pi}{\omega}$ and which vanishes at the origin, the function (5.103) (viewed as a function of x) , is a sine function with the same period but shifted to the left by $\Delta x = \frac{\omega}{k}\Delta t$. Using (5.101) we have

$$\Delta x = v\Delta t. \tag{5.104}$$

Since this holds for an arbitrary interval Δt we can visualise the solution z_+ as a wave which moves to the left with speed v, see Fig. (5.3).

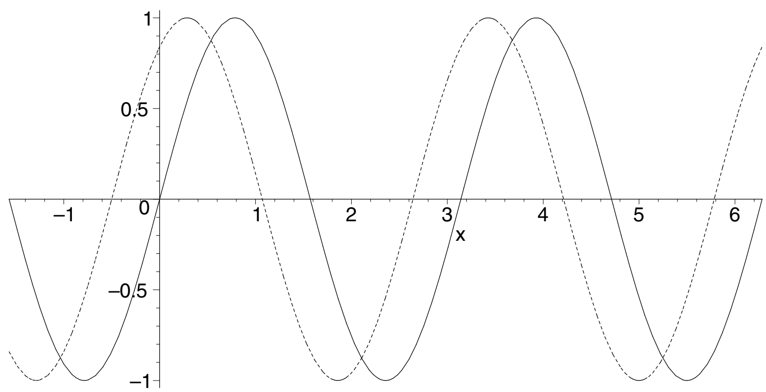

Figure 5.3: Left moving wave: plot of $\sin(2x + t)$ for $t = 0$ (solid line) and $t = 1$ (dashed line)

A similar analysis for z_- leads to the conclusion that it is a wave which moves to the right with speed v, see Fig. (5.4).

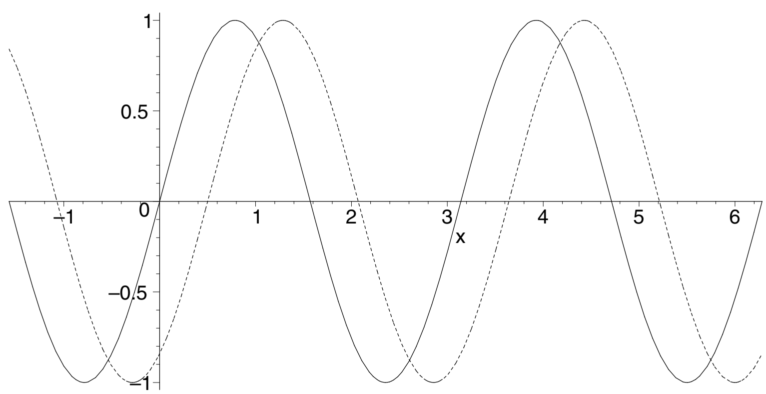

Figure 5.4: Right moving wave: plot of $\sin(2x - t)$ for $t = 0$ (solid line) and $t = 1$ (dashed line)

We thus arrive at the remarkable conclusion that the standing wave (5.97) is a sum of a left- and a right-moving wave. Moreover we found the physical interpretation of the parameter v: it is the speed of travelling wave solutions of the wave equation (5.96).

It is surprisingly easy to write down infinitely many travelling wave solutions, as shown by the following theorem.

Theorem 5.5.1 *Let $f : \mathbb{R} \to \mathbb{R}$ be an arbitrary twice differentiable function. Then $z_+(t, x) = f(x + vt)$ and $z_-(t, x) = f(x - vt)$ both satisfy the wave equation* (5.96).

Proof: Let $u = x + vt$. Then

$$\frac{\partial z_+}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} = v\frac{df}{du}$$

and

$$\frac{\partial^2 z_+}{\partial t^2} = v^2 \frac{d^2 f}{du^2}.$$

Also

$$\frac{\partial z_+}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = \frac{df}{du}$$

and

$$\frac{\partial^2 z_+}{\partial x^2} = \frac{d^2 f}{du^2}.$$

Hence

$$\frac{1}{v^2}\frac{\partial^2 z_+}{\partial t^2} = \frac{d^2 f}{du^2} = \frac{\partial^2 z_+}{\partial x^2}.$$

Similarly defining $w = x - vt$, we have

$$\frac{\partial z_-}{\partial t} = \frac{df}{dw}\frac{\partial w}{\partial t} = -v\frac{df}{dw}$$

and

$$\frac{\partial^2 z_-}{\partial t^2} = v^2 \frac{d^2 f}{dw^2}.$$

Also

$$\frac{\partial z_-}{\partial x} = \frac{df}{dw}\frac{\partial w}{\partial x} = \frac{df}{dw}$$

and

$$\frac{\partial^2 z_-}{\partial x^2} = \frac{d^2 f}{dw^2}.$$

Hence

$$\frac{1}{v^2}\frac{\partial^2 z_-}{\partial t^2} = \frac{d^2 f}{dw^2} = \frac{\partial^2 z_-}{\partial x^2}.$$

□

Using this theorem, we can construct waves of arbitrary shape. Taking for example the Gaussian $f(u) = \exp(-u^2)$ we can construct a right travelling wave

$$z_-(t, x) = \exp(-(x - vt)^2)$$

which, at any time t, looks like a “bell” centered at vt, see Fig 5.5.

We have already seen how to construct a standing wave by adding two travelling waves, one travelling to the right and one travelling to the left. In fact we can add any two solutions of the wave equation to obtain another solution. Although this result is easy to prove (see below), it is of fundamental importance in the study of the wave equation. It is called the **superposition principle**. We state in the following form:

Theorem 5.5.2 *Suppose $z_1(t, x)$ and $z_2(t, x)$ satisfy the wave equation* (5.96) *and A, B are real constants. Then $z(t, x) = Az_1(t, x) + Bz_2(t, x)$ also satisfies the wave equation.*

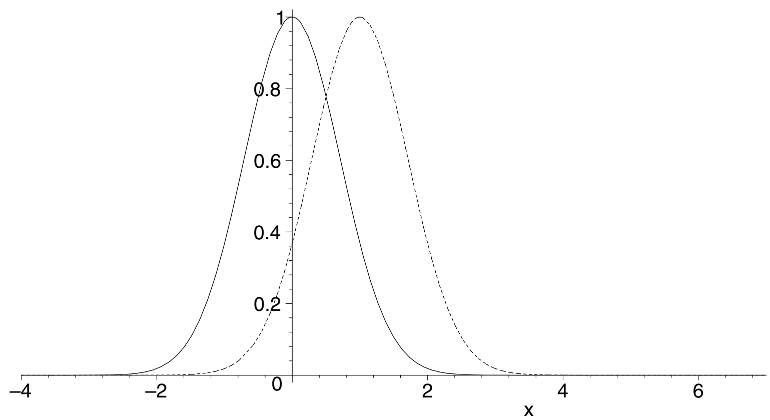

Figure 5.5: Right moving bell shape: plot of $\exp(-(x-t)^2)$ for $t = 0$ (solid line) and $t = 1$ (dashed line)

Proof: The result follows from the linearity of the wave equation. With $z(t, x)$ as defined in the theorem we deduce

$$\frac{\partial z}{\partial t} = A\frac{\partial z_1}{\partial t} + B\frac{\partial z_2}{\partial t}$$

and

$$\frac{\partial^2 z}{\partial t^2} = A\frac{\partial^2 z_1}{\partial t^2} + B\frac{\partial^2 z_2}{\partial t^2}.$$

Similarly

$$\frac{\partial z}{\partial x} = A\frac{\partial z_1}{\partial x} + B\frac{\partial z_2}{\partial x}$$

and

$$\frac{\partial^2 z}{\partial x^2} = A\frac{\partial^2 z_1}{\partial x^2} + B\frac{\partial^2 z_2}{\partial x^2}.$$

Hence

$$\begin{aligned} \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2} &= A\frac{1}{v^2}\frac{\partial^2 z_1}{\partial t^2} + B\frac{1}{v^2}\frac{\partial^2 z_2}{\partial t^2} \\ &= A\frac{\partial^2 z_1}{\partial x^2} + B\frac{\partial^2 z_2}{\partial x^2} \\ &= \frac{\partial^2 z}{\partial x^2}, \end{aligned} \tag{5.105}$$

where we used that both z_1 and z_2 satisfy the wave equation in the intermediate step. □