Waring's Theorem

Angus S. Macdonald

Volume 3, pp. 1749–1750

In

Encyclopedia Of Actuarial Science
(ISBN 0-470-84676-3)

Edited by

Jozef L. Teugels and Bjørn Sundt

Waring's Theorem

Waring's theorem gives the probability that exactly r out of n possible events should occur. Denoting the events $A_1, A_2, \ldots, A_n$, the required probability is

$$\sum_{t=0}^{t=n-r} (-1)^t \binom{r+t}{t} S_{r+t}, \qquad (1)$$

where $S_0 = 1$, $S_1 = \sum_i \mathrm{P}[A_i]$, $S_2 = \sum_{i<j} \mathrm{P}[A_i \cap A_j]$, $S_3 = \sum_{i<j<k} \mathrm{P}[A_i \cap A_j \cap A_k]$, and so on. This generalizes the inclusion–exclusion theorem of elementary probability theory, which is the case $r = 1$. See [2] (Chapter IV) for a proof of Waring's result, and [1] for a generalization, known as the Schuette–Nesbitt formula. By summation of equation (1), the probability that at least r out of the n possible events will occur is

$$\sum_{t=0}^{t=n-r} (-1)^t \binom{r+t-1}{t} S_{r+t}. \qquad (2)$$

(with the understanding that $\binom{r-1}{0} = 1$ identically). This and similar results have applications in the valuation of assurances and annuities contingent upon the death or survival of a large number of lives, although it must be admitted that such problems rarely arise in the mainstream of actuarial practice. For example, given n people, now age $x_1, x_2, \ldots, x_n$ respectively, the standard actuarial notation (*see* **International Actuarial Notation**) for life table survival probabilities is extended as follows:

$$ {}_tp_{\overline{x_1x_2\ldots x_n}}^{\,[r]} = P[\text{Exactly } r \text{ survivors after } t \text{ years}] \qquad (3)$$

$$ {}_tp_{\overline{x_1x_2\ldots x_n}}^{\,r} = P[\text{At least } r \text{ survivors after } t \text{ years}]. \qquad (4)$$

Assuming independence among the lives, the first of these may be evaluated using Waring's theorem (equation 1) in terms of $S_0 = 1$, $S_1 = \sum_i {}_tp_{x_i}$, $S_2 = \sum_{i<j} {}_tp_{x_i}\ {}_tp_{x_j}$, $S_3 = \sum_{i<j<k} {}_tp_{x_i}\ {}_tp_{x_j}\ {}_tp_{x_k}$, and so on. The second can be similarly evaluated using equation (2).

Older textbooks (see [3]) describe the so-called Z-method for computing these probabilities, which simply amounts to the fact that the binomial coefficients in Waring's theorem are the same as those of the first $n - r + 1$ terms in the expansion of $Z^r/(1+Z)^{r+1}$, and likewise when the probability in equation (4) is expressed in terms of $S_0, S_1, \ldots, S_n$, the binomial coefficients are the first $n - r + 1$ in the expansion of $Z^r/(1+Z)^r$.

References

[1] Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. & Nesbitt, C.J. (1986). *Actuarial Mathematics*, The Society of Actuaries, Itasca, IL.

[2] Feller, W. (1968). *An Introduction to Probability Theory and its Applications*, Vol. 1, 3rd Edition, John Wiley, New York.

[3] Neill, A. (1977). *Life Contingencies*, Heinemann, London.

ANGUS S. MACDONALD