Commutation Functions

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The basis of most calculations in **life insurance** is the expected present value (EPV) of some payments made either on the death of the insured person, or periodically, as long as the insured person survives. The primary computational tool of the actuary was (and often still is) the **life table** that tabulates l_x at integer ages x, representing the expected number (in a probabilistic sense) of survivors at age x out of a large **cohort** of lives alive at some starting age (often, but not necessarily at birth). From this starting point, it is simple to develop mathematical expressions for various EPVs, assuming a deterministic and constant rate of interest i per annum effective. For example, defining $v = 1/(1+i)$ for convenience:

$$A_x = \sum_{t=0}^{\infty} v^{t+1} \frac{l_{x+t} - l_{x+t+1}}{l_x} \tag{1}$$

is the EPV of a sum assured of \$1 payable at the end of the year in which a person now aged x dies, and

$$a_x = \sum_{t=0}^{\infty} v^{t+1} \frac{l_{x+t+1}}{l_x} = \sum_{t=1}^{\infty} v^{t} \frac{l_{x+t}}{l_x} \tag{2}$$

is the EPV of an **annuity** of \$1 per annum, payable at the end of each future year provided someone now aged x is then alive. These are the simplest examples of the **International Actuarial Notation** for EPVs. We make two remarks:

- Although the summations are taken to ∞, the sums of course terminate at the highest age tabulated in the life table.
- In the probabilistic setting, the life table l_x is just a convenient way to compute survival probabilities, and we would most naturally express EPVs in terms of these probabilities. It will become clear, however, why we have chosen to express EPVs in terms of l_x.

Although it is simple to write down such mathematical expressions, it is only since modern computers became available that it has been equally simple to compute them numerically. *Commutation functions* are an ingenious and effective system of tabulated functions that allow most of the EPVs in everyday use to be calculated with a minimal number of arithmetical operations. We list their definitions and then show how they may be used:

$$D_x = v^x \, l_x \tag{3}$$

$$N_x = \sum_{y=x}^{\infty} D_y \tag{4}$$

$$S_x = \sum_{y=x}^{\infty} N_y \tag{5}$$

$$C_x = v^{x+1}(l_{x+t} - l_{x+t+1}) \tag{6}$$

$$M_x = \sum_{y=x}^{\infty} C_y \tag{7}$$

$$R_x = \sum_{y=x}^{\infty} M_y. \tag{8}$$

The most elementary calculation using commutation functions is to note that

$$\frac{D_{x+t}}{D_x} = \frac{v^{x+t}}{v^x} \frac{l_{x+t}}{l_x} = v^t \;{}_t p_x, \tag{9}$$

which is the EPV of \$1 payable in t years' time to someone who is now aged x, provided they are then alive. It is clear that an annuity payable yearly is a sum of such contingent payments, so by the linearity of expected values we can write equation (2) as

$$a_x = \sum_{t=0}^{\infty} v^{t+1} \frac{l_{x+t+1}}{l_x} = \sum_{t=0}^{\infty} \frac{D_{x+t+1}}{D_x} = \frac{N_{x+1}}{D_x}. \tag{10}$$

Moreover, annuities with limited terms are easily dealt with by simple differences of the function N_x, for example;

$$\begin{aligned} a_{x:\overline{n}|} &= \sum_{t=0}^{n-1} v^{t+1} \frac{l_{x+t+1}}{l_x} \\ &= \sum_{t=0}^{n-1} \frac{D_{x+t+1}}{D_x} = \frac{N_{x+1} - N_{x+t+1}}{D_x} \end{aligned} \tag{11}$$

is the EPV of an annuity of \$1 per annum, payable in arrear for at most n years to someone who is now aged x.

Assurances and annuities whose amounts increase at an arithmetic rate present an even greater computational challenge, but one that is easily dealt with by commutation functions. For example, consider an annuity payable annually in arrear, for life, to a person now aged x, of amount \$1 in year 1, \$2 in year 2, \$3 in year 3, and so on. The EPV of this annuity (giving it its symbol in the International Actuarial Notation) is simply

$$(Ia)_x = \frac{S_{x+1}}{D_x}. \tag{12}$$

The commutation functions C_x, M_x, and R_x do for assurances payable at the end of the year of death, exactly what the functions D_x, N_x, and S_x do for annuities payable in arrear. For example, it is easily seen that equation (1) can be computed as

$$A_x = \sum_{t=0}^{\infty} v^{t+1} \frac{l_{x+t} - l_{x+t+1}}{l_x} = \sum_{t=0}^{\infty} \frac{C_{x+t}}{D_x} = \frac{M_x}{D_x}. \tag{13}$$

Assurances with a limited term are also easily accommodated, for example,

$$A^{1}_{x:\overline{n}|} = \sum_{t=0}^{n-1} v^{t+1} \frac{l_{x+t} - l_{x+t+1}}{l_x} = \sum_{t=0}^{n-1} \frac{C_{x+t}}{D_x} = \frac{M_x - M_{x+n}}{D_x} \tag{14}$$

is the EPV of a temporary (or term) assurance payable at the end of the year of death, for a person now aged x, if death occurs within n years. EPVs of increasing assurances can be simply computed using the function R_x.

Combinations of level and arithmetically increasing assurances and annuities cover most benefits found in practice, so these six commutation functions were the basis of most numerical work in life insurance until the advent of computers. However, the assumptions underlying them, that annuities are paid annually and that insurance benefits are paid at the end of the year of death, are not always realistic, rather they reflect the simplicity of the underlying life table l_x, tabulated at integer ages. There is no theoretical objection to setting up a life table based on a smaller time unit, in order to handle annuities payable more frequently than annually, or assurances payable soon after death; but in practice, the EPVs of such payments were usually found approximately starting with the EPVs based on the usual life table. Therefore, the arithmetic was reduced to a few operations with commutation functions and then some simple adjustments. These approximate methods are described in detail in textbooks on life insurance mathematics; see [1–3].

We may mention two specialized variants of the classical commutation functions described above (see [3] for details).

- They may be adapted to a continuous-time model instead of the discrete-time life table, allowing annuity payments to be made continuously and death benefits to be paid immediately on death.
- They may be extended to the valuation of pension funds, which requires a multiple-decrement table (e.g. including death, withdrawal from employment, age retirement, ill-health retirement, and so on) and payments that may be a multiple of current or averaged salaries.

Modern treatments have downplayed the use of tables of commutation functions, since clearly the same EPVs can be calculated in a few columns of a spreadsheet. As Gerber points out in [2], their use was also closely associated with the obsolete interpretation of the life table in which l_x was regarded as a deterministic model of the survival of a cohort of lives. We have presented them above as a means of calculating certain EPVs in a probabilistic model, in which setting it is clear that they yield numerically correct results, but have no theoretical content. Gerber said in [2]: “It may therefore be taken for granted that the days of glory for the commutation functions now belong in the past.”

References

[1] Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A. & Nesbitt, C.J. (1986). *Actuarial Mathematics*, The Society of Actuaries, Itasca, IL.

[2] Gerber, H.U. (1990). *Life Insurance Mathematics*, Springer-Verlag, Berlin.

[3] Neill, A. (1977). *Life Contingencies*, Heinemann, London.

(*See also* **International Actuarial Notation**)

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