AN OPTION PRICING APPROACH TO CHARGING FOR MATURITY GUARANTEES GIVEN UNDER UNITISED WITH-PROFITS POLICIES

By

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I hereby declare that the work presented in this thesis was carried out by myself at Heriot-Watt University, Edinburgh, except where due acknowledgement is made, and has not been submitted for any other degree.

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Abstract

Many insurance contracts pay a maturity benefit which depends in some way on the performance of some underlying assets subject to some minimum guaranteed benefit. Brennan and Schwartz (1976) were the first to apply the modern financial approaches of Black-Scholes-Merton to unit-linked policies with maturity guarantees. Wilkie (1987) was the first to extend this approach to with-profits policies. Since their pioneering research there have been many papers concerned with the pricing and reserving of investment guarantees. Interest has intensified in recent years as falling share prices and lower bond yields have made the guarantees very valuable. In this thesis we provide an extensive review of the literature of investment guarantees.

For the majority of the thesis we concentrate on unitised with-profits contracts. These contracts contain a guarantee made up of units which grow at a guaranteed growth rate (possibly zero) plus a variable reversionary bonus rate. We describe how to charge for these guarantees by the deduction of the cost of matching options in a similar way to that introduced by Wilkie (1987) for conventional with-profits. The approach for unitised with-profits has the advantage that options can always be found to match the guarantee, whereas it is possible for the guarantee to become uncovered when the approach is applied to conventional with-profits.

We simulate the payouts of unitised-with profits and unit-linked (without guarantees) policies. Projections are performed using firstly Geometric Brownian Motion and secondly the Wilkie Model as models of the real world return on investments. The existence of guarantees on the unitised with-profits policies reduces the variability of the payout compared with the unit-linked policy. However, the cost of matching these guarantees with options leads to a lower mean payout for the unitised with-profits policies. Higher guaranteed growth rates or higher bonuses reduce

the variability of the payout further, but at the cost of a lower expected payout.

At first we use a simple model for the risk-free return and bonuses. However modifications are introduced in later chapters to make the model more realistic. The risk-free return used to price the options is initially assumed to be a constant, but the use of a stochastic risk-free rate derived from the Wilkie model is found to increase the variability of the unitised with-profits payout.

We review the bonus algorithms used in with-profits simulations in the literature. The payouts on the unitised with-profits policies are then compared for a variety of different bonus algorithms. It is found that moving from a simple bonus algorithm, where the insurer attempts to declare the same bonus rate each year, to a dynamic bonus rate linked to the investment return during the year, leads to both a reduction in the variability of the payout and an increase in the mean payout. Smoothing the reversionary bonus rate is then considered, which has the desired effect of reducing the variability of the guarantees, but also leads to lower guarantees on average. As lower guarantees require less to be invested in options and hence more to be retained in equities, we find that the payouts are larger on average, but are more variable when reversionary bonuses are smoothed.

Finally, we create a portfolio of unitised with-profits policies with different issue dates. Firstly it is assumed that the insurer purchases options to match the guarantees. Then we assume that the insurer charges the policyholders an amount equal to the cost of the matching options, but actually invests these charges in the risk-free asset. The cashflows resulting from mismatching the assets and liabilities are calculated. There is a risk here that the charges collected are insufficient to make good the guarantee. Many simulations show substantial losses for the insurer. However, in the majority of simulations the charges received by the insurer exceed the cost of meeting the guarantee at maturity. In many simulations the guarantees never bite, leading the accumulated charges to form an excessively large estate, causing a problem for a mutual insurer as to how this estate can be equitably returned to policyholders.

The size of the free estate required to support this mismatched strategy is investigated. We find that free assets in excess of 50% of the asset share can be distributed to the owners of the insurer without significantly affecting the with-profits fund's ability to honour the guarantees.

Introduction

0.1 Introduction

Participating policies have been sold in the U.K. for many years, but the guarantees inherent in these products have sometimes been poorly understood by the offices selling them. Insurers have not accurately charged policyholders for their guarantees. Indeed Lister et al. (2000) states that the majority of U.K. companies made no deduction from policyholders' asset shares in respect of the cost of guarantees.

However, in recent years interest in the cost of these guarantees has grown. There have been two reasons for this interest. Firstly, the transition to a lower inflation environment has led to lower investment returns, and hence the guarantees have become more valuable. Secondly, there has been a growing appreciation of the financial economic techniques which can be used to price these guarantees.

In this thesis we consider unitised with-profits policies, a type of participating policy common in the U.K.. We use the option pricing technique introduced by Wilkie (1987) which he applied to U.K. conventional with-profits policies. Using options to price unitised with-profits policies was first considered by Yap (1999) in a MSc project under my supervision, and by Hare et al. (2000). However, neither of these papers have considered the effect of declaring bonuses in each future year using the option pricing approach.

Our aim in this thesis is to simulate the payouts and guarantees of unitised with-profits policies using the option pricing approach, and compare them to unit-linked and risk-free investments. Secondly, insurers in the U.K. have not traditionally invested in options, so we want to investigate whether it is beneficial for the insurer selling a portfolio of unitised with-profits contracts through time to invest in assets which do not match the guarantee.

0.2 Thesis Outline

Chapter 1 describes the operation of the insurance contracts and financial derivatives considered later in this thesis and defines the terminology used in their discussion.

Chapter 2 describes some of the different forms of guarantees available on both unit-linked and participating policies throughout the world and reviews the literature on pricing and reserving for these guarantees.

Chapter 3 begins with a description of the option pricing mechanism suggested by Wilkie (1987) for conventional with-profits policies. We then describe how Wilkie's methodology can be applied to unitised with-profits policies. The chapter concludes with a comparison of the approach as applied to conventional and unitised policies.

Chapters 4 and 5 project the payouts on the unitised with-profits policies and compare their mean and variance with unit-linked policies without guarantees. To perform the projections we need a 'real world' model of the investment returns through time which may be different from the model that the market uses at any given point in time to determine derivative prices. Chapter 4 uses geometric Brownian motion as the real world model, which is consistent with the assumptions used in the Black-Scholes formula to value options. Chapter 5 contrasts these results with those obtained using the Wilkie model to simulate the real world, when options are still priced using geometric Brownian motion.

Chapter 6 extends the work in Chapter 5 to use a more realistic risk-free rate of return when pricing options. Following the approach in Yang (2001) and Wilkie et al. (2003) we show how a zero-coupon yield curve can be fitted to the base rates and consol yields derived by the Wilkie model. We then value the options in the projections using a risk-free rate of return equal to the yield on a zero-coupon bond with the same term to expiry as the unitised with-profits policy.

In Chapters 4, 5 and 6 we consider a very simple bonus strategy. In Chapter 7 we review the bonus algorithms used in the literature when simulating both conventional and unitised with-profits policies. We then consider the effectiveness of a number of different bonus strategies for use with the option pricing technique.

Chapter 8 introduces a cohort of policies with different start dates. We begin by investigating the hedging portfolio. We then investigate an alternative approach. We assume that the insurer charges the policyholder in the same way according to the cost of matching options. These charges are passed to a guarantee account which actually invests in the risk-free asset rather than the matching options. We simulate the resulting mismatching profits and losses using the model introduced in Chapter 5. We then improve the model to introduce a stochastic risk-free rate as in Chapter 6, and a dynamic bonus algorithm as in Chapter 7.

In Chapter 9 we apply the multiple generation model of Chapter 8 to a number of issues that have affected the with-profits industry in recent years. We begin by considering the effects of new business growth. We then consider the effect of transfers from the guarantee account to the insurer's owners when mismatching profits have been particularly large. Finally we consider the effects of a transition to a low inflation environment.

Chapter 10 contains the conclusions and some suggestions for further research.

Chapter 1

Background to Life Insurance Policies and Financial Derivatives

In this chapter we define the terminology required in the remainder of the thesis. We begin in Section 1.1 by introducing with-profits and unit-linked policies. Then in Section 1.2 we will describe financial options. This material is well-known and so we give only a summary of the main ideas.

1.1 Types of Insurance Policy

In this thesis we consider endowment policies with financial guarantees. The main purpose of these endowment policies is as a savings vehicle. In return for either regular premiums or a single premium, the insurer will pay out a sum of money on maturity of the policy. If the policyholder dies before the end of the term of the policy their estate will instead receive a sum of money at that time. Alternatively, the policyholder may choose to end the contract early and receive a surrender payment instead of any future death or maturity benefits.

There are a number of ways in which the benefits may be calculated. We will consider three different forms of such endowment policies. In Section 1.1.1 we consider conventional with-profits policies, in Section 1.1.2 we consider unit-linked policies, and finally in Section 1.1.3 we consider unitised with-profits policies.

1.1.1 Conventional With-Profits

Conventional with-profits (CWP) policies were a very common form of participating policy in the U.K.. Very few CWP policies are sold today in the U.K., although there are many still in-force. We will describe below how CWP policies operate in the U.K..

At the outset of the policy a sum assured is set which is guaranteed to be paid at maturity or on earlier death. Typically the sum assured is calculated by equating the value of the premiums with the value of the benefits and expenses using a very low rate of interest. This sum assured is a much lower amount than would be paid on a non-participating policy with the same premium.

In return for accepting a low sum assured the policyholder is entitled to participate in the profits of the insurer. The profits to which the policyholder is entitled will vary from insurer to insurer.

In a mutual company there are no shareholders. The with-profits policyholders have voting rights at the annual general meeting. The policyholders would be entitled to any surplus emerging due to better experience than allowed for in the premium basis. This surplus may arise from the investment, mortality, or expense experience, including such profits from non-participating policies.

Proprietary companies are owned by shareholders. These companies may still sell with-profits business, although the profits will be split between the with-profits policyholders and the shareholders. Often all surpluses from business sold within the with-profits fund are split so that 90% goes to the policyholders and 10% goes to the shareholders. Another common approach is for all investment surplus to belong to the with-profits policyholders, while the shareholders take all remaining profits.

The profits of the insurer are returned to the policyholder in the form of bonuses. Once these bonuses are added to the policy they are guaranteed to be paid on maturity or earlier death. There are two forms of bonus: reversionary bonus, and terminal bonus.

Reversionary bonuses are so called because the policyholder will not receive them until some future date. They are typically declared on an annual basis and so are also called regular bonuses. The reversionary bonuses may be simple, compound, or super compound. Simple bonuses are declared as a proportion of the sum assured. Compound bonuses are declared as a proportion of the sum assured and previously declared bonuses. Super compound bonuses are declared at two different rates — one rate is declared as a proportion of the sum assured, whilst a second rate is declared as a proportion of the previously declared bonuses.

The insurer will not want to use all the surplus from the year to declare reversionary bonuses. They will want to retain some surplus to protect against losses in future years. For example, policyholders expect that the reversionary bonus rates will not change much from year to year. The insurer therefore retains some of the surplus in good years, in order to be able to declare bonuses in bad years. The process of changing the reversionary bonus rate more slowly each year than the experience would suggest is called smoothing.

Typically when the policy reaches maturity, the bonuses which have been added to the policy will have been less than could have been paid by the surpluses which have arisen. The insurer will then add a terminal bonus as a final payment. A terminal bonus is often also payable on death.

The starting point for calculating the final payout is typically the asset share. The asset share represents the part of the insurer's assets which has been contributed by the policyholder. We can calculate the asset share as the premiums paid, less expenses incurred, less the cost of life cover, less taxation incurred, all accumulated at the actual return earned on the assets. Hence, if the insurer paid each maturing policy its asset share, the insurer would make neither a profit nor a loss. The asset share is often modified by adding profits made from other policies, deducting transfers made to shareholders, and deducting charges for the cost of financial guarantees and the use of capital. It is these charges for financial guarantees that will be investigated in this thesis.

Therefore, at maturity the insurer will begin by calculating the asset share. However, in the same way that reversionary bonuses were smoothed from year to year, the insurer will also smooth the final payouts so that the maturity proceeds from policies maturing in consecutive years change more slowly than the asset shares. Hence, if the asset share is less (more) than the payout on comparable policies last year, then the actual payout will be a little more (less) than the asset share, but less (more) than the previous year's payout. However, the payout is subject to a minimum of the sum assured plus reversionary bonuses declared.

Once the insurer has determined the final maturity payout, the terminal bonus is calculated so that the sum assured plus reversionary bonuses plus the terminal bonus equals the payout.

The sum assured and reversionary bonuses declared to date are guaranteed to be paid on both maturity and earlier death. However, these guarantees do not apply to policies which are surrendered. Some policies offer guaranteed surrender values, although these will be lower than the guarantees on maturity and death. Other policies will pay a surrender value based on the asset share on surrender. However, many policies have paid less than asset share on surrender in order to discourage surrenders and generate surpluses.

An important feature of U.K. with-profits business is that the insurer retains considerable discretion in the management of the business. For example, we have already seen that the insurer retains the right to vary bonuses and payouts from year to year. In addition, the insurer's management has control of the investment strategy of the with-profits fund.

1.1.2 Unit-Linked

Unit-linked (UL) policies continue to be a popular means of saving. UL policies are sold by insurers and operate in a similar way to unit trusts in the U.K. and mutual funds in the U.S.A..

The policyholder can use their premiums, less any charges to cover expenses and the profits of the insurer, to buy units in a variety of unit funds. The value of the units is adjusted up and down in line with the performance of the underlying assets on at least a daily basis. At maturity the policyholder will receive the value of their units, and so their payout will exactly reflect the performance of their chosen investment.

Typically there are no guarantees on UL policies, so it is possible for the policyholder to get back less than they put in. However, some policies do offer guarantees, but these guarantees are fixed at outset and do not increase with bonuses.

On death the policyholder normally receives the value of their units or a guaranteed sum assured if greater. The guarantee is charged for by a regular deduction from the policyholder's units. Note that this sum assured is only payable on death and not on maturity.

On surrender the policyholder generally receives the value of their units as this reflects the accumulated value of their premiums less charges. However, this amount may be reduced by a surrender penalty to allow the insurer to recoup some of the charges and profits it would have made if the policy had continued in force.

1.1.3 Unitised With-Profits

Unitised With-Profits (UWP) policies operate in a similar way to UL, but are participating policies in the same way as CWP. They have now replaced CWP as the main type of participating policy sold in the U.K..

The policyholder uses their premium, less any charges to cover expenses and the profits of the insurer, to buy units in the with-profits fund. However, the value of the units does not directly vary with the value of the underlying assets. Typically the units grow with both a guaranteed growth rate and regular bonuses.

Firstly, the units are guaranteed to grow at some minimum rate. Some insurers express this as a guaranteed minimum rate of bonus. As long as the units are guaranteed to grow at least at 0% p.a. then the value of the units cannot fall. Some contracts guarantee that both existing units and units purchased with future premiums will grow at the guaranteed rate. Other contracts reserve the right to vary the guaranteed growth rate for units purchased with future premiums. The accumulation of the units rolled up at the guaranteed growth rate is analogous to the sum assured on CWP policies.

The guaranteed growth rate will be set at a conservatively low level. The policyholder participates in the profits of the insurer via regular bonuses. These bonuses increase the unit price on typically a daily basis. From time to time the insurer will change the bonus rate to be added in the future to reflect the performance of the underlying assets.

In the same way as for CWP, the insurer will use only part of the annual surplus each year to enable it to declare smoothed regular bonuses. Therefore at maturity a terminal bonus is declared to bring the payout up to the asset share subject to any smoothing of maturity payouts. However, the insurer must always pay out at least the value of the units at maturity, and so will make a loss if the asset share is less than this amount.

On death the policyholder will receive the value of their units, possibly with the addition of a terminal bonus, or the sum assured if greater. The guaranteed sum assured is charged for in the same way as for a UL policy. The sum assured is only used to calculate death benefits and is not payable at maturity.

On surrender, many companies would pay the unit value, with possibly an adjustment up or down to reflect the actual value of the underlying assets. Some companies offer guaranteed surrender values.

1.2 Financial Derivatives

Financial derivatives are assets which derive their value from the value of other assets. Derivatives can often be used to match the financial guarantees given by insurers. Hence the price of derivatives can be used to determine the cost of the guarantees. A description of financial derivatives can be found in textbooks such as Hull (1997).

In Section 1.2.1 we consider two simple financial derivatives, namely put options and call options. In Section 1.2.2 we consider how derivatives may be priced.

1.2.1 Put Options and Call Options

In this thesis we will show how financial options can be used to match the financial guarantees inherent in conventional with-profits and unitised with-profits policies.

The purchaser of an equity option has the right, but not the obligation, to trade a share at a fixed price called the exercise price. A European option gives the purchaser the right to trade the option on a given date. An American option gives the purchaser the right to trade on any date before the end of a given time period.

A put option gives the purchaser the right to sell a share at the exercise price. A call option gives the purchaser the right to buy a share at the exercise price.

Throughout this thesis we will be using the European put option in particular. We will now consider the operation of such an option in more detail as follows.

At outset the term and exercise price of the option are agreed. The purchaser of the option then pays a premium to the writer of the option.

If the share price is below the exercise price at the end of the term, then the purchaser will exercise their option by selling a share to the writer for the exercise price. The purchaser will have gained an amount equal to the excess of the exercise price above the share price, with the writer losing a corresponding amount.

If the share price is above or equal to the exercise price at the end of the term, then the purchaser will not exercise their option, and the option expires worthless.

We can see how the payoff from an option with exercise price of £4 varies with the share price for the purchaser in Figure 1.1 and the writer in Figure 1.2.

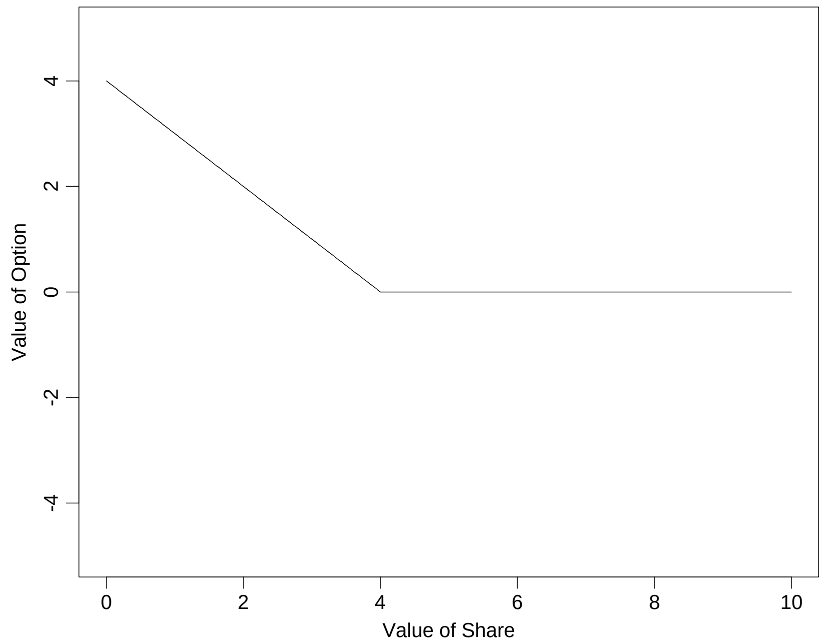

Figure 1.1: The Payoff from Holding One Put Option

One potential reason to buy put options is to protect a portfolio of shares from potential falls in the share price. For example, the value of an investment in one share and one put option at the expiry of the option is shown in Figure 1.3. We

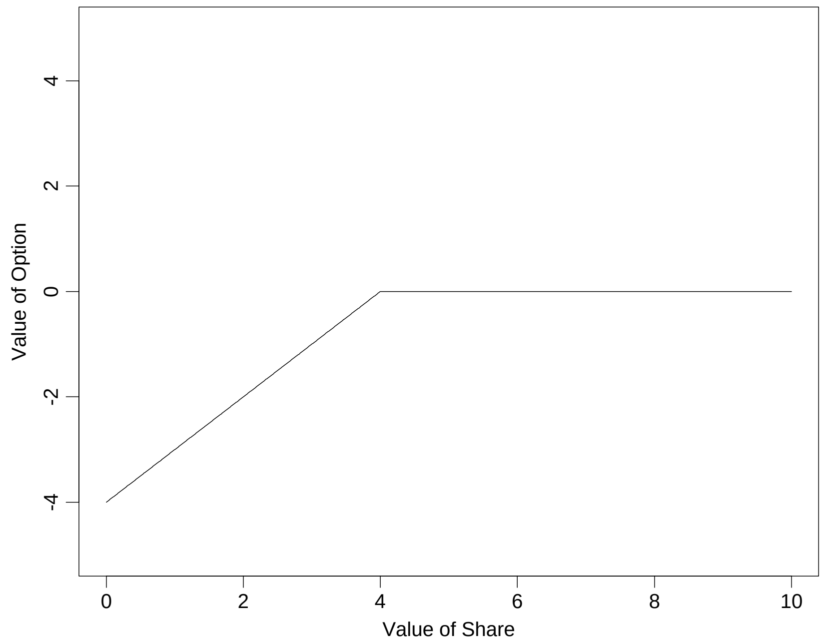

Figure 1.2: The Payoff from Writing One Put Option

can see that the portfolio is guaranteed to be worth at least as much as the exercise price regardless of the value of the share. It is this protection that we will be using throughout the remainder of this thesis.

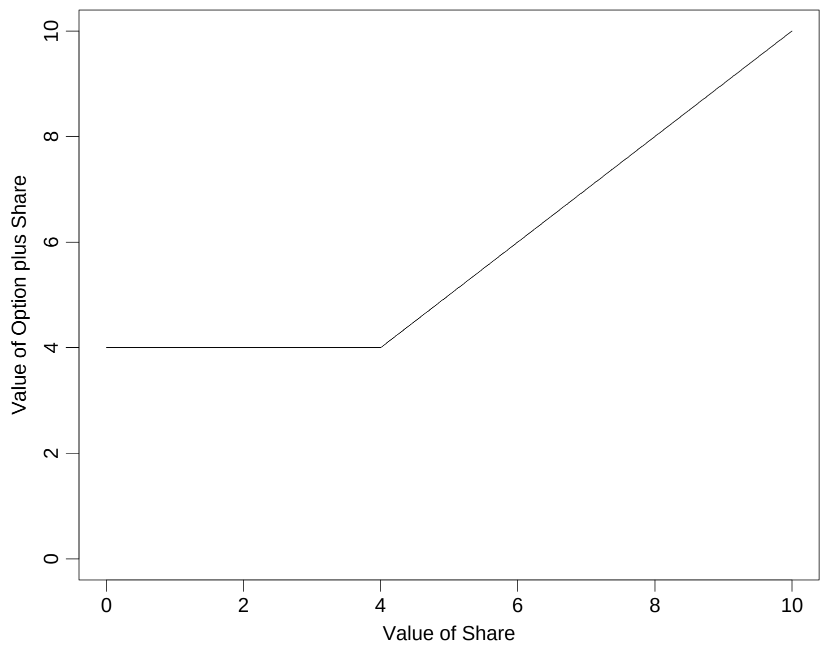

Figure 1.3: The Payoff from Holding One Put Option and One Share

1.2.2 Pricing Derivatives

The payoff from many derivatives, including the options described in Section 1.2.1, can be replicated by portfolios of other assets whose prices are known. Hence, by the no arbitrage principle, the price of the derivative must be equal to the value of the assets in the replicating portfolio.

It can be shown that this is equivalent to calculating the expected present value of the payoff at the risk-free rate. However, the expectation is not calculated using the real world probabilities of the payoffs, but using the equivalent martingale measure.

In the case of a European put option written on a non-dividend paying share, the above approaches lead to the Black-Scholes equation as follows:

$$\text{Put Price} = E\,e^{-r_f T}\,\Phi(-d_2) - S\,\Phi(-d_1)$$

$$d_1 = \frac{\ln(S/(E\,e^{-r_f T}))}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = \frac{\ln(S/(E\,e^{-r_f T}))}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$$

where

- T is the term of the option
- S is the value of a single share
- E is the exercise price of the put option
- r_f is the risk-free force of interest
- σ is the volatility of the share.

In Sections 1.1.1 and 1.1.3 we described conventional with-profits policies and unitised with-profits policies respectively. In Chapter 3 we will show how the charge for investment guarantees on these policies can be calculated using the cost of European put options given by the Black-Scholes formula above.

Chapter 2

Life Insurance Policies with Financial Options

The payout from many insurance contracts depends in some way upon the actual investment return earned on the assets subject to some minimum guarantee. Payouts may depend on the investment return in one of two ways. Firstly, we have unit-linked policies where the payout is directly linked to the performance of the underlying assets. Secondly, we have policies that participate in some way in the profits of the insurer. The full effect of the investment return is not immediately credited to these participating policies, but instead a series of bonuses is declared which will smooth the return credited to the policies.

The guarantee may take many forms. The following three types of financial guarantee are most common and can occur on both unit-linked and participating policies:

- **Maturity Guarantee** — Here the payout is guaranteed on maturity, but not on early surrender of the policy. The policy can be considered as containing a European type option.

- **Surrender Guarantee** — The guaranteed policy value can be taken at any time up to and including maturity. The guaranteed amount is generally larger for surrenders later in the policy term. The policy can be considered as containing an American type option.

- **Guaranteed Annuity Option** — Here the payout at maturity can be converted to an annuity on guaranteed terms if the policyholder so wishes. The maturity payout may also be guaranteed.

In this thesis we concentrate on policies with a guaranteed cash payout. We do not consider guaranteed annuity options further. Work on guaranteed annuity options can be found in Bolton et al. (1997), Bezooyen et al. (1998), Yang (2001), Ballotta and Haberman (2002), O'Brien (2002), Boyle and Hardy (2002), Wilkie et al. (2003) and Pelsser (2003).

In addition to the financial guarantees described above there may also be guaranteed death benefits. These may be the same in value as either the guaranteed maturity or surrender benefit, or may be separately defined. Guarantees on surrender and death bring very different risks to the insurer. Surrenders are in the control of the policyholder, who is most likely to exercise his option if it is in the money. The number and timing of deaths is much more predictable. In this thesis we concentrate on the investment risk and so ignore mortality for simplicity.

In this thesis we will concentrate on U.K. with-profits policies. However, it is interesting to contrast these policies with the financial options available throughout the world on both unit-linked and participating contracts. In Section 2.1 we will review the literature on the pricing and reserving for guarantees under unit-linked contracts. In Section 2.2 we do the same for participating policies.

2.1 Unit-Linked Policies with Guarantees

In this section we will review in turn the papers that have considered unit-linked policies with financial guarantees. Recall that unit-linked policies were introduced in Section 1.1.2. We begin in Section 2.1.1 by considering policies with guarantees on maturity, but no guaranteed surrender benefit. We then consider the addition of guaranteed surrender benefits in Section 2.1.2. Finally in Section 2.1.3 we compare the work of the different authors.

To give an indication of the cost of the guarantees, we quote for each paper, where possible, the typical cost of the option for a premium of £100. Note that these results are not directly comparable, as not only do the types of guarantee differ from paper to paper, but also the models and parameters used to value them.

We will use the following notation throughout Section 2.1:

- B_t is the benefit payable in the event of a claim at time t
- G_t is the guaranteed minimum benefit payable in the event of a claim at time t
- A_t is the value of the assets in the unit fund at time t
- T is the term of the policy.

2.1.1 Unit-Linked Policies with Maturity Guarantees

Brennan and Schwartz (1976)

Brennan and Schwartz (1976) were the first to consider life insurance policies with guarantees using the modern financial approaches introduced by Black-Scholes-Merton. The contracts considered are similar to unit-linked policies sold at the time, also known as segregated funds in Canada. The greater of the sum assured G_t (chosen at outset, but may be time dependent) or the value of the unit fund A_t is payable on either maturity or the end of the year of death. Hence the benefit payable in the event of a claim at time t is:

$$B_t = \max(A_t, G_t).$$

It is assumed that sufficient policies can be sold so that the mortality risk is removed and a deterministic proportion of policyholders will die each year.

The authors introduce the idea that a unit-linked policy with a guarantee can be matched by a combination of shares and European put options, or equivalently cash and European call options. Hence they can find the value of the insurance policy and the investment strategy to hedge the risk via the Black-Scholes formula. An analytic solution is found for the single premium case, while the differential equation for the regular premium case is solved numerically using finite differences.

Brennan and Schwartz (1976) provide numerical results for a policy guaranteeing a return of the investment component of the premiums paid to date on death or maturity. Using a risk-free rate of 4% p.a., volatility of the reference portfolio of 13.6% p.a., and ignoring mortality, the cost of the option adds a modest £1.8 to every premium for a 20-year regular premium contract which invests £100 each year into the reference portfolio. Shorter term contracts are more expensive — for example the cost of the option for a policy with a 10-year term adds £2.7 to every premium. Allowing for mortality reduces the average term of the policy and hence increases the cost — for example the cost of the option on a 20-year policy for a 50-year old is £1.9 p.a.. Increasing the risk-free rate of return to 8% p.a. considerably reduces the cost of the guarantee to the 50-year old to only £0.14 p.a.. Increasing the volatility to 20% p.a., while maintaining the risk-free rate at 8% p.a. considerably increases the cost of the guarantee to £1.02 p.a..

Boyle and Schwartz (1977)

Boyle and Schwartz (1977) follow the approach introduced by Brennan and Schwartz (1976). Brennan and Schwartz (1976) found that shorter term regular premium policies were more expensive. However, Boyle and Schwartz (1977) find that for single premium policies the cost of the guarantee initially increases with term before falling. Boyle and Schwartz (1977) give numerical examples using the same risk-free rate of 4% p.a. and volatility of 13.6% p.a. as Brennan and Schwartz (1976) and ignoring mortality. They find that the cost of the guarantee for a single premium investment of £100 is £3.58 for a 1-year term policy, rising to a peak of £4.30 for a 3-year policy, before falling to £1.73 for a 20-year policy.

The actual premium paid by the policyholder will be higher than the investment of £100 to cover expenses, profit and of course the cost of the guarantee. If the guarantee was £105, then the cost of the guarantee for the 20-year policy described above increases to £2.11. The authors then show how to calculate the cost of the premium for the policy if the guarantee is to return the full premium including the cost of the guarantee. If the guarantee was to return £105 plus the cost of the guarantee then the cost rises to £2.29.

Brennan and Schwartz (1979)

The work of Brennan and Schwartz (1976) was described in more detail in the monograph Brennan and Schwartz (1979). The monograph also considers the effect of transactions costs and the frequency of hedging. Premiums are calculated so that if the insurer continuously rebalances the hedge portfolio, and transactions costs are ignored, then the insurer breaks even with no risk. A naive investment strategy where the insurer invests the option premium in the risk-free asset leads to a positive expected profit, because the hedge portfolio does not short sell equities, but can lead to considerable losses. Allowing for transactions costs, hedging at regular intervals (the authors consider hedging every 1, 3, 6 and 12 months) leads to negative expected profits, but the potential downside is considerably reduced.

Ford et al. (1980)

The Maturity Guarantees Working Party was established in 1977 by the Faculty and Institute of Actuaries to consider reserving for the maturity guarantees which were common in the U.K. on unit-linked policies at that time. The working party's results were published in Ford et al. (1980).

Although the working party considered the option pricing approach of Brennan and Schwartz (1976), they decided to use the simulation method instead. Investment returns were projected for a large number of simulations. The present value of any claims under the maturity guarantee were then calculated. Quantile reserves were set up such that they would be sufficient in say 99% of cases.

The working party considered a number of investment models, but decided to use a combination of two models. Firstly they modelled dividends using a moving average model. They then modelled dividend yields using an auto-regressive model. The results were combined each year to derive the value of a unit investment in equities with the dividends reinvested.

Numerical results are given for regular premium policies which pay the larger of the value of the reference fund or a return of premiums at maturity. All policies are assumed to survive until maturity. Reserves are quoted for a total premium of £100 *i.e.* in the case of a 5-year policy the regular premium is £20.

The working party found that smaller reserves are needed for longer term policies. For example, to be sufficient in 99% of simulations, the quantile reserve required is £30 for a 5-year policy, but only £8 for a 20-year policy. The effect of discounting the loss over a longer time period reduces the reserve. There is also a lower probability of the guarantee biting for longer term policies. For example the guarantee bites in 23% of simulations for the 5-year policy, but only in 3% of simulations for the 20-year policy.

The working party also considered a portfolio of policies with different terms. Inevitably the guarantee bites more often, in 31% of simulations, because there are a range of dates at which the guarantee may be exercised. However, the reserves required are reduced showing that the risk has been diversified to some extent. The portfolio considered requires a reserve of only £5. This reserve is lower than the 20-year policy above, even though the portfolio has a lower average policy term of 18.62 years.

Some earlier work on maturity guarantees by Scott (1977) and Wilkie (1978) had also used simulations to obtain quantile reserves.

Collins (1980) and Collins (1982)

Collins (1982) considers reserving for unit-linked maturity guarantees by setting up a hedge portfolio. In the paper he usually refers to hedging as immunization. This work is expanded upon in a longer unpublished paper Collins (1980).

Collins (1982) starts by describing the hedging of single premium policies in the same way as Brennan and Schwartz (1976). Collins points out that the rebalancing of the portfolio required under the hedging strategy requires the insurer to regularly switch assets between shares and cash and thereby incur transactions costs. He then notes that the hedging strategy requires the largest rebalancing of assets when the value of the reference portfolio A_t is close to the value of the maturity guarantee $G_T\, e^{r_f(T-t)}$ discounted at the risk-free rate r_f. These switches between shares and cash become larger as maturity approaches.

To reduce the intensity of the asset reallocations, Collins suggests a technique called 'increasing the term' whereby we set up a hedge portfolio for a policy with a longer term $T' > T$ and larger guarantee $G_T\, e^{r_f(T'-T)}$. The initial reserves will be larger under this method, but the switches between shares and cash will be less extreme, and a surplus will emerge at maturity. Collins (1980) gives numerical results in the case where £95 is invested in the reference portfolio and the maturity guarantee is £100. He finds that for a 20-year policy the reserve required in excess of the reference portfolio is £3.67 if the term is not extended, but is £4.22 if the 'term' of the hedging portfolio is extended by 2 years. The difference is more noticeable for a 1-year policy where the excess reserve required is £7.63 if the term is not extended, but is £13.13 if the 'term' of the hedging portfolio is extended by 2 years. The surplus emerging at maturity depends on the ratio of the final value of the reference fund to the guarantee. For example, if the 'term' of the hedging portfolio has been extended by 2 years, then if the ratio equals 1 the surplus is £11.25, but the surplus falls below £0.10 if the ratio rises above 2 or falls below 0.5.

Collins (1982) then considers regular premium policies. The maturity guarantee is typically many times greater than the premium so that the assets in the early years would be inadequate to match the guarantee even if all the assets were invested at the risk-free rate. Brennan and Schwartz (1979) show that the full value of the guarantee can be hedged by a portfolio of three assets: the first two assets are shares and cash purchased from the past premiums and the third asset is the future premiums discounted at the risk-free rate.

Collins (1982) takes a different approach to Brennan and Schwartz (1979). Future premiums are ignored so that only the assets built up from past premiums are available in the hedge portfolio. Consequently only part of the total guarantee is actually hedged, with the amount of guarantee admitted increasing on premium payment dates. There are a range of ways in which the amount of the guarantee admitted to be hedged can be set. However, the author considers a 'postponement period' p such that during the period no guarantee is admitted and all assets are invested in shares. After the postponement period the guarantee is admitted in equal amounts $G_T/(T-p)$ on each premium payment date. Therefore the actual guarantee hedged on each premium payment date for an annual premium policy is:

$$
\begin{aligned}
AG_t &= 0 \quad \text{if } t < p \\
AG_t &= (t+1-p)\frac{G_T}{(T-p)} \quad \text{if } p \leq t < T.
\end{aligned}
$$

The advantage of the postponement period is that it reduces the intensity of the asset reallocations required for a portfolio of policies because hedging does not take place for the first p years of the contract. The disadvantage is that during the postponement period the insurer gains no protection from falls in the value of the reference portfolio.

Collins (1982) performs 500 simulations of a 20-year policy with annual premiums of £47.50 and a maturity guarantee of £1,000. He calculates the extra reserve that must be set up on each premium payment date due to the additional guarantee of £50 admitted at that time. With a postponement period of zero he finds that the average incremental reserve is £1.84 at time 0, £2.26 at time 10 and £0.85 at time 19. The maximum incremental reserve is £1.84 at time 0, £3.28 at time 10 and £3.82 at time 19. If instead we consider a postponement period of 5 years then no reserves are set up for the guarantee for the first 5 years. However, compared to the case of no postponement period, on average larger reserves are required for the 5-year postponement period of £2.56 at time 10 and £3.47 at time 19. Also, because the 5-year postponement period benefits less from hedging the investment risk, the maximum reserves observed increase substantially to £7.72 at time 10 and £16.28 at time 19.

Bacinello and Ortu (1993a) and Bacinello and Ortu (1993b)

Bacinello and Ortu (1993a) consider unit-linked policies with guarantees in the same framework as Brennan and Schwartz (1976) and Brennan and Schwartz (1979). The earlier papers find the premium for given guarantees. However, Bacinello and Ortu (1993a) consider endogenous guarantees *i.e.* guarantees given as a function of the premium. In the case of a single premium policy, the policyholder pays a premium P, of which only A_0 is invested in the reference fund. In the event of a claim at time t the policyholder receives the greater of the value of the reference fund and the guarantee which is a function of the premium $G_t(P)$ *i.e.* the benefit payable is:

$$B_t = \max(A_t, G_t(P)).$$

The authors show that the policy can be valued as the initial investment in the reference fund plus appropriate put options in the same way as described in the earlier papers. However, some functions of the premium are not eligible to describe the guarantee. Sufficient conditions are given for the premium to be well defined.

The authors then give numerical results assuming that the reference fund follows a geometric Brownian motion. For example, they consider a 10-year endowment policy which pays the greater of the reference fund and the single premium accumulated at 4% p.a. at the end of the year of death or earlier maturity *i.e.* $G_t(P) = P\,e^{0.04t}$. They find that for each £100 invested in the reference portfolio a premium of £114.37 is required. However, if the guarantee had been based on only the investment in the reference portfolio, *i.e.* $G_t = A_0\,e^{0.04t}$, then the premium reduces to £110.61. Hence the cost of 'endogenizing' the guarantee is £3.76. The authors find that the premium is far more sensitive to changes in the investment model parameters for guarantees based on the full premium rather than guarantees based only on the investment in the reference portfolio.

Boyle and Hardy (1997)

Boyle and Hardy (1997) compare two approaches for setting reserves for unit-linked policies with maturity guarantees, the stochastic simulation approach and the option pricing approach. The policies considered are Canadian segregated funds which offer a minimum payout at maturity equal to a proportion, often 75% or 100%, of the invested premium.

Under the stochastic simulation approach they use the Wilkie model to simulate the reference portfolio at maturity. Quantile reserves are then set so that the reference portfolio plus the reserve exceeds the guarantee with a given probability, say 95%. The authors first consider a 10-year contract with a single premium of £100, a guaranteed maturity payout of 100% of the premium and ignore expenses, mortality and lapses. They find that the reserve required to meet the guarantee with 95%

certainty is a modest £2.94. The expected cost is an even lower £0.61 implying that in the majority of cases the reserve can be returned to the insurer when the reference fund exceeds the guarantee. However, due to the skewness of the distribution of the maturity proceeds of the reference portfolio, we find that for longer term contracts the quantile reserve can be lower than the mean cost. This problem would have been avoided if reserves had been set as conditional tail expectations *i.e.* equal to the quantile reserve plus the expected value of the losses above the quantile reserve. The expected cost and reserve increase to £1.44 and £12.44 when 2% of the reference portfolio is deducted each year as a charge for expenses. The guarantee also applies on death, but lapses receive only the value of the reference fund, hence allowing for lapses of 5% p.a. and mortality of 0.5% p.a. and ignoring expenses leads to lower expected costs and reserves of £0.45 and £1.97 respectively. Similar results are obtained for regular premium policies.

Under the option pricing approach the insurer holds the hedging portfolio derived from the Black-Scholes equation. Whereas the simulation approach gives only say 95% security, the option pricing approach gives 100% security if the hedging can be performed continuously without transactions costs. The authors illustrate the costs for the policy described above, ignoring expenses, mortality and lapses. They assume a risk-free rate of 6% and volatility of the reference portfolio of 15%. The cost of the option is £1.69. This is more expensive than the expected cost of £0.61, but lower than the reserve of £2.94, found under the simulation approach. However, in practice continuous hedging is not possible and transaction costs must be paid. Assuming the insurer rebalances their hedge every month, then the expected transactions costs are £0.58, increasing the total cost to £2.27. The insurer is now also exposed to hedging error, although the authors do not set a reserve for this risk.

The authors then consider a move-based hedging strategy as an improvement on the time-based hedging strategy described above. Under the move-based strategy the portfolio is only rebalanced when the value of the reference portfolio changes by more than a given percentage. The authors find that if rebalancing takes place after moves of 5.4%, then the total cost is £2.27, which is exactly the same as under monthly rebalancing. The authors then consider the tracking error, *i.e.* the present value of the squared difference between the actual portfolio and the hedge portfolio summed over the times of rebalancing. The move-based strategy has a much lower tracking error of 0.18 compared to the monthly hedging tracking error of 0.35.

Boyle and Hardy (1996)

Boyle and Hardy (1996) is the working paper on which Boyle and Hardy (1997) was based. Boyle and Hardy (1996) provides more detail on the work described in Boyle and Hardy (1997) above.

In addition Boyle and Hardy (1996) describe segregated fund contracts with rollover options. Here the policyholder may choose to extend the term of their policy and reset the guarantee to the current value of the reference portfolio. This guarantee will only have value if the reference portfolio is larger than the guarantee. If the guarantee is larger then the policyholder should take the guaranteed maturity value and invest it in a new policy. The expected cost under the simulation method of a 10-year policy without a rollover option was £0.61, and the initial reserve was £2.94. Adding the rollover option to extend the term for a further 10 years increases the expected cost to £2.37, and the reserve to £16.75. Hence it is clear that the rollover option can be very expensive.

Hardy (1999) and Hardy (2001)

Hardy (1999) compares a number of different investment models in order to find the most suitable one for modelling the guarantees on segregated fund contracts. She finds that the best fit to U.S. and Canadian past equity returns is given by a regime switching lognormal (RSLN) model with two states. Under the RSLN model the stock market can be in one of two states. Each state is modelled by a lognormal model. The first state has relatively high expected return and low volatility, representing the usual state of the economy, while the second state has low expected return and high volatility, representing a period of uncertainty. At the end of each time interval, in this case monthly, the stock market can switch from one state to the other. The probability of transfer from the high return state to the low return state is lower than the probability of transfer in the opposite direction.

The author uses investment models fitted to Toronto Stock Exchange data to calculate quantile reserves at policy inception for contracts with a 10-year term, single premium of £100, maturity guarantee of a return of premiums and subject to a management charge of 2.75%. The expected cost of the guarantee is £1.081 under the lognormal model, which is similar to the expected cost of £1.278 under the RSLN model. However the fatter tails of the RSLN model means that the quantile reserve is much higher. For example, the reserve needed to be sufficient in 99% of cases is £18.960 under the RSLN model compared to just £8.959 under the lognormal model. This demonstrates that there is a danger of being under-reserved if the assets are modelled with an insufficiently fat tailed distribution.

Hardy (2001) compares a number of investment models with the RSLN model in a similar way to Hardy (1999). However, in the later paper, she also considers setting reserves using the conditional tail expectation. Again she finds that there is a danger of being under-reserved if the lognormal model is used instead of the heavier tailed RSLN model. In a similar example to the one described in Hardy (1999), but with different parameters, she finds that the conditional tail expectation reserve at the 95% level is £43.043 under the RSLN model compared to just £27.918 under the lognormal model.

A second type of guarantee, the 'guaranteed minimum accumulation benefit' (GMAB), is introduced in Hardy (1999). The term of a GMAB policy is divided into periods. At the end of each period the insurer guarantees to top up the value of the assets to equal the guarantee at the start of the period. The guarantee for the next period is then set equal to the value of the assets.

Hardy (2000)

Hardy (2000) builds on the work of Boyle and Hardy (1996) and Boyle and Hardy (1997). Hardy (2000) considers how to reserve for the guarantees attached to segregated funds, in the same way as the earlier papers *i.e.* by either the simulation approach or the option pricing approach.

Firstly the author compares the reserves required under the simulation approach using either the Wilkie model or the lognormal model. The parameters of the models have been chosen such that the expected cost of the guarantees are very similar. For example, a typical 10-year policy with a £100 single premium has expected cost of £1.1 and £1.0 under the lognormal and Wilkie model respectively. The 99%

quantile reserves are statistically different at the 5% level. For the same example, assuming no annual management charge, the reserves are £22.93 and £20.99 under the lognormal and Wilkie models respectively.

Smaller initial reserves are required if future management charges can be used to meet the cost of the guarantee. The size of the management charges depend on the future size of the reference fund and so their amount is uncertain. Hence the reserves only allow for the management charges they can expect to receive with say 95% certainty. In the case of a 1% p.a. management charge the initial reserves required are reduced to £18.76 and £16.91 under the lognormal and Wilkie models respectively.

The reserves considered above were initial reserves. Each subsequent year the same methodology could be used to recalculate the reserves. Hence each year the insurer would need to supply new capital to boost the reserves, or would be able to release some reserves. In an attempt to smooth these capital cashflows, the author considers using a corridor approach, whereby the reserves are only strengthened if the probability that they are insufficient falls below 92.5%, and may be weakened if the probability that they are insufficient rises above 99.8%. Using the corridor approach further capital is only required in 16% of simulations.

The option pricing approach is then considered. Firstly the author performs dynamic hedging with monthly rebalancing. This requires additional reserves, to cover future hedging error and transactions costs, which are calculated using the lognormal model. Secondly the author considers buying options directly from a third party who takes a margin of 5% in the volatility of the underlying when calculating the option price.

The expected cost of the guarantee is highest when options are bought (£6.56), followed by the dynamic hedging approach (£4.12), and is lowest for the simulation approach (£1.10). Ignoring counterparty risk, no further reserves are required under the option buying approach, and so the cost is a known £6.56. However the other two approaches are more risky and so additional reserves must be held. This raises the initial outgo to £17.65 under the simulation approach and £7.05 under the hedging approach. Hence the insurer has a choice between reducing the expected cost on the one hand and reducing the initial capital strain and risk on the other. Therefore the choice of method the insurer should use depends on their cost of capital and attitude to risk.

Hardy (2002)

Hardy (2002) considers how to allow for parameter uncertainty in the investment model used to set reserves for the guarantees under segregated fund contracts. The investment model focused on in the paper is the regime switching lognormal model.

Quantile reserves are usually set to be sufficient in say 95% of simulations of an investment model using best estimate parameters. Similarly the reserves can be set using the conditional tail expectation, *i.e.* the expected payout given that the payout exceeds say the 95% quantile, again using the best estimate parameters in the investment model. Hence we have allowed for process variability, but not parameter variability.

The author shows how to use a Bayesian approach to allow for the parameter variability. The joint distribution of the parameters is generated by Markov Chain Monte Carlo simulations.

The author gives numerical values for the reserves required for a variety of guarantees and at a variety of reserving levels. We will concentrate on the results for a typical segregated fund contract with a maturity guarantee and premium of £100. Reserves are set in this case as the conditional tail expectation at the 95% level. The RSLN model is fitted to Toronto Stock Exchange data.

The author considers two methods of reserving for the guarantees. Firstly, the simulation approach assumes that reserves are invested in risk-free bonds. Using best estimate parameters the required reserve is £1.23. However the reserve required more than doubles to £2.85 when we allow for parameter uncertainty.

The second method of reserving is based on option pricing. Here the reserve required equals the initial cost of buying the hedge portfolio plus a conditional tail expectation reserve to allow for hedging error and transactions costs. Using best estimate parameters the required reserve is £0.73. This reserve only increases slightly to £0.84 when parameter uncertainty is allowed for. In this case we see that the simulation approach is much more sensitive to parameter uncertainty than the option pricing approach. The author shows that this is also true for other guarantees and reserving levels.

Finally the author examines the effect of model error on the size of the reserves. A GARCH(1,1) model is now fitted to the data. Allowing for parameter uncertainty this gives a reserve of £0.68 under the simulation approach. This is substantially less than the reserve of £2.85 using the RSLN model. Hardy (2001) shows that the RSLN model is a better fit to Canadian data than the GARCH(1,1) model. Hence there is a considerable risk of under-reserving if the wrong model is used under the simulation approach.

Using the option pricing approach the reserve calculated by the GARCH model is £0.82 if parameter uncertainty is allowed for. This is only marginally less than the £0.84 obtained using the RSLN model. Hence the simulation approach is also more sensitive to model error than the option pricing approach. Again the author shows that this is also true for other guarantees and reserving levels.

Hardy (2002) also considers the 'guaranteed minimum accumulation benefit' (GMAB) which was introduced in Hardy (1999). The term of the GMAB policy is divided into three periods of length 8 years, 10 years and 10 years respectively. At the end of each period the insurer guarantees to top up the value of the assets to equal the guarantee at the start of the period. The guarantee for the next period is then set equal to the value of the assets. Using the simulation approach the author finds that the reserve required for the GMAB policy is £12.68. Using the hedging approach the required reserve is £3.78. These reserves are considerably higher than the reserves of £1.23 and £0.73 required for the 10-year contract with a maturity guarantee.

2.1.2 Unit-Linked Policies with Surrender Guarantees

Grosen and Jorgensen (1997)

Grosen and Jorgensen (1997) consider a single premium unit-linked policy which can be surrendered at any time before maturity for the greater of the value of the reference fund A_t and the initial investment A_0 accumulated at a guaranteed interest rate g. Hence at any time t before maturity the policyholder can choose to take the following benefit:

$$B_t = \max(A_t, A_0 e^{gt}).$$

The authors show that if the reference fund follows a geometric Brownian motion, and the policyholder surrenders the policy at the time to optimise the policy value, then American option pricing theory can be used to calculate the premium required in excess of the initial investment.

The authors compare a policy which has an interest rate guarantee on surrender with a policy which only has an interest rate guarantee on maturity. Numerical results are given where the reference fund has volatility of 10% and the risk-free rate is 10%. A 10-year policy with a guaranteed minimum return of 4% exercisable at maturity has premium of £100.26 for an initial investment of £100. The same contract but with the surrender guarantee costs £102.93. Hence the cost of the surrender option of £2.93 is considerably more expensive than the cost of the maturity option of £0.26.

The authors extend this work to participating policies in Grosen and Jorgensen (2000) which we discuss in Section 2.2.2.

2.1.3 Summary of Unit-Linked Policies with Guarantees

All the papers reviewed in Sections 2.1.1 and 2.1.2 considered the same basic type of policy. These policies paid out the greater of the value of the reference fund and a guaranteed amount at maturity. The amount of the guarantee varied between the papers and could be equal to, greater than, or less than the investment in the reference fund, but it was always a fixed amount known at outset.

All the authors considered the basic policy described above, but some authors compared them with more complex products. Grosen and Jorgensen (1997) added a surrender guarantee, Boyle and Hardy (1996) included a rollover option, and Hardy (1999) and Hardy (2002) considered a guaranteed minimum accumulation benefit.

We can see in Table 2.1 that a number of authors have allowed for the extra complexity of paying guarantees on death. These papers all follow the approach introduced by Brennan and Schwartz (1976) who assumed that mortality rates were deterministic and that the insurer sold a sufficiently large number of policies to eliminate mortality risk. Hence the benefit to be valued is the weighted average of the benefits payable at known times where the weights are given by the probability of death in a given year or survival to maturity. Hence, if claims are paid at the end of the year of death or earlier maturity at time T, the benefit to be valued is

$$\sum_{t=0}^{T-2} {}_tp_x \, q_{x+t} \max(A_{t+1}, G_{t+1}) + {}_{T-1}p_x \max(A_T, G_T).$$

The remaining authors ignore mortality for simplicity.

Table 2.1: Unit-Linked Literature — Allowance for Mortality

Papers that Ignore Mortality	Papers that Include Mortality
Ford et al. (1980) Collins (1980) Collins (1982) Hardy (2000) Hardy (2001) Grosen and Jorgensen (1997)	Brennan and Schwartz (1976) Boyle and Schwartz (1977) Brennan and Schwartz (1979) Bacinello and Ortu (1993a) Boyle and Hardy (1996) Boyle and Hardy (1997) Hardy (1999) Hardy (2002)

We can see in Table 2.2 which authors consider single premium policies, which consider regular premium policies, and which consider both. The single premium case is the simplest to consider. The authors that consider regular premium policies have all assumed that the amount of future premiums are known at outset, although in practice the policyholder can often choose to vary their premiums.

Ford et al. (1980), Boyle and Hardy (1996), and Boyle and Hardy (1997) all use simulations to set reserves for their regular premium policies and include the future premiums as additional cashflows in their projections.

Brennan and Schwartz (1976), Boyle and Schwartz (1977), Brennan and Schwartz (1979), and Bacinello and Ortu (1993a) all use option pricing techniques to calculate the fair value of a regular premium contract. Boyle and Schwartz (1977) show

Table 2.2: Unit-Linked Literature — Premium Frequency

Papers that Consider Single Premium Only	Papers that Consider Regular Premium Only	Papers that Consider Single and Regular Premiums
Hardy (1999) Hardy (2000) Hardy (2001) Hardy (2002) Grosen and Jorgensen (1997)	Ford et al. (1980)	Brennan and Schwartz (1976) Boyle and Schwartz (1977) Brennan and Schwartz (1979) Collins (1980) Collins (1982) Bacinello and Ortu (1993a) Boyle and Hardy (1996) Boyle and Hardy (1997)

how a regular premium policy can be valued using the Black-Scholes equation for a dividend paying stock where the premium is considered as a negative dividend. Brennan and Schwartz (1979) show how to construct a hedging portfolio constructed from three assets: the reference portfolio, the risk-free asset, and the value of the future premiums discounted at the risk-free rate. The discounted future premiums can also be considered as a risk-free asset, as we have assumed that they are known fixed cashflows.

Collins (1980) and Collins (1982) also use an option pricing approach. However, whereas Brennan and Schwartz (1976), Boyle and Schwartz (1977), Brennan and Schwartz (1979), and Bacinello and Ortu (1993a) value the total guarantee by allowing for the receipt of future premiums, Collins (1980) and Collins (1982) allow for only a part of the guarantee and hedge it using only the current assets ignoring future premiums. Collins (1980) and Collins (1982) also modify this approach by using a 'postponement period' during which all assets are invested in the reference portfolio which reduces the intensity of asset reallocations at the cost of increased risk.

Tables 2.3 and 2.4 show the problems that the papers are trying to solve. Table 2.3 shows the papers which calculate a charge for the guarantee. The charge considered is always an initial charge such that the premium paid by the policyholder is higher than the investment into the reference fund. Charges made throughout the policy and at the end of the policy are not considered in any of these papers. The method for calculating the charge in each case is the cost of buying appropriate options.

Table 2.3: Unit-Linked Literature — Calculation of Initial Charge

Papers that do Not Calculate an Initial Charge	Papers that Calculate an Initial Charge
Ford et al. (1980) Hardy (1999) Hardy (2001)	Brennan and Schwartz (1976) Boyle and Schwartz (1977) Brennan and Schwartz (1979) Collins (1980) Collins (1982) Bacinello and Ortu (1993a) Boyle and Hardy (1996) Boyle and Hardy (1997) Hardy (2000) Hardy (2002) Grosen and Jorgensen (1997)

Table 2.4 shows the authors which have considered hedging error, transactions costs, and reserves. We discussed above how to charge for the guarantee by buying appropriate options. However the writer of the options must hedge the risk and this introduces hedging error and transactions costs. Brennan and Schwartz (1979) consider the distribution of profits and losses caused by hedging error and transactions costs for an insurer that hedges at discrete time intervals. Collins (1980) and Collins (1982) use a technique called 'increasing the term' in an attempt to reduce the effect of hedging error and transactions costs by setting up larger reserves based on the hedge portfolio for an option with a longer term than the insurance contract. Boyle and Hardy (1996) and Boyle and Hardy (1997) compare the distribution of the transactions costs when hedging is performed by time-based and move-based hedging strategies. Hardy (1999) and Hardy (2000) calculate the quantile reserve which is sufficient to cover the transactions costs and hedging error with a given probability. Similarly Hardy (2002) calculates a conditional tail expectation reserve for transactions costs and hedging error.

Collins (1980), Collins (1982), Hardy (1999), Hardy (2000), and Hardy (2002)

Table 2.4: Unit-Linked Literature — Hedging Error, Transactions Costs, and Reserves

Papers that Consider Hedging Error	Papers that Consider Transactions Costs	Papers that Consider Reserves
Brennan and Schwartz (1979)	Brennan and Schwartz (1979)	
		Ford et al. (1980)
Collins (1980)	Collins (1980)	Collins (1980)
Collins (1982)	Collins (1982)	Collins (1982)
	Boyle and Hardy (1996)	Boyle and Hardy (1996)
	Boyle and Hardy (1997)	Boyle and Hardy (1997)
Hardy (1999)	Hardy (1999)	Hardy (1999)
Hardy (2000)	Hardy (2000)	Hardy (2000)
		Hardy (2001)
Hardy (2002)	Hardy (2002)	Hardy (2002)

set up reserves in excess of the cost of the matching options in order to allow for transactions costs and hedging error. An alternative approach for an insurer who either cannot buy or does not wish to buy such options is to set up large reserves to protect against the mismatching risk. Ford et al. (1980), Boyle and Hardy (1996), Boyle and Hardy (1997), Hardy (1999), Hardy (2000), Hardy (2001), and Hardy (2002) invest the premiums in the reference portfolio and hold additional quantile reserves in cash which are sufficient to meet any excess of the guarantee over the value of the reference fund with given probability. Similarly Hardy (2001) and Hardy (2002) also calculate such reserves based on conditional tail expectations.

Table 2.5 shows the methodology used by the authors. The majority of papers have used an option pricing technique to derive the cost of the guarantees or a hedging portfolio. Maturity guarantees lead to European options, while the surrender guarantees discussed by Grosen and Jorgensen (1997) lead to American options. The guaranteed minimum accumulation benefit considered by Hardy (1999) and Hardy (2002) leads to a more complex combination of European options.

Simulations have been used to set reserves by Ford et al. (1980), Boyle and Hardy (1996), Boyle and Hardy (1997), Hardy (1999), Hardy (2000), Hardy (2001), and Hardy (2002). Brennan and Schwartz (1979) use simulations to obtain the

Table 2.5: Unit-Linked Literature — Methodology Used

Papers that Use Option Pricing Only	Papers that Use Simulation Only	Papers that Use Option Pricing and Simulation
Brennan and Schwartz (1976)	Ford et al. (1980)	Brennan and Schwartz (1979)
Boyle and Schwartz (1977)	Hardy (2001)	Collins (1980)
Grosen and Jorgensen (1997)		Collins (1982)
		Bacinello and Ortu (1993a)
		Boyle and Hardy (1996)
		Boyle and Hardy (1997)
		Hardy (1999)
		Hardy (2000)
		Hardy (2002)

distribution of profits and losses if discrete hedging is used subject to transactions costs. Collins (1980) and Collins (1982) use simulations to obtain the distribution of the reserves that will be required in the future, although the reserves themselves are calculated using an option pricing approach. Bacinello and Ortu (1993a) use simulations to calculate the cost of the guarantees for regular premium policies. A variety of investment models have been used to simulate the rates of return as can be seen in Table 2.6.

2.2 Participating Policies with Guarantees

In many countries participating savings and insurance policies are available which include a guaranteed element to any payouts. The initial guarantee is supplemented by some form of bonus which can be varied to reflect the performance of the underlying investments. Once declared, bonuses become part of the guarantee. However the exact structure of these policies varies substantially between countries. These policies may share in profits from other sources, for example mortality or other lines of business, but we will concentrate solely on the surplus arising from the investments.

Participating policies can be divided into two broad types: policies with maturity guarantees where only the payout at maturity is guaranteed and increased with

Table 2.6: Unit-Linked Literature — Investment Model used in Simulations

	Investment Model Used			
Paper	Lognormal	RSLN	Wilkie Model	Other
Brennan and Schwartz (1979)	yes	no	no	
Ford et al. (1980)	no	no	no	dividends - MA dividend yield - AR
Collins (1980)	yes	no	no	
Collins (1982)	yes	no	no	
Bacinello and Ortu (1993a)	yes	no	no	
Boyle and Hardy (1996)	no	no	yes	
Boyle and Hardy (1997)	no	no	yes	
Hardy (1999)	yes	yes	yes	
Hardy (2000)	yes	no	yes	
Hardy (2001)	yes	yes	no	
Hardy (2002)	no	yes	no	GARCH

bonuses, and the payout on early surrender is at the insurer's discretion; and policies with interest rate guarantees where the guaranteed value of the fund can be taken at any time. The way bonuses are added under both policy types can vary from systems with total discretion given to the insurer when setting bonuses, to systems where bonuses are strictly calculated by a formula set at outset.

We will now review the literature for participating policies. We begin in Section 2.2.1 by considering policies with guarantees on maturity, but no guaranteed surrender benefit. We then consider the addition of guaranteed surrender benefits in Section 2.2.2. Then in Section 2.2.3 we allow for the possibility of reduced payouts if the insurer becomes insolvent. In Section 2.2.4 we consider policies where the insurer has discretion over the investment of the policyholder's assets. Finally in Section 2.2.5 we compare the work of the different authors.

To give an indication of the cost of the guarantees, we quote for each paper, where possible, the typical cost of the option for a premium of £100. Note that these results are not directly comparable, as not only do the types of guarantee differ from paper to paper, but also the models and parameters used to value them.

We will use the following notation throughout Section 2.2:

- B_t is the benefit payable in the event of a claim at time t
- G_t is the guaranteed minimum *maturity* benefit declared to date at time t
- A_t is the value of the assets in the reference portfolio at time t
- X_t is the value of the assets in the reference portfolio allocated to policyholders at time t
- Z_t is the value of the assets in the reference portfolio allocated to shareholders at time t
- Y_t is the value of the assets in the reference portfolio which are not yet allocated to policyholders or shareholders at time t
- T is the term of the policy.

2.2.1 Participating Policies with Maturity Guarantees

Wilkie (1987)

Wilkie (1987) is the first to consider pricing with-profits guarantees using modern financial economics. He considers a conventional with-profits policy which pays the greater of the value of the assets and a guaranteed amount. The initial guarantee is equal to the sum assured which is then increased on a regular basis by reversionary bonuses.

Wilkie notices that, from the policyholder's point of view, a with-profits policy is the same as a portfolio of shares and put options. The put options are based on the portfolio of shares and have the same maturity date as the insurance policy. He shows how the number of options required N_t and their exercise price E_t must be recalculated to match the guarantees after every bonus declaration and premium payment as follows:

$$G_t = N_t\,E_t.$$

Wilkie's approach is very different to that of other authors in that the *policyholder's* assets are actually considered to be invested in shares *and* put options. The work of the other authors which we describe in the remainder of this chapter all assume that the policyholder's assets are held in a reference portfolio of shares (or in the case of Persson and Aase (1997) the portfolio is made up entirely of cash). The other authors assume that it is the insurer who buys the options or performs the hedging on their own account. Under Wilkie's approach, the cost of buying the options is an implicit charge for the guarantee. Even if the insurer did not invest the policyholder's assets in shares and put options, this approach could be used to calculate the asset share after deduction of charges for the guarantee. It is this reduced asset share, rather than the value of the actual assets, which should be used as a starting point for determining the terminal bonus.

A second important difference between Wilkie's approach and other authors is that to use Wilkie's approach in practice requires no assumptions about bonus methodology. The policyholders could even be given a free choice in how much bonus they wanted each year, subject to the maximum affordable. A high bonus simply leads to a greater proportion of assets being invested in options. All other authors require a bonus mechanism to be determined at outset, although the bonuses themselves may be stochastic, for example by relating them to the investment return in that year.

Wilkie obtains numerical results using both stochastic simulations and historic past data. For example, consider a 20-year policy with a single premium of £100 sold in June 1965. Shares follow the Financial Times — Actuaries All Share Index with dividends reinvested. Options are priced using the Black-Scholes formula with risk-free rate of interest of 7% p.a., and volatility of shares of 20% p.a.. If the initial sum assured is £100 with reversionary bonuses of 2% declared at the start of each following year then the maturity payout is £1,198, made up of £100 sum assured, £46 reversionary bonus and £1,052 of terminal bonus. This compares to a payout of £1,218 from a unit-linked policy invested in the same shares with no guarantee. Hence the guarantee has cost £20 in this case. The cost of the guarantee can be much larger if either the sum assured or reversionary bonuses are increased. For example, the maturity payout is only £840 for a policy with sum assured of £200 and reversionary bonuses of 4% each year, and so the cost of the guarantees is £378.

Wilkie's methodolgy for regular premium policies is described in detail together with numerical results in Section 3.2 of this thesis.

Wilkie's methodolgy has also been used in MSc projects under my supervision by Yap (1999), Kouloumbos (2000), and Miranda (2001).

Yap (1999) considers Wilkie's option pricing approach with a slightly more complex bonus algorithm whereby a bonus is only declared if the new guarantee is less than say 70% of the maximum possible guarantee. Yap also extends Wilkie's approach to unitised with-profits policies. These two methods are compared with Wilkie's original approach for a 20-year policy using the actual investment returns between 1965 and 1985.

Kouloumbos (2000) considers the same three methods as Yap (1999): conventional with-profits with bonuses allowed up to the maximum, conventional with-profits with bonuses allowed up to 70% of the maximum, and unitised with-profits. Koulombos compares the payouts under the three methods for policies sold between 1950 and 1979.

Miranda (2001) uses the option pricing approach to hedge the guarantees for conventional with-profits policies rather than buying the options. She considers the hedging error introduced by rebalancing the portfolio at discrete time intervals rather than continuously. Finally, she compares the hedging investment strategy with a more traditional strategy where assets are gradually switched from equities to gilts as the policy approaches maturity.

Two further MSc projects under my supervision by Abbey (2003) and Lal (2003) used the option pricing approach to unitised with-profits first suggested by Yap (1999).

Abbey (2003) uses the option pricing approach to hedge the guarantees for unitised with-profits policies rather than buying the options. In the same way as Miranda (2001) did for conventional with-profits, he considers the hedging error introduced by rebalancing the portfolio at discrete time intervals rather than continuously.

Lal (2003) simulates the profits and losses from unitised with-profits policies where the insurer does not use the guarantee charges to invest in matching options or a hedge portfolio.

Persson and Aase (1997)

Persson and Aase (1997) show how to calculate a market price for two different types of Norwegian participating policy. For both policy types the benefits are payable on either death or maturity. Under the first type the policyholder receives their investment accumulated at the greater of the actual investment return and the guaranteed investment return. Hence if r_t and g_t are the actual return and guaranteed return respectively in year t, then the benefit payable at the end of year T under the first type of guarantee for a unit investment at time 0 would be:

$$B_T^1 = \exp\left[\max\left(\sum_{t=1}^{T} r_t, \sum_{t=1}^{T} g_t\right)\right]. \tag{2.1}$$

The second type of policy is more valuable as it applies the guarantee to each year separately. The policyholder's account at the end of the year is the start of year account accumulated with the greater of the guaranteed rate or actual return over the year as follows:

$$B_T^2 = \exp\left[\sum_{t=1}^{T} \max(r_t, g_t)\right]. \tag{2.2}$$

These policies are assumed to be invested in cash, and so are quite unlike U.K. with-profits policies which have sizeable equity investments. No-arbitrage prices are found assuming that the cash investment follows an Ornstein-Uhlenbeck process. Taking for example a policy with a 20-year term, the authors find that the cost of the first type of guarantee is £1.63 for an investment of £100. However, for the second type of policy where the guarantee is applied to each year separately, the cost of the guarantee is greatly increased to £5.64.

Miltersen and Persson (1999)

Miltersen and Persson (1999) use a Heath-Jarrow-Morton approach to modelling the term structure of interest rates and hence derive formulae for prices and hedging portfolios for policies with guarantees. The authors consider the same two types of guarantee as Persson and Aase (1997) although the guarantee is here only paid on maturity and not death *i.e.* the guaranteed rate of return is either applied over the entire term of the contract, or is applied annually.

The reference portfolio can either be invested in shares or cash. Shares are modelled as a geometric Brownian motion. The authors show how both the Vasicek and Cox-Ingersoll-Ross interest rate models fit into the Heath-Jarrow-Morton framework. They then show how a combination of the reference portfolio and a zero coupon bond (with the same outstanding term as the policy) can be used to hedge the guarantee. The ability to stochastically model the term structure of interest rates when calculating the value of the guarantees is an important improvement over the deterministic risk-free rate used in other papers.

The authors give numerical results for a reference portfolio invested in shares and a guarantee that is applied over the entire period of the contract (*i.e.* the policy has the first type of guarantee described in Persson and Aase (1997)). These results show that the value of the guarantee for terms of less than seven years is similar under the assumption of deterministic or stochastic short-term interest rates. For example, the value of the guarantee is approximately £5.5 per £100 investment for 2-year contracts in both cases. However, for the longer term contracts which would typically be offered by insurers, the cost of the option is much higher under the assumption of stochastic interest rates than under deterministic interest rates. For example, the guarantee costs approximately £3.5 under stochastic interest rates, compared to just £2 under the deterministic interest assumption.

The authors also give numerical results for the second type of guarantee described in Persson and Aase (1997) where the guarantee is applied to each year individually. They only consider policies with a 2-year term. Using the same parameters as used for the examples above they find that the guarantee costs £10.15 under stochastic interest rates and £10.53 under the deterministic interest assumption. Hence they confirm the result of Persson and Aase (1997) who found that the second type of guarantee was very much more expensive than the first type. Again the authors find that the cost of the guarantee for short term policies is similar under the assumption of deterministic and stochastic short-term interest rates.

Miltersen and Persson (2003)

Miltersen and Persson (2003) compare participating policies with and without a terminal bonus. They assume that all policies continue to maturity. At the outset the policyholder invests a single premium A_0 into a reference portfolio. At the end of any given year t, the assets of the reference portfolio A_t are divided between a number of accounts: the policyholder account X_t, the shareholder account Z_t, and in the case of policies with a terminal bonus entitlement, the buffer account Y_t. The assets backing these different accounts are all invested in the same reference portfolio. The total investment in the reference portfolio A_t is assumed to follow a geometric Brownian motion.

At the end of each year the policyholder account is credited with a guaranteed rate of return g_t and a proportion α of any excess of the actual investment return r_t over the guarantee. Hence the policyholder account evolves as follows:

$$X_t = A_0 \exp\left[\sum_{\tau=1}^{t} \left(g_\tau + \max\{\alpha\,(r_\tau - g_\tau), 0\}\right)\right].$$

If the policy is not entitled to a terminal bonus then the remainder of the reference portfolio is held in the shareholder account as a charge for the guarantee *i.e.* $Z_t = A_t - X_t$.

If the policy is entitled to a terminal bonus then the shareholder account is credited with a proportion β of any excess of the actual investment return over the guarantee based on the *policyholder* account as follows:

$$Z_t = Z_{t-1} + X_{t-1}\left(\exp[\max\{\beta\,(r_t - g_t), 0\}] - 1\right).$$

Notice the unusual feature that the shareholder account is not accumulated with investment return and so only increases by the annual charge.

If the policy is entitled to a terminal bonus then the balance of the reference portfolio is held in the buffer account so that $Y_t = A_t - X_t - Z_t$. If the buffer account is positive at maturity it is paid in full to the policyholder as a terminal bonus. If the buffer account is negative the policyholder only receives the policyholder account, with the insurer meeting the cost from the shareholder account and, if necessary, other funds.

The authors then find combinations of g_t, α and β, that are fair in the sense that, under the equivalent martingale measure, the expected present value of the payoffs to the policyholder and shareholder are equal to their initial investment of A_0 and zero respectively.

The authors find that the term of the contract has no effect on the choice of g_t and α for policies with no terminal bonus. For example, assuming a risk-free rate of 10% p.a. (high by today's standards) and volatility of the reference portfolio of 20% p.a., they find that a guaranteed return g_t of 5% p.a. and a participation in excess returns α of 50% represents a fair contract. Under the equivalent martingale measure the expected present value of the charges for such a policy with a 5-year term is approximately £32 for every £100 of premium.

If the policyholder wants their policy to be eligible for a terminal bonus then for a given guaranteed growth rate they will have to accept a lower participation in excess returns. The terminal bonus buffer also reduces the losses for the insurer. Hence the insurer will charge less. An equivalent terminal bonus policy to the example above with a participation rate α of 30% would reduce the expected present value of charges to just £15. The charges could be decreased further if the policyholder picked a policy with a lower value of g_t or α.

Hare et al. (2000)

Hare et al. (2000) consider maturity guarantees on both conventional and unitised with-profits policies. Initially they consider the reserves required for such policies, although the main theme of the paper is charging for the guarantees.

Throughout the paper they model the return on shares using the Wilkie model with low inflation parameters. Gilts could also have been modelled using the Wilkie model, however to simplify the interpretation of the results the authors choose to model gilts as earning a deterministic 5% p.a.. In the single premium case this is equivalent to saying that the insurer buys zero coupon bonds, with the same maturity date as the policy, which will return a known 5% p.a..

The authors give numerical results for a 10-year single premium conventional with-profits endowment. They consider policies that have been in-force for differing numbers of years. The investment performance to date has been a return of 7.6% p.a. on equities and 5% p.a. on gilts, with the future modelled stochastically as above. The sum assured is equal to the premium accumulated with interest of 1% p.a.. Super compound bonuses of 1% of the sum assured and 2% of the attaching bonuses have been declared at the end of each year, but it is assumed that no further bonuses will be declared in the future. Expenses and mortality are ignored throughout.

Reserves are set under the then current U.K. reserving regulations including both the resilience reserve and minimum solvency margin. The authors find that although the policies initially require capital support from the insurer, that in later years the policies actually supply capital. For example, if the policy described above is invested entirely in shares, then the reserves at outset are 111.3% of the asset share. Lower reserves are required if the insurer invests more in gilts, for example the reserve is only 96.9% of the asset share if 70% is invested in shares and 30% in gilts. By the time the policy has been in-force for 5 years the reserve as a percentage of the asset shares has fallen to 95.2% and 86.9% for an equity backing ratio of 100% and 70% respectively.

However, the statutory reserves are inadequate if the insurer wants to be 99% sure that it can meet its liabilities. It is now assumed that the insurer sets reserves such that they are adequate in 99% of simulations of the investment model described above. The reserves are now larger and the policy requires capital support for longer. For example, in the case of an equity backing ratio of 70%, the reserves are now 110.3% and 106.1% of the asset shares at outset and after 5 years respectively. In this case the policy does not supply capital until time 7.

In the examples above the reserves are assumed to be invested in the same proportions of equities and gilts as the asset share. The additional reserves are required to meet the shortfall when equities have fallen below the guaranteed amount. However if these additional reserves are also, at least in part, invested in equities they will fall in value at precisely the time they are needed. It would have been better if the excess of the reserves over the asset share were invested entirely in gilts, as this would result in the need for smaller reserves.

The authors now consider charging for the guarantee and the cost of capital required to set up the reserves. They consider three different methods of charging.

Firstly they consider the 'asset share charging approach'. Under this method a constant proportion of the asset share is deducted each year. The authors acknowledge that this is a broad brush approach which is unlikely to reflect the actual cost of guarantees and capital support for a given individual policy. However, it is simple to calculate and explain and was the most commonly used method in the U.K. according to Tillinghast (1997). In their examples the authors set the cost equal to 0.15% p.a.. The authors have chosen an arbitrary value for the charge in contrast to Persson and Aase (1997), Miltersen and Persson (1999), Miltersen and Persson (2003), Bacinello (2001), Grosen and Jorgensen (2000), Miltersen and Hansen (2002), Miltersen and Persson (2000), Grosen and Jorgensen (2002), and Jorgensen (2001) who all set charges that are fair in the sense that the expected value of the guarantees and charges are equal under the equivalent martingale measure.

Secondly the authors introduce the 'capital support charging approach'. Here the charge is related to the excess of the reserve over the asset share. Hence this approach is better targeted than the asset share charging approach because higher guarantees will result in higher reserves and hence higher charges. The charge deducted has two parts: a) the part of the capital support invested in gilts is charged for the expected excess return of equities over gilts, and b) the full capital support is charged say 1% for its use. The justification for charge a) is that if the insurer was not required to use its assets for capital support, it would have invested in equities. However, this ignores the fact that gilts have lower expected return because they are less risky. Again the charges are not set equal to the value of the guarantee under the equivalent martingale measure.

The final method is the 'put spread strategy'. Here, in a similar way to Wilkie (1987), put options are bought so that the guarantee could be met by the known value of the gilts plus the sale of the the equities at the exercise price. However, this method gives 100% security whereas the first two methods only charged for the reserves required to give 99% security. Therefore the authors decided that put options would also be sold with an exercise price equal to the equity price that could be exceeded with probability of 99%. Hence there is still a 1% chance of the guarantee becoming uncovered. The charge for the guarantee is equal to the cost of the option purchased less the receipts from the option written.

The authors compare the payout the policyholder would receive under the three different charging mechanisms. The same policy and investment model as described above were used. They only consider the case where no reversionary bonuses are declared throughout the policy. The authors find that the asset share charging approach overcharges policies with low equity backing ratio and undercharges policies with high equity backing ratio. This is as expected because the same charge is deducted irrespective of the size of the guarantee or the riskiness of the assets. The capital support charging approach gave very similar results to the put spread strategy for any given equity backing ratio. For example, if the insurer invests entirely in equities, the policyholder's return on his investment is 7.6% p.a. under the asset share charge, but only 7.1% p.a. under the put spread strategy and 7.2% p.a. under the capital support charge. However the position is reversed when the equity backing ratio is 60% because the policyholder's return on his investment is only 6.8% p.a. under the asset share charge, but is 6.9% p.a. under both the put spread strategy and the capital support charge.

The authors also provide many of the above results for regular premium conventional with-profits policies and single premium unitised with-profits policies. The results are similar in both cases.

Bruskova (2001), in an MSc project under my supervision, extends the results of Hare et al. (2000). For example, she models gilts using consols from the Wilkie model whereas Hare et al. assume a deterministic return of 5% p.a.. She also investigates the effect of using the put spread strategy when bonuses are declared every year whereas Hare et al. assume that no bonuses will be paid in the future.

Bacinello (2001)

Bacinello (2001) considers Italian style participating policies. The benefit B_t is paid on death or maturity in year t. The initial guaranteed benefit B_0 and premium P_0 are set using a traditional equation of value using a low cautious interest rate

i. For example, for a regular premium endowment policy of term n, we have the equation of value:

$$P_0\, \ddot{a}^i_{x:\overline{n}|} = B_0\, A^i_{x:\overline{n}|}.$$

The policyholders are entitled to participate in a proportion α of the actual investment return r_t earned in year t. Hence at the end of each year the guaranteed benefit is increased if αr_t exceeds the guaranteed rate of return i implicit in the pricing formula. Hence we have

$$B_t = B_{t-1}\left[1 + \max\left(\frac{\alpha\, r_t - i}{1+i}, 0\right)\right] = B_{t-1}\,(1 + b_t). \tag{2.3}$$

An interesting feature of Italian contracts is that their premiums increase at the same rate of bonus as the guarantees *i.e.* $P_t = P_{t-1}\,(1 + b_t)$. No terminal bonus is added to these policies.

The reference portfolio is assumed to follow geometric Brownian motion, and the risk-free rate of return is constant. The author then finds a relationship between the four parameters governing the policy such that the policy is fair. The fairness relation is the same for both regular and single premium contracts and does not depend on the term of the contract. One parameter, the risk-free interest rate, is beyond the control of the insurer. The other three parameters, the interest rate in the premium basis i, the participation rate α and the volatility of the reference portfolio σ, are all within the control of the insurer. For example, if the risk-free rate is 3%, and the insurer chooses to set the participation rate α equal to 20% and invests in a reference fund with volatility σ of 15%, then for the contract to be fair it must be priced using an interest rate i of 2.48%.

2.2.2 Participating Policies with Surrender Guarantees

Grosen and Jorgensen (2000)

Grosen and Jorgensen (1997) considered the value of unit-linked policies with guaranteed surrender values. Grosen and Jorgensen (2000) extend this work to single premium participating policies.

At the outset of the policy, an amount A_0 is invested in the reference fund. The reference fund is split into two parts: the policy account X_t, and the bonus reserve Y_t. The policy can be surrendered at any time t before maturity for the value of the policy account. Each year bonuses are added to the policy account subject to some minimum guaranteed interest rate g. The authors use a fixed bonus mechanism (although they recognise that in practice the insurer can change the formula) such that if the bonus reserve to policy account ratio exceeds a given target buffer ratio γ, then a defined proportion α of the excess is used to declare bonuses. The bonus reserve is a balancing account between the assets and the policy account. No terminal bonus is given and any remaining assets in the bonus reserve at maturity are retained by the insurer as a charge for the guarantee. Hence the policy account and bonus reserve evolve as follows:

$$\begin{aligned} X_t &= X_{t-1} \max\left(1+g, 1+\alpha\left[\frac{Y_t}{X_t}-\gamma\right]\right) \\ A_t &= X_t + Y_t. \end{aligned}$$

Although the authors are attempting to model Danish participating policies, their contract is also very similar to a U.K. unitised with-profits policy with the policy account behaving in a similar way to the unit value. The target buffer ratio γ works in a very similar way to a target terminal bonus rate. However, Danish policies have a much lower proportion of assets invested in equities than has been the case in the U.K. in the past, typically around 30%, and so a fixed bonus mechanism with no terminal bonus is a more appropriate model than for the U.K..

The investments backing the reference fund follow a geometric Brownian motion. The bonuses declared, and hence the value of the option, is path dependent. The authors show how to calculate a risk-neutral price for these contracts using Monte Carlo simulations for the case where the guarantee can only be taken at maturity and binomial tree methods for the case where the guarantee can be taken on surrender or maturity.

The authors give numerical results where the risk-free rate is 6% and the volatility of the reference portfolio is 15%. They consider a 20-year endowment with target buffer ratio γ of 10%, and participation rate α of 25%. The premium required for a policy with a surrender guarantee is £105.17 per £100 invested in the reference fund. If the policy has a maturity guarantee but no surrender guarantee, then the premium falls to £95.47 *i.e.* the insurer actually pays £4.53 of their own funds into the reference fund at outset. Higher participation rates and lower target buffer ratios all lead to higher premiums.

Miltersen and Hansen (2002)

Miltersen and Hansen (2002) combine the work of Grosen and Jorgensen (2000) and Miltersen and Persson (2003). They use exactly the same mechanism to declare reversionary bonus as was introduced in Grosen and Jorgensen (2000). However, Grosen and Jorgensen (2000) declare no terminal bonus, so that any remaining funds at maturity are kept by the insurer as a charge for offering the guarantee. Instead Miltersen and Hansen (2002) pay a terminal bonus so that the final payout is equal to the asset share in a similar way to with-profits policies in the U.K.. Miltersen and Hansen (2002) charge for this guarantee in the same way as Miltersen and Persson (2003) by introducing a shareholder account which receives annual charges from the policy account and meets any shortfall between the guarantees and the policy account at maturity. The total assets A_t are therefore split between the policy account X_t, shareholder account Z_t, and the balance is held in a buffer account Y_t.

Two methods of charging are considered. Direct charges are deducted as a proportion η of the guarantee. Indirect charges operate by crediting the shareholder account with a proportion β of any distributed surplus.

Hence the total assets are divided between the three accounts as follows. Firstly the assets A_t are credited with the investment return over the year. Then, the policyholder account is credited with a proportion α of any excess of the buffer at the start of the year over the target ratio γ. This bonus is subject to a minimum of g. A direct charge η is deducted from the policyholder account. Similarly the sum of the policyholder and shareholder accounts are credited with a proportion $\alpha + \beta$ of any excess of the buffer at the start of the year over the target ratio γ, again subject to a minimum of g. The shareholder account can then be calculated as the balance of these two accounts. Finally the buffer account is calculated as the balance of the total assets.

$$
\begin{aligned}
X_t &= X_{t-1} \max\left(1+g, 1+\alpha\left[\frac{Y_{t-1}}{X_{t-1}+Z_{t-1}}-\gamma\right]\right) - \eta\, X_{t-1} \\
Z_t &= (X_{t-1}+Z_{t-1}) \max\left(1+g, 1+(\alpha+\beta)\left[\frac{Y_{t-1}}{X_{t-1}+Z_{t-1}}-\gamma\right]\right) - X_t \\
Y_t &= A_t - X_t - Z_t.
\end{aligned}
$$

The assets backing the contract are assumed to follow a geometric Brownian motion. The payouts under these policies are path dependent and so Monte Carlo simulations are used to project the assets and the values of the policyholder and shareholder accounts. The authors then find what guarantees can be offered given the participation rate and level of charges so that the expected present value of the final shareholder account is zero under the equivalent martingale measure. For example, if the policyholder wanted a minimum rate of bonus g of 2%, with a participation rate α of 50%, then a fair contract would either have a direct charge η of approximately 0.7%, or an indirect charge β of approximately 30%.

To conclude their paper, Miltersen and Hansen (2002) consider the effect of pooling different contracts. They first calculate the charges as above for a given level of guarantees. Hence the expected present value of the final payout should equal the premium under the equivalent martingale measure. They then consider two different policies in the same pooled fund. If the asset share to policy account ratio exceeds a given value then a proportion of the excess is credited to the two policies via identical bonus rates. This bonus distribution mechanism is identical to that used in the single policy case except we now compare a pooled asset share to the sum of the policy accounts. The two policies are chosen so that they have different guarantees or different terms to maturity. The authors show that such pooling can result in cross subsidies between groups of policies. They also show that in certain circumstances the insurer's profit increases indicating that pooling could be used to reduce the charges of all the pooled policies.

Miltersen and Persson (2000)

Miltersen and Persson (2000) also build on the work of Miltersen and Persson (2003). They use the same reversionary and terminal bonus mechanism as Miltersen and Persson (2003), except that the guaranteed return g_1 added to the original investment may be different than the guaranteed return g_2 added to the attaching bonuses. The indirect charging approach is followed. Hence the policy accounts X_t^1 and X_t^2, shareholder account Z_t, and the balance Y_t, evolve as follows:

$$\begin{aligned}
X_t^1 &= A_0\, e^{g_1 t} \\
X_t^2 &= X_t^1 \left(\exp\left[\max\{\alpha\,(r_t - g_1), 0\}\right] - 1\right) + X_{t-1}^2 \exp\left[g_2 + \max\{\alpha\,(r_t - g_2), 0\}\right] \\
Z_t &= Z_{t-1} + X_{t-1}^1 \left(\exp[\max\{\beta\,(r_t - g_1), 0\}] - 1\right) \\
&\quad + X_{t-1}^2 \left(\exp[\max\{\beta\,(r_t - g_2), 0\}] - 1\right) \\
Y_t &= A_t - X_t^1 - X_t^2 - Z_t.
\end{aligned}$$

The authors develop the work of Miltersen and Persson (2003) further by considering the effect of a non-negativity constraint on the buffer fund. Miltersen and Persson (2003) allow the buffer to become negative during the term of the contract, with any shortfall at maturity met by the shareholder account. Miltersen and Persson (2000) also consider the case where the shareholder account is used to reset the buffer to zero if it is found to be negative at the end of any year:

$$\begin{aligned}
Y_t &= \max(Y_t, 0) \\
Z_t &= Z_t + \min(Y_t, 0).
\end{aligned}$$

The authors then, in the same way as Miltersen and Persson (2003), find what guarantees can be offered given the participation rate and level of charges so that the expected present value of the final shareholder account is zero under the equivalent martingale measure.

The authors give numerical results for a 5-year policy with guarantee rates g_1 and

g_2 both equal to 3% and a participation rate α of 50%. For a policy without the non-negativity constraint on the buffer fund the insurer's charge β is only approximately 5%. However, once the non-negativity constraint is introduced the insurer's charge is increased to approximately 10%.

Changes to the guaranteed rate of return on the attaching bonuses g_2 make little difference to the value of the policy, because the term is too short to allow substantial bonuses to build up.

2.2.3 Allowing for the Possibility of the Insurer becoming Insolvent

Briys and de Varenne (1997)

Briys and de Varenne (1997) stochastically model participating contracts allowing for the possibility of the insurer becoming insolvent. The shareholders of the insurer are assumed to supply a limited amount of capital, $(1-\rho)A_0$, at the outset of the policy. The policyholder invests an amount $\rho\, A_0$. The maturity benefit accumulates at a minimum guaranteed rate of return g. This guaranteed maturity payout is further augmented by a proportion α of the excess of the policyholder's assets over the minimum guarantee. No reversionary bonuses are added during the term of the policy. Ignoring the possibility of insurer insolvency, the policyholder receives the following at maturity:

$$B_T = \rho\, A_0\, e^{gT} + \alpha \max(0, \rho\, A_T - \rho\, A_0\, e^{gT}).$$

However, if the total assets are less than the maturity benefit, then the insurer defaults and the policyholder receives whatever assets remain *i.e.* $\min(A_T, B_T)$. Hence the policyholder's position is equivalent to holding a zero coupon bond, plus a call option entitling them to a share in the excess assets, *less* a put option representing the insurer defaulting when total assets fall below the guarantee.

The authors are interested in the duration of the liabilities allowing for the possibility of default of the insurer, rather than pricing or reserving for the guarantee. The case when $\alpha = 0$ and $\rho = 0$ corresponds to a non-profit contract sold by an insurer with infinite capital. In this case the duration of a 10-year contract is 10 years as expected. However, the authors show that under their model, the duration falls as either α or ρ increases. For example, when $\alpha = 0.4$ and $\rho = 0.7$ the duration falls to 7 years.

Grosen and Jorgensen (2002)

Grosen and Jorgensen (2002) extend the work of Briys and de Varenne (1997). They consider the same guarantees and terminal bonus mechanism. They also allow for the possibility of default of the insurer in the same way as Briys and de Varenne (1997). Briys and de Varenne (1997) only check the solvency of the insurer at maturity and determine the distribution of assets between policyholder and shareholder at that date. Grosen and Jorgensen (2002) improve this model by introducing a regulatory constraint such that the insurer is closed down and its assets distributed if the assets ever fall below the liabilities multiplied by a parameter λ representing the strictness of the solvency regulations *i.e.* the insurer is closed down at time t if $A_t \leq \lambda \, \rho \, A_0 \, e^{gt}$. On closure the policyholder receives the smaller of the remaining assets and the guaranteed amount $\rho \, A_0 \, e^{gt}$. The authors then show how to express the liabilities in terms of barrier options.

The authors then determine a fair bonus mechanism and level of guarantees given a level of initial shareholder investment such that the expected present value, under the equivalent martingale measure, of the policyholder benefits equals their premium. For example, consider a 20-year policy with guaranteed rate of return g of 2%. The proportion ρ of the assets invested by the policyholder is 80%. The risk-free rate of return is 5% and volatility of the assets is 20%. Then the authors find that the fair value for the participation in profits α is 0.951 if there is no check on solvency prior to maturity. The possibility of regulatory intervention before maturity reduces the value of the default option to the insurer and so the policyholder is prepared to accept a lower participation in profits. For example if λ equals 1 the fair participation in profits falls to 0.569.

Jorgensen (2001)

Jorgensen (2001) extends the work of Briys and de Varenne (1997) and Grosen and Jorgensen (2002). He considers the same guarantees and terminal bonus mechanism as in the other two papers and allows for the possibility of regulatory intervention during the contract in the same way as Grosen and Jorgensen (2002). However Jorgensen (2001) uses the Vasicek model for the risk-free rate of return whilst Grosen and Jorgensen (2002) used a constant risk-free rate.

The author begins, in a similar way to Grosen and Jorgensen (2002), by showing how to determine a fair bonus mechanism and level of guarantees. If the policyholder invests £75 into such a fair contract then it must have an initial fair value of £75.

The author then considers what would happen to the value of the policy if the regulatory regime were changed without a corresponding change to the policy conditions. The author considers the case where the regulator closes down the insurer if the assets are less than 75% of the guarantee *i.e.* $\lambda = 0.75$. As λ falls the value of the policy falls because the value of the insurer's assets is allowed to fall further before the regulator intervenes and distributes these assets to the policyholder. As λ rises then the value of the policy increases above £75 because policyholder protection is added which is not allowed for in the pricing of the contract. However, for some value of λ greater than one the value of the policy reaches a maximum. Any further increase in λ becomes counterproductive because the policyholder only receives the guaranteed amount on wind-up and loses the rights to the potential terminal bonus. The changes in policy value are most extreme for large guarantees given for funds invested in assets with high volatility because these policies are most likely to hit the regulatory boundary.

The author then considers how the value of the policy changes if we change the parameters of the investment model again without a corresponding change to the policy conditions. The author concludes that the value of the policy is less sensitive to changes in the volatility of shares than a standard option because of the ability of the insurer to default. For example, the value of a typical 20-year policy increases from £75 to only £76.9 when the volatility is increased from 10% to 15%.

2.2.4 Allowing for the Insurer's Discretion in Investing the Policyholder's Assets

Hibbert and Turnbull (2003)

Hibbert and Turnbull (2003) describe how to calculate fair values, charges and reserves, and how to hedge the risks inherent in participating policies. The authors illustrate their methods with a number of graphs based on an example U.K. conventional with-profits policy. However, their approach could be applied equally as well to other policies.

Their methodology is as follows. An asset model is chosen and calibrated to market prices. A liability model is also chosen which describes how bonuses, investment strategy, lapses *etc.* are affected by different asset model scenarios. Unlike all the other papers discussed in this chapter, the insurer has discretion over the choice of investments to back the asset share. In their examples the authors assume that this discretion is used to switch a proportion of the policyholders' funds from equities to gilts whenever guarantees are too large in relation to the asset share. The fair value is then the expected present value of the simulated policy proceeds. Discounting can be performed either via state price deflators or, in the case of a risk neutral model, at the risk-free rate. The fair value includes the value of both the potential terminal bonus and the cost of the guarantees. Hence the fair value is £648 in excess of the asset share of £13,000 in the example policy with no charges.

Deducting a charge for the guarantees from the policyholder's fund increases the fair value of the guarantee above £648 because the fund will now increase more slowly. The fair value of the guarantee and the charges are both equal to a little less than £800 when a charge of 0.6% p.a. is deducted from the asset share.

The fair value of the guarantee of £648 is based upon the restriction that no more than 10% of the assets can be switched between equities and gilts in a single year and that the bonus rate cannot be more than 1% smaller or greater than the previous year. The authors show that giving discretion to insurers over the investment of policyholders' assets can substantially reduce the fair value of the policy. For example, allowing the proportion of assets that can be switched during a year to be increased to 40% leads the fair value to drop to £500. Increasing the bonus flexibility to allow changes of greater than 1% has little affect. However, restricting the bonuses to change by no more than 0.5% increases the fair value of the guarantee to over £800.

The authors show that due to the investment discretion, the fair value of the guarantee is less sensitive to changes in the asset share than for a policy with a fixed equity backing ratio. They find that a collar option, whereby the insurer is long a put option and short a call option with a higher exercise price, forms a better match than a put option alone.

The authors suggest that reserves should be held so that the insurer is say 99% confident of having sufficient funds to meet the fair value in one year's time. They justify the one year time horizon by suggesting that at any time the insurer can move to a more matched position. Hedging the guarantees or buying options can substantially reduce the potential losses for the insurer and the size of the theoretical mismatching reserve required. For example, the total reserve required of around £700 when the collar hedge is used is only slightly greater than the fair value of £648. However, the total reserve for the guarantees rises to around £1,200 when the reserves are invested in the same assets as the asset share.

2.2.5 Summary of Participating Policies with Guarantees

In this section we compare the different approaches taken by the authors discussed in Sections 2.2.1, 2.2.2, 2.2.3, and 2.2.4.

As can be seen in Table 2.7, the authors have considered a variety of different participating policies. Some policies receive only regular bonuses, others receive only a terminal bonus, while the remainder receive both regular bonuses and a terminal bonus. Persson and Aase (1997) and Miltersen and Persson (1999) each consider two different policies, one of which has only regular bonuses, and the other has only a terminal bonus. Miltersen and Persson (2003) consider two different policies, both of which receive regular bonuses, but only one of which receives a terminal bonus. In the U.K. participating policies typically receive both regular bonuses and a terminal bonus.

In Table 2.8 we see that only three authors have allowed for mortality. Persson

Table 2.7: Participating Literature — Regular and Terminal Bonuses

Papers that Consider Regular Bonuses Only	Papers that Consider Terminal Bonuses Only	Papers that Consider Both Regular and Terminal Bonuses
Persson and Aase (1997)	Persson and Aase (1997)	Wilkie (1987)
	Briys and de Varenne (1997)	Miltersen and Persson (2000)
Miltersen and Persson (1999)	Miltersen and Persson (1999)	Hare et al. (2000)
Grosen and Jorgensen (2000)		Jorgensen (2001)
Bacinello (2001)		Miltersen and Hansen (2002)
		Grosen and Jorgensen (2002)
Miltersen and Persson (2003)		Miltersen and Persson (2003)
		Hibbert and Turnbull (2003)

and Aase (1997) consider policies where the benefit increases by the greater of the actual investment return or the guaranteed return. This calculation can either be performed separately for each year up to the time of death or maturity, or can be performed once by taking the duration of the policy as a single period (see Equations 2.2 and 2.1 respectively). Bacinello (2001) considers policies where the benefit payable on death or maturity is increased each year if a proportion of the actual return exceeds the interest rate implicit in the pricing formula (see Equation 2.3). Hibbert and Turnbull (2003) allows for both mortality and surrenders. On maturity or earlier death the policyholder receives the greater of the asset share and the sum assured plus reversionary bonuses. On surrender only the asset share is paid. Therefore the authors have allowed for a smaller number of lapses when the asset share is low because policyholders will want to retain their more valuable guarantee. The remaining authors assume that all policies reach maturity.

We can see from Table 2.9 that most authors consider only the single premium case. Wilkie (1987) and Hare et al. (2000) also consider policies with level regular premiums. Hibbert and Turnbull (2003) only consider policies with level regular premiums. Bacinello (2001) considers policies with single premiums and policies where the regular premium increases at the same rate as the benefit increases.

Wilkie (1987) calculates the ratio of the maximum guarantee that could be purchased with the current assets excluding future premiums, divided by the maximum

Table 2.8: Participating Literature — Allowance for Mortality

Papers that Ignore Mortality	Papers that Include Mortality
Wilkie (1987) Briys and de Varenne (1997) Miltersen and Persson (1999) Miltersen and Persson (2000) Hare et al. (2000) Grosen and Jorgensen (2000) Jorgensen (2001) Miltersen and Hansen (2002) Grosen and Jorgensen (2002) Miltersen and Persson (2003)	Persson and Aase (1997) Bacinello (2001) Hibbert and Turnbull (2003)

Table 2.9: Participating Literature — Premium Frequency

Papers that Consider Single Premium Only	Papers that Consider Regular Premium Only	Papers that Consider Single and Regular Premiums
Persson and Aase (1997) Briys and de Varenne (1997) Miltersen and Persson (1999) Miltersen and Persson (2000) Grosen and Jorgensen (2000) Jorgensen (2001) Miltersen and Hansen (2002) Grosen and Jorgensen (2002) Miltersen and Persson (2003)	Hibbert and Turnbull (2003)	Wilkie (1987) Hare et al. (2000) Bacinello (2001)

guarantee that could be purchased with the sum of the current assets and future premiums. He then purchases options to match only the declared guarantee multiplied by this ratio. This approach is similar to that used by Collins (1980) and Collins (1982) for unit-linked policies in that only part of the guarantee can be matched by current assets.

Hare et al. (2000) consider both single premium and regular premium policies when they use the simulation methodology. However they only investigate single premium policies using the derivative based approach.

Bacinello (2001) equates the expected present value of the benefits and premiums under the equivalent martingale measure to set a fair price for the contract. Similarly Hibbert and Turnbull (2003) calculates the fair value of the contract as the expected present value of the benefits less the future premiums and charges under the equivalent martingale measure.

Next we will look at the problems the papers are trying to solve. We saw in Table 2.4 that many authors had considered for unit-linked policies the problems of hedging the guarantee *i.e.* hedging error and transactions costs. The published papers described in Section 2.2 have not investigated transactions costs for participating policies. Hibbert and Turnbull (2003) is the only one of these papers to have looked at hedging error, although the MSc projects by Miranda (2001) and Abbey (2003) also looked at hedging error.

We also saw in Table 2.4 that many authors had calculated reserves for unit-linked policies. Hare et al. (2000) and Hibbert and Turnbull (2003) are the only papers to consider reserves for participating policies. Hare et al. (2000) perform simulations to calculate the probability of insolvency for an insurer which held the minimum reserves allowed by U.K. regulations. They also calculate the quantile reserve such that the insurer remains solvent at maturity in 99% of simulations. Hibbert and Turnbull (2003) calculate the quantile reserve such that in one year's time the insurer has assets in excess of the fair value in 99% of simulations.

We see in Table 2.10 that each author considered in Section 2.2 has calculated a charge for the guarantee in some way. We saw in Table 2.3 that only initial charges were calculated for unit-linked policies. However, for participating policies regular charges and exit charges have also been considered.

Table 2.10: Participating Literature — Calculation of Charges

Papers that Calculate an Initial Charge	Papers that Calculate a Regular Charge	Papers that Calculate an Exit Charge
	Wilkie (1987)	
Persson and Aase (1997)		
		Briys and de Varenne (1997)
Miltersen and Persson (1999)		
	Hare et al. (2000)	
	Miltersen and Persson (2000)	
Grosen and Jorgensen (2000)		Grosen and Jorgensen (2000)
	Bacinello (2001)	
		Jorgensen (2001)
	Miltersen and Hansen (2002)	
		Grosen and Jorgensen (2002)
	Miltersen and Persson (2003)	
	Hibbert and Turnbull (2003)	

All papers calculate the charge using an option pricing approach. The charge can be calculated in one of two ways. Firstly the charge can be calculated directly as the cost of purchasing matching options (Wilkie (1987), Persson and Aase (1997), Miltersen and Persson (1999) and Hare et al. (2000)). Persson and Aase (1997) and Miltersen and Persson (1999) purchase a complex option at outset and hold it to maturity, whereas Wilkie (1987) and Hare et al. (2000) use only plain vanilla put options such that whenever a bonus is declared the existing options are sold and new options bought with exercise prices to match the new guarantees. Secondly the charge can be calculated such that the expected present values of the charges and guarantees are equal under the equivalent martingale measure (Briys and de Varenne (1997), Miltersen and Persson (2000), Grosen and Jorgensen (2000), Bacinello (2001), Jorgensen (2001), Miltersen and Hansen (2002), Grosen and Jorgensen (2002), Miltersen and Persson (2003) and Hibbert and Turnbull (2003)).

Hare et al. (2000) also introduce two simpler methods for calculating the charges. They show that a charge as a percentage of asset share does not match the option pricing approach well, but a charge of 1% p.a. of the capital support required gives similar results to the option pricing approach.

Table 2.11 shows the methodology used in each paper. All the authors have used an option pricing approach to derive the cost of the guarantees, to set charges, or in the case of Briys and de Varenne (1997) to calculate the duration of the liabilities. However in the case of Miltersen and Persson (2000), Grosen and Jorgensen (2000), Jorgensen (2001), Miltersen and Hansen (2002), Miltersen and Persson (2003) and Hibbert and Turnbull (2003) the options required are of a complex nature and so simulations have been used to calculate the expected present value under the equivalent martingale measure. Wilkie (1987) uses simulations to calculate the expected guarantee at maturity and the expected payout while charging for the guarantee using option pricing. In addition to valuing the guarantee using options Hare et al. (2000) have calculated quantile reserves using simulations under the real world measure.

Table 2.11: Participating Literature — Methodology Used

Papers that Use Option Pricing Only	Papers that Use Option Pricing and Simulation
Persson and Aase (1997) Briys and de Varenne (1997) Miltersen and Persson (1999) Bacinello (2001) Grosen and Jorgensen (2002)	Wilkie (1987) Hare et al. (2000) Miltersen and Persson (2000) Grosen and Jorgensen (2000) Jorgensen (2001) Miltersen and Hansen (2002) Miltersen and Persson (2003) Hibbert and Turnbull (2003)

The investment return for participating policies has been simulated by either the lognormal model or the Wilkie model as can be seen in Table 2.12.

In the remainder of this thesis we will look at the guarantees inherent in unitised with-profits policies. In keeping with the majority of the literature on participating policies we will ignore mortality, hedging error, transactions costs, and reserves, in order to concentrate on the investment guarantees. We will consider both single premium and regular premium policies, which receive both regular bonuses and a

Table 2.12: Participating Literature — Investment Model used in Simulations

	Investment Model Used	
Paper	Lognormal	Wilkie Model
Wilkie (1987)	no	yes
Hare et al. (2000)	no	yes
Miltersen and Persson (2000)	yes	no
Grosen and Jorgensen (2000)	yes	no
Jorgensen (2001)	yes	no
Miltersen and Hansen (2002)	yes	no
Miltersen and Persson (2003)	yes	no
Hibbert and Turnbull (2003)	yes	no

terminal bonus. We will only consider regular charges for the investment guarantee. Simulations will be performed using both the lognormal model and the Wilkie model.

Chapter 3

Maturity Guarantee Charging Mechanisms for Conventional and Unitised With-Profits Policies

In this chapter we consider both conventional and unitised with-profits policies with maturity guarantees. The insurer will make a loss on these policies if the guaranteed payout is greater than the asset share. Therefore the insurer needs to charge the policyholder for providing these guarantees.

In this chapter we discuss the types of charges that could be made in Section 3.1. We then describe the option pricing approach to charges described by Wilkie (1987) for conventional with-profits policies in Section 3.2. We then extend Wilkie's approach to unitised with-profits policies in Section 3.3. Finally in Section 3.4 we compare the option pricing approach for conventional and unitised with-profits policies, and discuss how the option pricing approach could be applied in practice.

3.1 Methods of Charging for Guarantees

Charges for guarantees can be broken down into three main types based on the timing of the charges:

- Charges at outset of the policy.
- Charges during the policy term.

- Charges on termination of the policy.

The insurer can use more than one type of charge. There are many different ways these charges can be calculated. We discuss below the methods used for the participating policies in the papers described in Section 2.2 and shown in Table 2.10.

The simplest method is to charge the policyholder at the outset of the policy. The premium the policyholder pays is higher than the amount invested on their behalf. The excess is retained by the insurer as a charge for the potential cost of the guarantee. This method is used by Persson and Aase (1997), Miltersen and Persson (1999), and Grosen and Jorgensen (2000).

There are numerous ways that regular charges can be deducted during the term of the policy. Wilkie (1987) charges the policyholder for the cost of buying put options that match the guarantee. Hare et al. (2000) consider three different methods of regular charges: charges as a proportion of asset share, charges related to the excess of the required reserves over the asset share, and charges equal to the cost of purchasing options that match the risk with 99% security. Hibbert and Turnbull (2003) also consider charges as a proportion of asset share. Miltersen and Persson (2003), Bacinello (2001), and Miltersen and Persson (2000) credit the insurer with a proportion of any excess investment return over the guaranteed rate of return. Miltersen and Hansen (2002) use two regular charges: direct charges are deducted as a proportion of the guarantee, and indirect charges credit the insurer with a proportion of any distributed surplus.

The calculation of the charge can be deferred until the end of the contract. In addition to taking an initial charge, Grosen and Jorgensen (2000) pay no terminal bonus so that the insurer retains any excess funds on exit. Briys and de Varenne (1997), Grosen and Jorgensen (2002), and Jorgensen (2001) retain a predetermined proportion of any excess of assets over the guaranteed payout, with the remainder paid out as terminal bonus.

We have seen that a number of different mechanisms have been suggested to price guarantees. However in this thesis we will consider only one mechanism — the option pricing technique proposed by Wilkie (1987) that he applied to conventional with-profits endowments. We consider this method applied to unitised with-profits policies as first considered by Yap (1999). We will see in Section 3.4.5 why we consider this method to be the most appropriate for participating policies issued in the U.K..

The basic idea with this mechanism is that, whenever a bonus is declared or a premium is paid, put options are bought with exercise date equal to the maturity date. At all times the policyholder will have (or be nominally allocated) an equal number of options and units of an equity index. The combination of options and equities ensures that the guarantee can be met by, if necessary, exercising the option and receiving the exercise price in exchange for the equities. At the end of each year the options are sold, and new options are bought at an exercise price determined by the new guarantee.

There are a number of combinations of equities and options that would meet the guarantee. However, holding an equal number of units in the equity index and options (all with the same exercise price) is the cheapest method. Hence, the maximum possible investment in equities is retained.

3.2 A Maturity Guarantee Charging Mechanism for Conventional With-Profits Policies

In this section we will consider conventional with-profits policies. These policies have a guaranteed payment on death and maturity. The guarantee is initially set equal to the sum assured, but this is increased from year to year by adding reversionary bonuses. Finally, if the assets perform well, a terminal bonus may be added.

We will ignore the possibility of death and assume that all policies reach maturity.

Wilkie (1987) notices that the payoff from a conventional with-profits policy is the same as a portfolio of shares and put options. Hence we can use shares and options to replicate the conventional with-profits policy. The charge for the guarantees is given by the cost of the options. As premiums are paid and bonuses are declared it becomes necessary to rebalance the portfolio of shares and put options.

We begin in section 3.2.1 by describing how Wilkie's option charging mechanism works. Then in Sections 3.2.2, 3.2.3, and 3.2.4 we consider some numerical examples.

3.2.1 Description of the Conventional With-Profits Charging Mechanism

It will be helpful to make the following definitions before showing the option charging mechanism in detail:

- T is the term of the policy
- P_t is the premium payable at time t
- G_t is the guarantee attaching at time t just after the bonus declaration at that date *i.e.* the sum assured plus reversionary bonuses declared to date
- A_t is the value of the portfolio of equities and put options at time t
- S_t is the value of a single unit of an equity index at time t where all dividends are immediately reinvested into the index
- E_t is the exercise price of the put option bought at time t
- O_t^- and O_t^+ are the prices of the put options at time t with exercise prices E_{t-1} and E_t respectively
- N_t is the number of units of the equity index held after rebalancing at time t, hence it is also the number of options held after rebalancing at time t
- r_f is the risk-free force of interest
- σ is the volatility of the equity index.

We will demonstrate the charging mechanism that Wilkie (1987) used for an annual premium policy. This method can also be used for single premium policies by setting all premiums after the first to zero. We will ignore expenses, mortality and lapses in order to concentrate on the maturity guarantee.

The insurer only invests in equities and options. Whenever a dividend is paid it is immediately reinvested in equities. S_t represents the value at time t of a single unit of an equity index, where dividends are reinvested, with an initial value of £100.

At the outset of the policy the policyholder pays a premium P_0. In return the insurer guarantees that the maturity value will be at least as large as the sum assured. Hence the initial guarantee is simply the sum assured *i.e.*

$$G_0 = \text{Sum Assured.}$$

The initial assets 'belonging' to the policyholder A_0, are equal in value to the premium if we ignore expenses. The insurer now needs to decide how to invest these assets.

The maximum present guarantee MPG_t is the maximum guarantee that can be made given the asset value at time t and ignoring future premiums. The maximum present guarantee is therefore the accumulated value of the current assets if they were invested in a risk-free asset until maturity. Assuming that such a risk-free asset exists and guarantees a force of interest r_f, then the maximum present guarantee at time zero is given by:

$$MPG_0 = A_0\, e^{r_f T} = P_0\, e^{r_f T}.$$

The sum assured on a regular premium policy in the U.K. is usually at least as large as 75% of the sum of all the premiums payable throughout the policy term in order to qualify under Inland Revenue rules. Hence the premium at time zero will be insufficient to purchase the full guarantee. Therefore we must consider that part of the guarantee is purchased by future premiums. Wilkie (1987) assumes that the risk-free rate of return is constant throughout the term of the policy. Hence the maximum future guarantee MFG_0 obtained by investing all future premiums in the risk-free asset is given by:

$$MFG_0 = \sum_{\tau=1}^{T-1} P_\tau\, e^{r_f(T-\tau)}.$$

There are many different speeds at which the guarantee can be purchased. One possibility suggested by Wilkie (1987) is that the maximum proportion of the guarantee affordable is purchased each year until the full guarantee is matched by appropriate assets. However, we will use the mechanism that Wilkie uses in all his calculations as follows. First we calculate the maximum present guarantee as a proportion of the sum of the maximum present and future guarantees. We then arrange the assets to purchase this proportion of the total guarantee. Hence the guarantee actually purchased is:

$$AG_0 = \left(\frac{MPG_0}{MPG_0 + MFG_0} \right) G_0.$$

We must now decide how to invest the assets. There are many combinations of units of the equity index and put options written upon them which would ensure that the actual guarantee AG_0 was met. The approach used in Wilkie (1987) requires that the options all have the same exercise price and that the same number of units are held as options. If we had held options with a lower exercise price, then we would not be able to afford sufficient options and equities to be certain of meeting the guarantee. If we had held options with a higher exercise price, then we would be certain of meeting the guarantee, but at the cost of selling more of the equity upside. Options with a mixture of exercise prices could be held which would ensure that the guarantee was met, but again at the cost of losing more of the equity upside.

Hence, the insurer now invests in an equal number of units of the equity index and put options written on the index. Therefore, if necessary it can sell the units at maturity for a known price which matches the guarantee purchased so far AG_0. We have two equations to enable us to find a solution. Firstly the value of the assets must equal the value of the units and options purchased as follows:

$$A_0 = P_0 = N_0 \left(S_0 + O_0^+ \right) \tag{3.4}$$

where the value of the option is a function of the exercise price E_0 and is given by the Black-Scholes formula as follows:

$$O_0^+ = E_0 \, e^{-r_f T} \, \Phi(-d_2) - S_0 \, \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0 / (E_0 \, e^{-r_f T}))}{\sigma \sqrt{T}} + \frac{\sigma \sqrt{T}}{2}$$

$$d_2 = \frac{\ln(S_0/(E_0\, e^{-r_f T}))}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}$$

where $\Phi(x) = P(X < x)$ where X has unit normal distribution. Notice that we have used the Black-Scholes formula for a non-dividend paying share index S_t, because all dividends are immediately reinvested back into the share index. We do not buy options written directly on the shares themselves, and so do not use the Black-Scholes formula for dividend paying stocks.

Secondly the exercise price must be such that we are guaranteed to have assets at least equal to the guarantee at maturity *i.e.*

$$AG_0 = N_0\, E_0. \tag{3.5}$$

The equity index value S_0, guarantee to be purchased AG_0, and asset value A_0 are all known. O_0^+ is the cost of the option at time zero with exercise price E_0, and so is a function of the unknown exercise price and the known equity index value, risk-free rate r_f and volatility σ. Hence we have two equations, 3.4 and 3.5, with the two unknowns E_0 and N_0. Hence we need to find the unique solution E_0, using numerical methods, to the following equation

$$f(E_0) = \frac{A_0\, E_0}{S_0 + O_0^+} - AG_0 = 0$$

and hence find N_0.

In each future year before maturity we work through the same steps. We consider below the calculations required at time t.

The first step is to work out the value of the assets at time t. No buying or selling of assets takes place during the year. Hence the assets held are N_{t-1} units of the equity index, plus an equal number of put options with exercise price E_{t-1}, plus the new premium P_t. Their value is as follows:

$$A_t = N_{t-1}\,(S_t + O_t^-) + P_t.$$

The put option is valued using Black-Scholes as follows:

$$O_t^- = E_{t-1}\, e^{-r_f(T-t)}\, \Phi(-d_2) - S_t\, \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_t/(E_{t-1}\, e^{-r_f(T-t)}))}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$$

$$d_2 = \frac{\ln(S_t/(E_{t-1}\, e^{-r_f(T-t)}))}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}.$$

The next step is to work out whether the bonus we wish to declare is affordable *i.e.* are the current assets and future premiums sufficient to purchase options to back the guarantee? The maximum present guarantee can be purchased by investing all the current assets at the risk-free rate of return. Hence

$$MPG_t = A_t\, e^{r_f(T-t)}.$$

Similarly the maximum future guarantee obtained by investing all future premiums in the risk-free asset is given by:

$$MFG_t = \sum_{\tau=t+1}^{T-1} P_\tau\, e^{r_f(T-\tau)}.$$

Hence the bonus we declare will be affordable if the sum assured plus reversionary bonuses declared to date are less than the sum of the maximum present and future guarantees.

The third step is to calculate the bonus we would like to declare and hence the size of the guarantee we have declared to date. There are many different ways in which we could determine the reversionary bonus rate. However we will consider the simple strategy outlined in Wilkie (1987). A desired bonus rate b is determined at the outset of the policy. Hence the new desired guarantee is given as follows:

$$DG_t = G_{t-1}\,(1+b).$$

We must then check that the desired guarantee is less than the maximum affordable guarantee. If the desired guarantee is unaffordable then we declare no new bonus. Therefore the maturity guarantee becomes:

$$
\begin{aligned}
G_t &= G_{t-1}\,(1+b) = DG_t \quad &\text{if} \quad DG_t \leq MPG_t + MFG_t \\
G_t &= G_{t-1} \quad &\text{if} \quad DG_t > MPG_t + MFG_t.
\end{aligned}
$$

The maturity guarantee G_t is the amount that the insurer has promised to pay the policyholder. However, the insurer only matches a proportion of this guarantee given by the ratio of the maximum guarantee affordable given the value of the current assets to the maximum guarantee affordable given the value of the total assets including the future premiums. Hence the actual guarantee at time t is given by:

$$
AG_t = \left(\frac{MPG_t}{MPG_t + MFG_t} \right) G_t. \tag{3.6}
$$

If the value of the assets has fallen particularly sharply it is possible that the actual guarantee AG_t is unaffordable even though no bonus has been declared this year. In this case the insurer maintains their current investment portfolio of N_{t-1} units and options with exercise price of E_{t-1}, which will still match the guarantee from last year AG_{t-1}. The new premium is invested in an equal number of units and put options with exercise E_{t-1}. Hence the guarantee actually backed by options is a little larger than AG_{t-1}, but falls short of the required AG_t. Hence the number of units held and the revised actual guarantee AG_t are given by:

$$
A_t = N_t\,(S_t + O_t^-)
$$

and

$$
AG_t = N_t\,E_{t-1} < \left(\frac{MPG_t}{MPG_t + MFG_t} \right) G_t.
$$

Therefore it is possible using this methodology that the assets at maturity, even if the options are exercised, will fall short of the declared guarantee. We will discuss this problem further in Section 3.4.

The final step, if the actual guarantee AG_t is affordable, is to rebalance the investment portfolio. We know that the the cost of the options to meet the guarantee will be lowest if we hold an equal number of units and options each with the same exercise price. Therefore we sell the current holdings of units of the equity index and put options and buy the appropriate new amounts of units and put options with a new exercise price such that they match the above guarantee AG_t.

We again have two equations. Firstly the value of the assets we sell must equal the value of the equities and options that we buy as follows:

$$A_t = N_{t-1}\,(S_t + O_t^-) + P_t = N_t\,(S_t + O_t^+)$$

where O_t^+ is the value of the new put option with exercise price E_t and is given by the Black-Scholes formula as follows:

$$O_t^+ = E_t\, e^{-r_f(T-t)}\,\Phi(-d_2) - S_t\,\Phi(-d_1)$$

$$d_1 = \frac{\ln(S_t/(E_t\, e^{-r_f(T-t)}))}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$$

$$d_2 = \frac{\ln(S_t/(E_t\, e^{-r_f(T-t)}))}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}.$$

Secondly the exercise price must be such that we are guaranteed to have assets at least equal to the guarantee at maturity *i.e.*

$$AG_t = N_t\,E_t.$$

Again we have two equations with two unknowns E_t and N_t. Hence we need to find the unique solution E_t to the following equation

$$f(E_t) = \frac{A_t\,E_t}{S_t + O_t^+} - AG_t = 0$$

and hence find N_t.

The above steps are repeated each year prior to maturity. Finally at maturity the policyholder receives the value of their assets. Either the value of the equity index is at least as great as the exercise price, so that the options expire worthless, or the exercise price exceeds the value of the equity index, so that the options are exercised and the policyholder receives only the guaranteed amount *i.e.*

$$\text{CWP Maturity Payout} = B_T = N_{T-1} \max(S_T, E_{T-1}).$$

In the numerical results which follow, it is interesting to compare the payout on a conventional with-profits policy with the payouts from a unit-linked and risk-free policy with the same premium and policy term.

The unit-linked (UL) policy represents one extreme of the spectrum of possible policies where the policyholder takes all the investment risk and no guarantees are given. The premiums are entirely invested in the equity index so that the maturity payout is:

$$\text{UL Maturity Payout} = B_T^{UL} = \sum_{\tau=0}^{T-1} P_\tau \frac{S_T}{S_\tau}. \tag{3.7}$$

The risk-free (RF) policy represents the opposite extreme where the policyholder takes no risks. Each premium is invested at the risk-free rate of return prevailing at the time. We assume here that the risk-free rate is constant so that the maturity payout is known at outset to be as follows:

$$\text{Risk-Free Maturity Payout} = B_T^{RF} = \sum_{\tau=0}^{T-1} P_\tau \, e^{r_f (T-\tau)}. \tag{3.8}$$

3.2.2 Results for the Conventional With-Profits Charging Mechanism

We will now illustrate the conventional with-profits charging mechanism described in Section 3.2.1 by re-constructing some of the numerical results shown in Wilkie (1987). In Sections 3.3.2, 3.3.3, and 3.3.4 we calculate comparable figures for the unitised with-profits charging mechanism.

The policies considered are 20-year conventional with-profits endowment policies with an annual premium of £50. The insurer's bonus policy is to declare a constant rate of reversionary bonus b, if affordable, at the end of each year except the last. A terminal bonus is paid at maturity so that the maturity payout equals the value of the assets. We consider two examples in Sections 3.2.3 and 3.2.4, each with a sum assured of £1,200, together with desired reversionary bonuses b of 2% and 5% respectively.

Wilkie (1987) considers the results of following the conventional with-profits charging mechanism using a number of different experience bases. In this section we will illustrate the method with just one of these experience bases as follows.

The value of the equity index S_t is based on values taken from the Financial Times — Actuaries All Share Index with dividends reinvested, subject to tax at 35%. The values of the index are taken on the last day in June from 1965 to 1985.

The price of options is assumed to be given by the Black-Scholes equation. The volatility of the equity index S_t is 20% p.a.. The risk-free rate of interest r_f is a constant 7% p.a., corresponding to a force of interest of 6.76586% p.a..

3.2.3 Example 1 — Conventional With-Profits with Low Bonuses

Figure 3.4 shows how the assets and guarantees evolve through time for the policy described above with sum assured of £1,200 and desired reversionary bonus rate of 2%. These results are in agreement with Table 5.3 in Wilkie (1987).

The first thing to notice is that the 20 year period starting in June 1965 was a period of strong equity growth. The return on shares over the period was 13% p.a.. However, there was a substantial drop of 43% between June 1973 and June 1974.

The assets grow each year with investment return and the addition of new premiums. The only time the assets decrease in value is in June 1974, due to the stock market crash over the previous 12 months. Other smaller falls in the value of shares, for example in 1967, are more than compensated for by the additional premiums. The final value of the assets is £5,024.

The sum assured of £1,200 and the desired bonus rate of 2% are modest in comparison to the performance of the assets. Therefore the bonus of 2% is comfortably affordable at the end of each year. Recall that the insurer does not declare reversionary bonuses at the end of the final year. The maturity guarantee after the final reversionary bonus declaration is £1,748. This is substantially less than the value of the assets, so that the options expire worthless, and the policyholder receives the

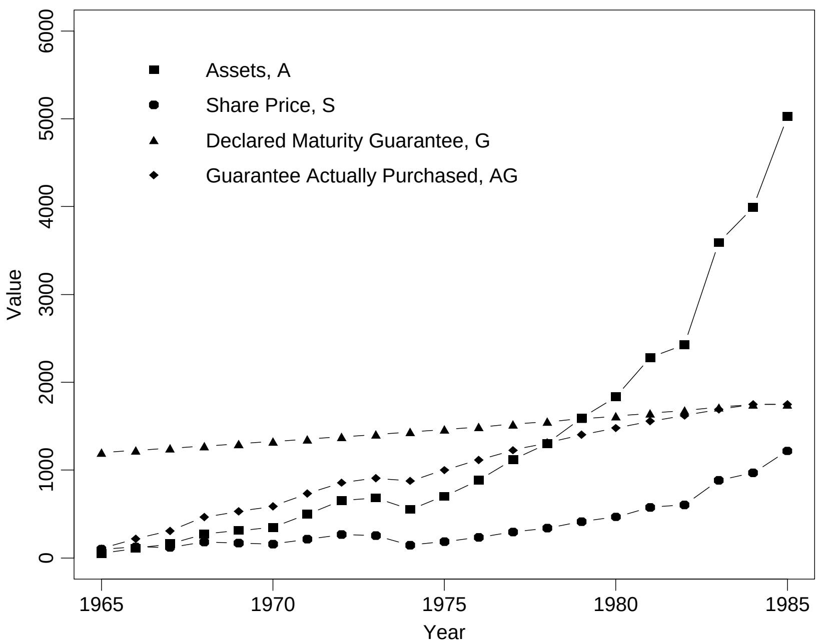

Figure 3.4: The Value of the Assets and Guarantees under a Conventional With-Profits Policy (Sum Assured of £1200, Reversionary Bonus of 2% each year)

value of the assets of £5,024, possibly subject to smoothing and charges for capital.

Finally, we look at the actual guarantee purchased each year. Recall from Equation 3.6 that only a fraction of the declared maturity guarantee is actually purchased each year. We can see that at outset only a guarantee of £106 is purchased. In most years we increase the guarantee we buy for two reasons. Firstly the guarantee declared is increasing. Secondly the ratio of the maximum present guarantee to the maximum present and future guarantees usually increases because the assets are growing. However in June 1974 shares fall so sharply that even when the premium is added the value of the assets has still fallen. Hence the maximum guarantee affordable in June 1974 is actually less than in June 1973. The maximum present guarantee falls sharply enough so that the actual guarantee purchased also falls, despite the increase in the guarantee declared. The possibility that the actual guarantee purchased could fall is a feature that will be discussed further in Section 3.4.

The final payout for this policy was £5,024. If the policyholder had instead invested in a unit-linked policy with no guarantees the payout would have been £5,464 by Equation 3.7. Hence the accumulated cost of the guarantees was £440.

The payout in this example is considerably higher than the risk-free payout of

£2,193 given by Equation 3.8. However the conventional with-profits policy is more risky. If assets had fallen very sharply in the final year then the with-profits policyholder would only have received the guaranteed value of £1,748, whereas the full value of the risk-free contract is guaranteed. Table 3.13 compares the final guarantee and payout under the unit-linked and risk-free contracts with this low bonus conventional with-profits policy.

3.2.4 Example 2 — Conventional With-Profits with High Bonuses

We now turn to our second example in Figure 3.5 with sum assured of £1,200 and desired reversionary bonus rate of 5%. Again these results are in agreement with Table 5.3 in Wilkie (1987).

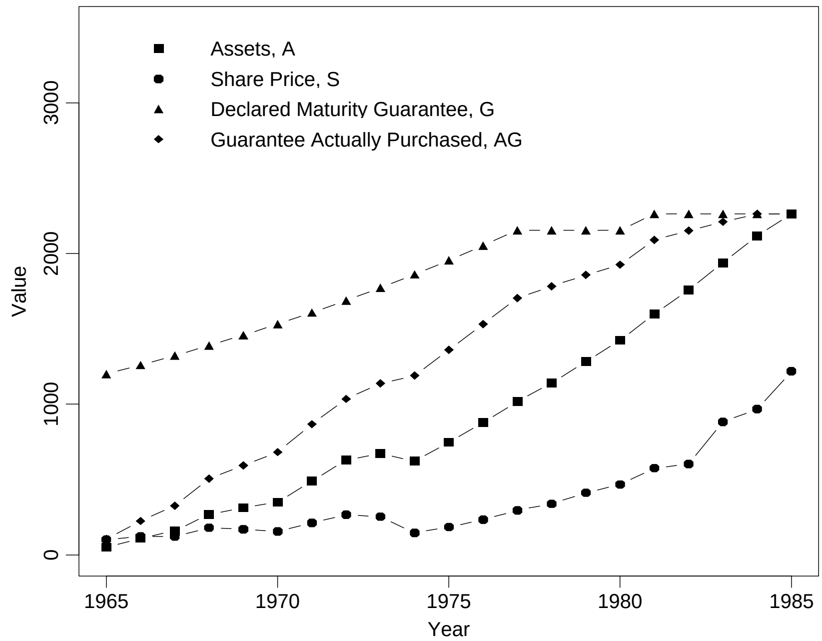

Figure 3.5: The Value of the Assets and Guarantees under a Conventional With-Profits Policy (Sum Assured of £1200, Reversionary Bonus of 5% each year)

In this second example we consider the same period as before. Hence the value of the equity index is identical in Figures 3.4 and 3.5. However, the desired bonus rate is higher at 5% as opposed to 2%. The main consequence of the higher bonus rate is that a larger proportion of assets is invested in put options rather than the equity index. Hence the value of the assets grows more slowly in the good investment years. On the other hand, the fall in assets in June 1974 is now less severe as the rise in the value of the options cancels out some of the fall in the value of the equity index.

We can see that the maturity guarantee rises more rapidly than in the first example. However, in June 1978 the desired guarantee is unaffordable so the insurer decides to declare no bonus at all that year. In fact we find that the desired guarantee is only affordable on one further occasion, which is June 1981. Despite the desired guarantee being unaffordable on a number of occasions, the final maturity guarantee of £2,263 is still higher than in the first example where the final maturity guarantee was £1,748. However, Wilkie (1987) does record circumstances where a higher sum assured or desired bonus rate results in a lower maturity guarantee because too many of the reversionary bonuses are unaffordable.

In Figure 3.4 we see that the actual guarantee purchased is never much more than the value of the assets. In fact from June 1979 onwards the value of the assets is greater than the actual guarantee purchased. In contrast in Figure 3.5 we see that the guarantee purchased is always bigger than the assets. In fact the guarantee purchased becomes so much larger than the value of the assets that almost all the shares must be sold to buy options. Most of the future premiums need to be invested in options too. Hence from June 1978 the insurer has insufficient investment in shares to generate the high returns required to fund future reversionary bonuses.

At maturity the exercise price of the options is £1,597 which is greater than the share price of £1,218. Hence the option is exercised and the policyholder receives the guaranteed sum of £2,263 with no terminal bonus.

Notice that in this second example the final guaranteed sum of £2,263 is greater than the risk-free payout of £2,193. The strong performance of the shares has allowed the guarantee to be increased significantly throughout the term. However the with-profits payout of £2,263 is substantially less than the unit-linked payout of £5,464, showing that the accumulated cost of the guarantees is £3,201. The final guarantee and payout under the unit-linked, risk-free, low bonus and high bonus conventional with-profits policies are shown in Table 3.13.

Table 3.13: Maturity Payout and Final Guarantee for 20 year Policies Taken Out in 1965

Type of Policy	Maturity Payout	Final Guarantee
Unit-Linked	£5,464	£0
CWP — Low Bonus	£5,024	£1,748
CWP — High Bonus	£2,263	£2,263
Risk-Free	£2,193	£2,193

3.3 A Maturity Guarantee Charging Mechanism for Unitised With-Profits Policies

In this section we will consider unitised with-profits policies. Premiums buy units in the with-profits fund. (The policyholder's units in the with-profits fund represent a liability to the insurer and should not be confused with the units in the equity index which are the insurer's assets.) The units have a guaranteed minimum growth rate which may also be augmented by bonuses. On maturity or earlier death the policyholder is guaranteed to receive the value of the units. In addition, if the assets perform well, a terminal bonus may be added.

Again we will ignore the possibility of death and assume that all policies reach maturity.

Wilkie (1987) does not consider unitised with-profits policies. However in an MSc project under my supervision Yap (1999) shows how to extend Wilkie's conventional with-profits approach to unitised with-profits. In a similar way to conventional with-profits, the payoff from a unitised with-profits policy is the same as a portfolio of shares and put options. The only difference between the two policy types is the rate at which the maturity guarantees build up.

We begin in Section 3.3.1 by describing how the option charging mechanism works for unitised with-profits policies. Then in sections 3.3.2, 3.3.3, and 3.3.4 we consider some numerical examples.

3.3.1 Description of the Unitised With-Profits Charging Mechanism

We will demonstrate the charging mechanism that Yap (1999) used for an annual premium policy. Again we will ignore expenses, mortality and lapses in order to concentrate on the maturity guarantee. We will use the same notation as in Section 3.2.1 with a slight alteration to the definition of the guarantee G_t and some additions as follows:

- G_t is the guarantee attaching at time t just after the bonus declaration at that date *i.e.* the premiums accumulated at the guaranteed growth rate to *maturity* plus reversionary bonuses declared to date
- U_t is the face value of the units at time t *i.e.* the premiums accumulated at the guaranteed growth rate to *time* t plus reversionary bonuses declared to date
- y_t is the minimum guaranteed growth rate of units purchased at time t
- z_t is the bonus rate the insurer wants to declare at time t.

In practice bonuses accrue to the policy on a daily basis. The bonus rate is typically reviewed on an annual basis. However the insurer retains the right to alter bonuses more frequently if the bonus rate is seriously out of line with emerging experience. In this model we will set bonuses retrospectively at the end of the year as it simplifies the model and represents the insurer's discretion in changing the bonus rates. To be consistent with the conventional with-profits case we will attempt to declare the same rate of bonus z each year.

At the outset of the policy the policyholder pays a premium P_0. We ignore expenses so that the entire premium is invested in units. Hence

$$U_0 = P_0 = A_0.$$

Therefore the guaranteed death benefit at this stage would be U_0. However we are interested in maturity guarantees. We must declare the compulsory guarantee y_t each year. However, although y_t is declared annually to the policyholder and credited to their unit value, it is promised in the contract to be paid in all future years too. Hence we must in fact purchase options to back all future declarations as soon as the premium is paid. (If the compulsory guarantee is fixed for all future premiums, then we would need a substantial additional reserve, much in excess of the early premiums, to cover the risk of the risk-free interest rate falling below the compulsory guarantee. However, many contracts allow the compulsory guarantee to be changed for future premiums, so that the insurer only needs to allow for the guaranteed growth rate on premiums already paid. In our examples we will assume that the insurer retains the right to change the guaranteed growth rate on future premiums.) Therefore the maturity guarantee after the first premium is paid is as follows:

$$G_0 = P_0 \, (1 + y_0)^T . \tag{3.9}$$

The insurer must check that this guarantee is affordable. We proceed in a similar way to the conventional with-profits case. As in Section 3.2.1, the maximum present guarantee MPG_t is the maximum guarantee that can be made given the asset value at time t and ignoring future premiums. Hence the maximum present guarantee at time zero is given by:

$$MPG_0 = A_0 \, e^{r_f T} = P_0 \, e^{r_f T} . \tag{3.10}$$

Comparing Equations 3.9 and 3.10 we see that the guarantee is affordable as long as the minimum guaranteed rate of return y_t is less than the risk-free rate. There is no need to consider the guarantee that can be bought with future premiums. Hence the actual guarantee purchased AG_0 is always the same as the declared maturity guarantee G_0.

We must now decide how to invest the assets. There are many combinations of units of the equity index and put options written upon them that would match the guarantee G_0. We will follow the same approach as in Section 3.2.1 where the options all have the same exercise price and we hold the same number of units as options. This maintains the maximum investment in equities which meets the guarantee.

We need to calculate the exercise price of the put option which will match the guarantee G_0, and hence the number of options and units of the equity index that we buy. We require two equations to hold. Firstly we need to meet the guarantee:

$$G_0 = N_0 \, E_0.$$

We must have one unit for every option so that we can exercise the option if necessary at maturity. We also need to ensure that we can buy the appropriate number of options and units. Hence

$$A_0 = N_0 \, (S_0 + O_0^+)$$

where, in exactly the same way as in the conventional with-profits case, the value of the option is a function of the exercise price E_0 and is given by the Black-Scholes formula for a non-dividend paying stock as follows:

$$O_0^+ = E_0 \, e^{-r_f T} \, \Phi(-d_2) - S_0 \, \Phi(-d_1)$$

$$d_1 = \frac{\ln(S_0/(E_0 \, e^{-r_f T}))}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2}$$

$$d_2 = \frac{\ln(S_0/(E_0 \, e^{-r_f T}))}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2}.$$

The equity index value S_0, total guarantee G_0 and asset value A_0 are all known. O_0^+ is the cost of the option at time zero with exercise price E_0, and so is a function of the unknown exercise price and the known equity index value, risk-free force of interest r_f and volatility σ. Hence we have two equations with the two unknowns E_0 and N_0. Hence we need to find the unique solution E_0 to the following equation

$$f(E_0) = \frac{A_0 \, E_0}{S_0 + O_0^+} - G_0 = 0$$

and hence find N_0.

In each future year before maturity we work through the same steps. We consider below the calculations required at time t.

The first step is to work out the value of the assets at time t. No buying or selling of assets takes place during the year. Hence the assets held are N_{t-1} units of the equity index, plus an equal number of put options with exercise price E_{t-1}, plus the new premium P_t. Their value is as follows:

$$A_t = N_{t-1}\,(S_t + O_t^-) + P_t.$$

The put option is valued using Black-Scholes as follows:

$$O_t^- = E_{t-1}\,e^{-r_f(T-t)}\,\Phi(-d_2) - S_t\,\Phi(-d_1)$$

$$d_1 = \frac{\ln(S_t/(E_{t-1}\,e^{-r_f(T-t)}))}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$$

$$d_2 = \frac{\ln(S_t/(E_{t-1}\,e^{-r_f(T-t)}))}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}.$$

The second step is to calculate the bonus we would like to declare and hence the size of the guarantee we have declared to date.

The insurer promises a minimum rate of return y_t on the premium paid at time t. We let G_t^y represent the increase in the maturity guarantee due to the guaranteed return on the premium paid at time t. We call G_t^y the compulsory new guarantee because we must increase the guarantee by this amount. Therefore when the new premium is paid we must purchase the additional guarantee as follows:

$$G_t^y = P_t\,(1+y_t)^{T-t}.$$

We also desire to declare an additional bonus z_t if it is affordable. In this chapter we declare the same bonus z each year when affordable and zero if not. However the mechanism described below can be applied to a variety of different bonus mechanisms. This bonus is declared as a proportion of the total guarantee from the previous year G_{t-1}. We let G_t^z represent the increase in the maturity guarantee due to the additional bonus declared at time t. We call G_t^z the desired new guarantee because the insurer is not obliged to declare a bonus if it is unaffordable. The desired new guarantee can be calculated as follows:

$$G_t^z = G_{t-1}\, z_t.$$

Hence we obtain the total desired guarantee as the sum of the guarantee from the previous year, plus the compulsory guarantee on any new premium, plus the desired new guarantee:

$$DG_t = G_{t-1} + G_t^y + G_t^z.$$

The next step is to work out whether the guarantee described above is affordable. The maximum present guarantee is given by

$$MPG_t = A_t\, e^{r_f(T-t)}.$$

We must then check that the desired guarantee is less than the maximum affordable guarantee. If the desired guarantee is unaffordable then we declare no desired new guarantee. The previous year's guarantee has already been backed by options and so is always affordable — we could hold the options to maturity. The new compulsory guarantee will be affordable if y_t is less than the risk-free rate (or an adequate terminal bonus cushion has been built up) — this will always be the case when the risk-free rate is constant as in this chapter. Therefore the actual total guarantee declared is

$$\begin{array}{lcll} G_t & = & G_{t-1} + G_t^y + G_t^z = DG_t & \text{if} \quad DG_t \leq MPG_t \\ G_t & = & G_{t-1} + G_t^y & \text{if} \quad DG_t > MPG_t. \end{array}$$

Note however that G_t represents the maturity guarantee if no more premiums are paid and no more bonuses are declared, so that the policyholder's units increase at the guaranteed minimum rate of return. Assuming that all the policyholder's units have the same minimum guaranteed rate of return y, the actual unit value that the policyholders will see is:

$$\begin{array}{lcll} U_t & = & U_{t-1}\,(1+y)\,(1+z_t) + P_t & \text{if} \quad DG_t \leq MPG_t \\ U_t & = & U_{t-1}\,(1+y) + P_t & \text{if} \quad DG_t > MPG_t. \end{array}$$

Again assuming that all units have the same minimum guaranteed rate of return y, we see that the maturity guarantee and unit value are linked as follows:

$$G_t = U_t\,(1+y)^{T-t}. \tag{3.11}$$

The final step is to rebalance the investment portolio. We know that the the cost of the options to meet the guarantee will be lowest if we hold an equal number of units of the equity index and options each with the same exercise price. Therefore we must sell the current holdings of units and put options, and buy the appropriate new number of units and put options with a new exercise price such that they match the above guarantee.

We again have two equations. Firstly the value of the assets we sell must equal the value of the units and options that we buy as follows:

$$A_t = N_{t-1}\,(S_t + O_t^-) + P_t = N_t\,(S_t + O_t^+)$$

where O_t^+ is the value of the new put option with exercise price E_t and is given by the Black-Scholes formula as follows:

$$O_t^+ = E_t\,e^{-r_f(T-t)}\,\Phi(-d_2) - S_t\,\Phi(-d_1)$$

$$d_1 = \frac{\ln(S_t/(E_t\,e^{-r_f(T-t)}))}{\sigma\sqrt{T-t}} + \frac{\sigma\sqrt{T-t}}{2}$$

$$d_2 = \frac{\ln(S_t/(E_t\,e^{-r_f(T-t)}))}{\sigma\sqrt{T-t}} - \frac{\sigma\sqrt{T-t}}{2}.$$

Secondly the exercise price must be such that we are guaranteed to have assets at least equal to the guarantee at maturity *i.e.*

$$G_t = N_t\,E_t.$$

Again we have two equations with two unknowns E_t and N_t. Hence we need to find the value of E_t which solves the following equation:

$$f(E_t) = \frac{A_t\, E_t}{S_t + O_t^+} - G_t$$

and hence find N_t.

The above steps are performed each year prior to maturity. Finally at maturity the policyholder receives the value of their assets. Either the value of the equity index is at least as great as the exercise price, so that the options expire worthless, or the exercise price exceeds the value of the equity index, so that the options are exercised and the policyholder receives only the guaranteed amount *i.e.*

$$\text{Maturity Payout} = N_{T-1} \max(S_T, E_{T-1}).$$

We can compare the payout on a unitised with-profits policy with the payouts from a unit-linked and risk-free policy given by Equations 3.7 and 3.8 respectively.

3.3.2 Results for the Unitised With-Profits Charging Mechanism

In this section we provide new numerical results to illustrate the unitised with-profits charging mechanism described in Section 3.3.1. We will use the same experience basis as in Section 3.2.2 to enable comparisons with conventional with-profits policies.

The policies considered are 20-year unitised with-profits endowment policies with an annual premium of £50. The insurer guarantees that the units will grow each year at the guaranteed growth rate y. In addition bonuses are declared at rate z, if affordable, at the end of each year except the last. A terminal bonus is paid at maturity so that the maturity payout equals the value of the assets. We consider two examples in Sections 3.3.3 and 3.3.4, each with guaranteed growth rate y of 1.7055%, together with desired bonuses z of 3.5998% and 8.5035% respectively.

We use the same experience basis as in Section 3.2.2 which we repeat below for completeness.

The value of the equity index S_t is based on values taken from the Financial Times — Actuaries All Share Index with dividends reinvested, subject to tax at 35%. The values of the index are taken on the last day in June from 1965 to 1985.

The price of the options is assumed to be given by the Black-Scholes equation. The volatility of the equity index S_t is 20% p.a.. The risk-free rate of interest r_f is a constant 7% p.a., corresponding to a force of interest of 6.76586% p.a..

Yap (1999) also provides numerical results for the conventional with-profits policy described above, although his calculations use a risk-free rate of interest r_f of 5% p.a..

3.3.3 Example 1 — Unitised With-Profits with Low Bonuses

Figure 3.6 shows how the assets and guarantees evolve through time for the policy described above with guaranteed growth rate y of 1.7055% and desired reversionary bonus rate z of 3.5998%.

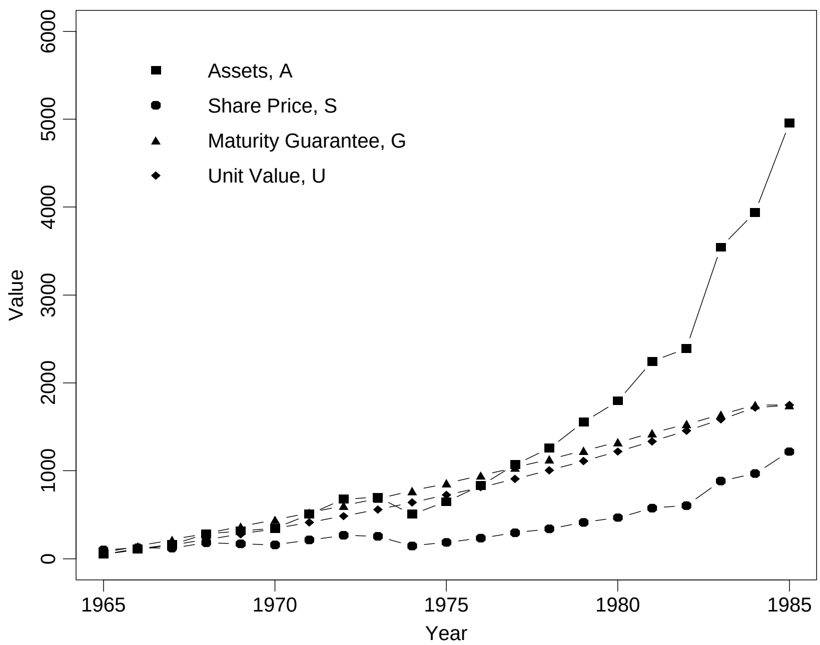

Figure 3.6: The Value of the Assets and Guarantees under a Unitised With-Profits Policy (Guaranteed Growth Rate 1.7055%, Desired Bonus Rate 3.5998%)

First of all note that we are using the same investment experience as in Figures 3.4 and 3.5. The assets grow with investment return and premiums added in the same way as the conventional with-profits case although the split between units and options is different. The value of the assets at maturity is £4,957.

The guaranteed growth rate and desired bonus rate are modest compared to the investment return so that the desired bonus rate is affordable each year. The guarantee at maturity is £1,748. Notice however that prior to maturity the guarantee that the policyholder sees is the value of the units U_t, which is less than the guarantee which has been actually purchased. Recall from Equation 3.11 that the maturity guarantee is equal to the unit value accumulated at the guaranteed growth rate to maturity.

If the policyholder had instead invested in a unit-linked policy with no guarantees the payout would have been £5,464 by Equation 3.7. The final payout for the unitised with-profits policy was £4,957, so that the accumulated cost of the guarantees was £507.

The payout in this example is considerably higher than the risk-free payout of £2,193 given by Equation 3.8. However the unitised with-profits policy is more risky. If assets had fallen very sharply in the final year then the with-profits policyholder would only have received the guaranteed value of £1,748, whereas the full value of the risk-free contract is guaranteed. Table 3.14 compares the final guarantee and payout under the unit-linked and risk-free contracts with this low bonus unitised with-profits policy.

3.3.4 Example 2 — Unitised With-Profits with High Bonuses

We now consider our second example shown in Figure 3.7 with guaranteed growth rate y of 1.7055% and desired reversionary bonus rate z of 8.5035%.

We have used the same investment experience as in 3.6. However, the larger guarantees in the second example mean that more units of the equity index are sold to purchase the options so that the final value of the assets is only £2,411.

We can see that the guarantee actually purchased increases smoothly each year up to 1979, because the desired bonus of 8.5035% is affordable each year. However in 1980, 1982, and 1984 the desired guarantee is unaffordable so that the only increase in the guarantee comes from the guaranteed growth on the new premium.

The value of the policyholder's units U_t behaves in a similar way to the actual

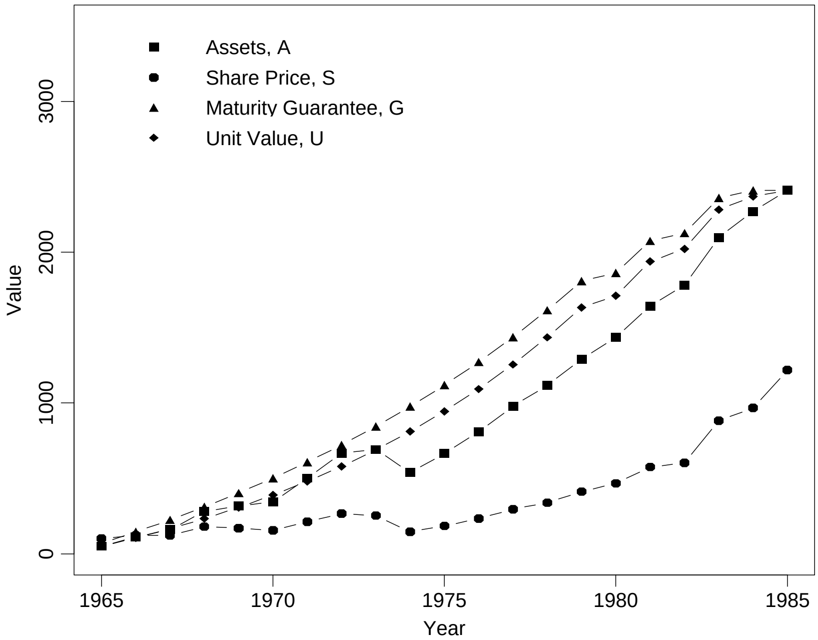

Figure 3.7: The Value of the Assets and Guarantees under a Unitised With-Profits Policy (Guaranteed Growth Rate 1.7055%, Desired Bonus Rate 8.5035%)

guarantee purchased. In each year except 1980, 1982, and 1984, the units grow with the guaranteed growth rate, the desired bonus, and the addition of a new premium. However in 1980, 1982, and 1984, no bonus is declared so that the units only grow with the guaranteed growth rate and the addition of a new premium.

At maturity the exercise price of the options of £1,348 is greater than the price of the shares of £1,218. Hence the policyholder receives the guaranteed amount of £2,411 with no terminal bonus.

Notice that in this second example the final guaranteed sum of £2,411 is greater than the risk-free payout of £2,193. The strong performance of the shares has allowed the guarantee to be increased significantly throughout the term. However the with-profits payout of £2,411 is substantially less than the unit-linked payout of £5,464, showing that the accumulated cost of the guarantees is £3,053. The final guarantee and payout under the unit-linked, risk-free, low bonus and high bonus unitised with-profits policies are shown in Table 3.14.

Table 3.14: Maturity Payout and Final Guarantee for 20 year Policies Taken Out in 1965

Type of Policy	Maturity Payout	Final Guarantee
Unit-Linked	£5,464	£0
UWP — Low Bonus	£4,957	£1,748
UWP — High Bonus	£2,411	£2,411
Risk-Free	£2,193	£2,193

3.4 Comparison of the Charging Mechanism for Conventional and Unitised With-Profits

In this section we will compare the conventional with-profits (CWP) mechanism described in Section 3.2 with the unitised with-profits (UWP) mechanism described in Section 3.3, and discuss some practical aspects of the application of the option pricing approach.

3.4.1 Comparison of the Guarantees for Conventional and Unitised With-Profits

The main difference between a CWP and UWP policy is the speed at which the guarantees build up. However, Yap (1999) showed how to set the bonus rates so that the guarantees are equivalent at maturity.

First of all assume that the insurer declares no bonuses at all under either policy. This is the smallest guarantee that the insurer can offer without being in breach of contract (although they may be in breach of policyholders' reasonable expectations). Under the CWP contract the maturity guarantee is then just the sum assured. Under the UWP contract the maturity guarantee is equal to the invested premiums accumulated at the guaranteed growth rate. Assuming that the guaranteed growth rate for all premiums is fixed at rate y, then we can set the maturity guarantees under the two policies to be equal as follows:

$$\text{CWP Sum Assured} = \sum_{\tau=0}^{T-1} P_\tau \left(1+y\right)^{T-\tau}$$

Using the above equation, we can see that our CWP example with sum assured of £1,200 is in a sense equivalent to our UWP example with guaranteed growth rate of 1.7055%.

Secondly let us assume that investment returns are sufficient to allow the insurer to declare the desired bonus rate every year. We can then set the maturity guarantees to be equal as follows:

$$\left(\text{CWP Sum Assured}\right)\left(1+b\right)^{T-1} = \sum_{\tau=0}^{T-1} P_\tau \left(1+y\right)^{T-\tau}\left(1+z\right)^{T-\tau-1}. \qquad (3.12)$$

Using the above equation, we can see that our CWP examples with desired bonuses of 2% and 5% are in a sense equivalent to our UWP examples with desired bonuses of 3.5998% and 8.5035% respectively.

We can compare the guarantees actually purchased in our examples in Figure 3.8. Recall that for conventional with-profits policies the guarantee actually purchased is only a fraction of the declared guarantee, as given by Equation 3.6. For unitised with-profits policies the guarantee actually purchased is the unit value accumulated at the guaranteed growth rate.

The first thing to notice about Figure 3.8 is that the guarantee at maturity is equal for both the low bonus CWP and the low bonus UWP examples. In the low bonus cases the guarantees were affordable every year and so the final guarantees were equal as given by Equation 3.12. The desired bonus is often unaffordable for the high bonus examples, so that the final guarantees are slightly different due to the differing bonuses foregone.

The next thing to notice is that the guarantees purchased in the early years are lower for UWP policies than for CWP policies. This fits with the fact that the guarantees declared under CWP policies are much higher than under UWP policies. However, the position may become reversed if we purchase a lower fraction of the declared CWP guarantee than given in Equation 3.6.

We can also see that the build up of guarantees is smoother under the UWP

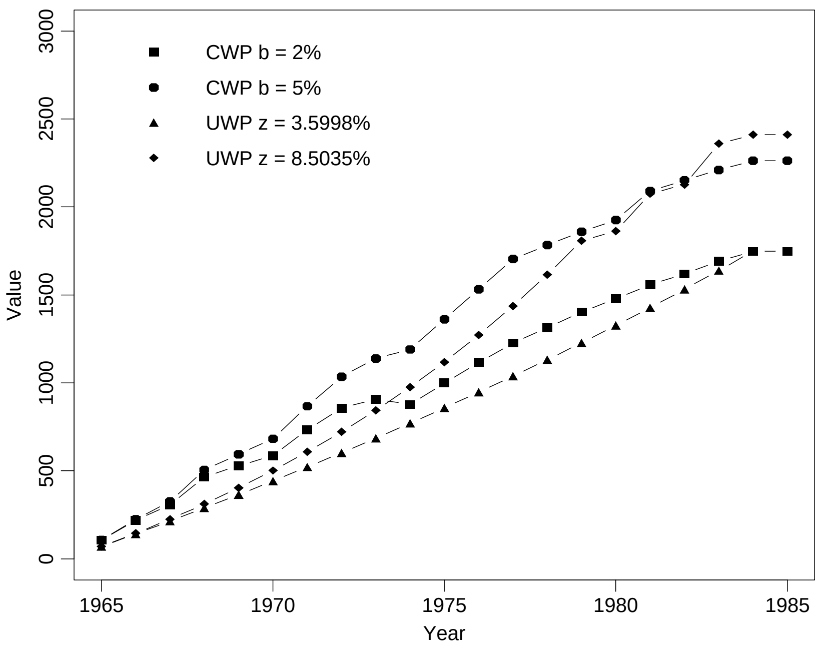

Figure 3.8: Comparison of the Guarantees Actually Purchased under a Conventional With-Profits Policy and a Unitised With-Profits Policy

method. UWP guarantees increase each year. However, the actual guarantee purchased for CWP policies can actually decrease. We discussed in Section 3.2.3 the reasons for the fall in the purchased guarantee for the CWP policy with low bonuses. We can see in Figure 3.8 that the sharp fall in assets in 1974 is accompanied by a fall in the purchased guarantee. The fall in assets means that the maximum guarantee that can be purchased with the current assets has fallen in relation to the maximum guarantee which can be purchased from future premiums. Equation 3.6 then shows that the actual guarantee purchased will be lower, and that therefore a greater proportion of the future premiums are expected to be invested in options.

The possibility of a fall in the purchased guarantee is an undesirable feature of the CWP strategy. Regulators and policyholders will be unhappy that the security of the liabilities is actually being further reduced by selling the protection of the options when the stock market has fallen sharply. However this strategy is highly beneficial if the stock market rapidly recovers from the crash, as it does in this example, because the asset value will benefit from the greater equity exposure during the recovery.

Although the possibility of a fall in the purchased guarantee is an unusual feature, it will not cause any problems in our model. We have assumed that the risk-free rate is constant. Hence we know that the maximum future guarantee can always be purchased by investing the future premiums in the risk-free asset. However, in practice the risk-free rate may decrease, so that it is possible that the declared guarantees can no longer be met by the current assets and future premiums. Hence at maturity it is possible that the assets will be insufficient to meet the guarantees, a problem that is exacerbated if the purchased guarantee has decreased during the term.

One way of reducing the probability that the declared guarantees become unaffordable under a CWP policy was suggested by Wilkie (1987). In Section 3.2.1 we declared a bonus only if the desired guarantee DG_t was less than the maximum present and future guarantees. Wilkie's suggested improvement was that a bonus should only be declared if the desired guarantee DG_t was less than say 80% of the maximum present and future guarantees. Yap (1999) found that this suggestion could indeed be effective at stopping the guarantees become too high too early. Yap found that in many cases even though the suggestion would stop bonuses being declared in some years, the extra investment freedom actually increased the number of bonuses that could be declared in later years and increased the final payout.

Finally, the guarantees shown in Figure 3.8 are the guarantees actually purchased, and are reasonably similar under the CWP and UWP approach. However the guarantee declared to the policyholder is very different. In the CWP case the declared guarantee is the full sum assured plus reversionary bonuses, which we can see in Figures 3.4 and 3.5 is very much larger in the early years than the actual guarantee. In the UWP case the declared guarantee is the value of the units, which we can see in Figures 3.6 and 3.7 is a little smaller in the early years than the actual guarantee. If all goes well, then at maturity the actual and declared guarantees are equal. However, as we have seen above, there is a danger in the CWP case that the declared guarantee becomes unaffordable, so that the insurer will make a loss at maturity if the value of the shares has fallen below the guaranteed value.

In conclusion, the UWP mechanism appears to be preferable to the CWP mechanism, because the purchased guarantee increases smoothly with no possibility that options cannot be purchased to meet the declared guarantee.

3.4.2 Comparison of the Assets for Conventional and Unitised With-Profits

We can now compare the value of the assets actually held in our four examples in Figure 3.9.

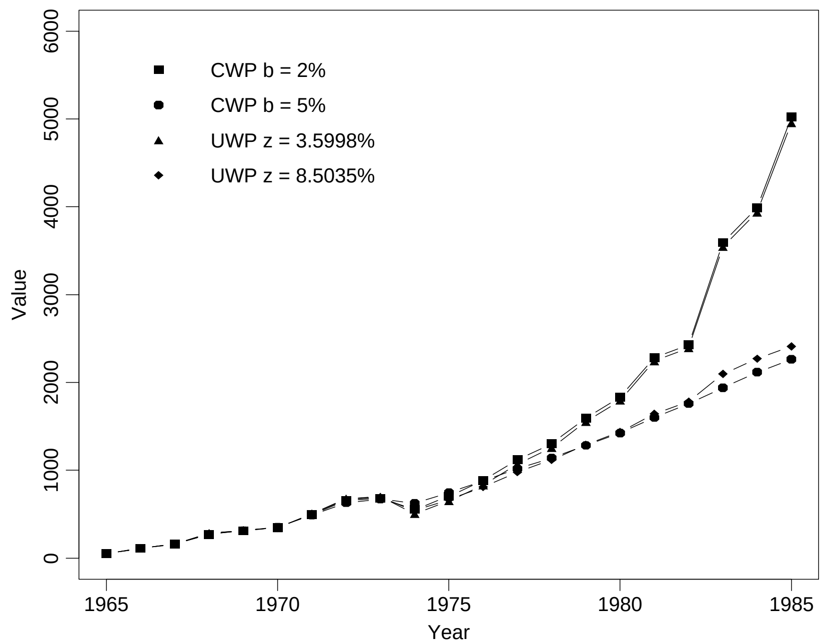

Figure 3.9: Comparison of the Value of the Assets under a Conventional With-Profits Policy and a Unitised With-Profits Policy

First of all we can see that in most years the value of the assets under the two low bonus strategies is higher than under the two high bonus strategies. High bonuses mean that more expensive options are required with a higher exercise price, and hence more units of the equity index need to be sold to buy the options. A smaller holding in shares will lead to a lower value of assets if the average equity return has been higher than the risk-free return to date.

However, a low bonus strategy is more risky. If the return on the equity index is lower than the risk-free rate then the low bonus strategy will lead to a lower value of assets. We can see this is the case after the stock market crash. The lowest value of assets in June 1974 is for the low bonus UWP policy, which is also the policy with the lowest guarantee at that time.

Finally comparing CWP with UWP policies no clear pattern emerges. The value of the assets depends not only on the size of the current guarantee, but also on when that guarantee was purchased as options are more expensive when the equity index has fallen in value. CWP builds up the guarantee more quickly and so is more sensitive to the value of the equity index in the earlier years. In this example, for the low bonus examples, the CWP policy has a higher value of assets at maturity (£5,024 as opposed to £4,957) because the larger holdings of options in the early years gives it more protection from the stock market crash. However, for the high bonus examples, the UWP policy has a higher value of assets at maturity (£2,411 as opposed to £2,263) because the CWP policy builds up the guarantee so quickly that the proportion of assets invested in equities becomes very small at an earlier stage, so that the CWP policy benefits less from the strong equity performance throughout the contract.

3.4.3 Actually Buying the Options, Hedging, or Notionally Buying the Options

There are three ways in which the option pricing approach to charging CWP and UWP policies for their maturity guarantees can be applied: the insurer can actually buy the options, the insurer can follow a hedging strategy, or the insurer notionally invests in the options. We describe each of the three methods below in greater detail.

1) **Buying the options**

The with-profits fund invests in a mixture of units of the equity index and put options as described in Sections 3.2 and 3.3. The put options ensure that the guarantees can always be met (*i.e.* they provide financial insurance that pays out if the equity returns are too low). Crucially each policyholder's asset share reflects the investment return on this portfolio of equities and options (as opposed to the more usual asset share invested in non-derivative assets such as equities and gilts). In effect each policyholder bears all his own investment risk, and other policyholders share only the mortality risk with each other (in addition we may choose to smooth payouts through time). The 'charge' deducted from the policyholder to pay for the guarantee is exactly the cost of buying the options needed to back their own policy. The options will cost more if either the guarantee is increased or share prices are low.

2) **Hedging**

As an alternative to actually buying the options, the insurer could choose the hedging strategy that replicates the maturity payoff.

We saw in Sections 3.2 and 3.3 that after each bonus declaration or premium payment the portfolio needed to be rebalanced so that the insurer held the same number N_t of units of the equity index and put options as follows:

$$A_t = N_t\,(S_t + O_t^+).$$

The number of units and put options would remain fixed until the next bonus declaration or premium payment at time $t+1$. So at any time τ between t and $t+1$ the value of the assets is given by

$$A_\tau = N_t\,(S_\tau + O_\tau^+) \quad \text{where} \quad t \leq \tau < t+1$$

where O_τ^+ is the value of the put option with exercise price E_t and is given by the Black-Scholes formula as follows:

$$O_\tau^+ = E_t\,e^{-r_f(T-\tau)}\,\Phi(-d_2(\tau)) - S_\tau\,\Phi(-d_1(\tau))$$

$$d_1(\tau) = \frac{\ln(S_\tau/(E_t\,e^{-r_f(T-\tau)}))}{\sigma\sqrt{T-\tau}} + \frac{\sigma\sqrt{T-\tau}}{2}$$

$$d_2(\tau) = \frac{\ln(S_\tau/(E_t\,e^{-r_f(T-\tau)}))}{\sigma\sqrt{T-\tau}} - \frac{\sigma\sqrt{T-\tau}}{2}.$$

However we can also use the above formulae to derive the hedging portfolio made up of units of the equity index and the risk-free asset needed at time τ as follows:

$$\begin{aligned}\text{Value of units held at time } \tau &= N_t\,S_\tau[1 - \Phi(-d_1(\tau))]\\ \text{Value of risk free asset held at time } \tau &= N_t\,E_t\,e^{-r_f(T-\tau)}\,\Phi(-d_2(\tau)).\end{aligned}$$

Notice that to hedge the option requires a negative number of units $-\Phi(-d_1(\tau))$. Therefore the total number of units held is less than N_t and varies during the year.

In theory the insurer can continuously rebalance the hedge portfolio to replicate the maturity guarantee. However in practice the insurer can only rebalance at discrete time intervals which introduces hedging error. In addition the trading of assets incurs transactions costs. Rebalancing more frequently will reduce hedging error but increase transactions costs. We do not consider hedging error and transactions costs further, but the effects on policies with maturity guarantees have been investigated by Brennan and Schwartz (1979), Boyle and Hardy (1996), Boyle and Hardy (1997), Hardy (1999), Hardy (2000), Hardy (2002) and Hibbert and Turnbull (2003).

3) **Not buying the guarantee**

The insurer could instead decide not to invest in options or the hedge portfolio, but follow some other investment strategy. For example, the insurer may decide to invest entirely in equities. The put options backing the guarantee are in this case hypothetically written by future generations of policyholders or by the free estate. Hence the with-profits fund bears the risk in a very real sense (*cf.* Equitable Life where the free estate was insufficient to cover the cost of the guarantees so that the policyholders without guarantees had to subsidise those with guarantees). Ultimately if equities fall to very low levels it is possible that the fund has insufficient assets to meet its guarantees — the liability then falls on either the shareholders for a proprietary company or the policyholders themselves (*i.e.* they cannot be paid the full value of their guarantee). Importantly however we must ensure that each policyholder is fairly charged for the risk they bring to the fund. These risks are represented by the cost of the matching options. Hence the asset share should reflect the notional investment in units and options even if the insurer follows a mismatched investment strategy.

We have seen above, ignoring the effect of transactions costs, that whether the insurer invests in options, performs continuous hedging, or mismatches, that the appropriate asset share is the same *i.e.* the asset share assuming the options are actually purchased. Hence for simplicity in Chapters 4, 5, 6, and 7 we assume that the options are actually purchased.

In Chapter 8 we will assume that the insurer mismatches the guarantee, but that payouts are still based on the asset share assuming a notional investment in options. We will then investigate the size of the free estate required so that it can safely write these uncovered guarantees in Chapter 9.

3.4.4 Surrendered and Paid-up Policies for Conventional and Unitised With-Profits

Calculating surrender and paid-up values can be very difficult for with-profits policies, and it might be felt that the introduction of the option pricing approach would further complicate their calculation. In fact it is very easy to calculate surrender and paid-up values for either the CWP or UWP policies under the option pricing approach.

The surrender value that returns the fair share of the assets to the policyholder is simply the value of the assets A_t for a policy surrendered at time t. Notice that the policyholder has been fairly charged for the maturity guarantees by the deduction of the cost of purchasing the options. Hence the policyholder cannot select against the insurer by surrendering the policy if the guarantees are unlikely to bite.

However, the policyholder may be giving up a valuable maturity guarantee and should be compensated for this. The surrender value A_t includes both the value of the equities and the value of the options. Hence the value of the future maturity guarantee is included in the surrender value.

In conclusion, the advantage of paying A_t as a surrender value is that the with-profits fund, and hence the other policyholders and the insurer, makes neither a profit nor a loss.

In practice the value of the assets A_t will be used as a starting point for the surrender value calculation. The insurer may want to reduce this payment to take account of additional expenses, or to discourage surrenders. Equally the insurer may wish to enhance the surrender value for competitive reasons.

If the policyholder wishes to cease paying premiums but retain the policy until maturity then the policy is made paid-up. The current value of the assets A_t can be considered as a single premium to a new policy with the same remaining term as the original regular premium policy.

For the regular premium CWP policy the sum assured plus reversionary bonuses are affordable if the future premiums are paid. However the actual guarantee AG_t has already been purchased from current assets without the need for future premiums. Hence the paid-up policy can be considered as a new single premium CWP policy with sum assured AG_t and assets A_t.

Making a UWP policy paid-up is even simpler. The UWP methodology makes no allowance for future premiums at any point in its calculation. Therefore the paid-up policy can be considered as a new single premium UWP policy with the same unit value U_t, guarantee G_t, and asset value A_t as the original regular premium policy.

The advantage of this approach to CWP and UWP paid-up policies is that not only does the with-profits fund make neither a profit nor a loss, but the split of the assets between units of the equity index and options does not need to be changed.

3.4.5 Flexibility of Bonus Declarations

In the numerical examples we assumed that the same bonus would be declared each year as long as it was affordable. In the CWP case the maximum guarantee was equal to the value at maturity of the current assets and future premiums if they were invested in the risk-free asset. In the UWP case the maximum guarantee was found by investing only the current assets at the risk-free rate.

In fact the insurer could have declared any bonus it wanted to, using the option pricing approach, as long as the resulting guarantee was less than the maximum. The method described in Sections 3.2.1 and 3.3.1 would still work. Higher bonuses simply lead to more equities being sold to buy options with a greater exercise price.

This is the great advantage of the option pricing approach used in this thesis. The charge for the guarantee is determined at the time the bonus is declared. Hence the bonus strategy can be changed during the contract without affecting the equity of the payout.

The approach taken in this thesis is in direct contrast to the approach of Persson and Aase (1997), Miltersen and Persson (1999), Miltersen and Persson (2003), Bacinello (2001), Grosen and Jorgensen (2000), Miltersen and Hansen (2002), Miltersen and Persson (2000), Grosen and Jorgensen (2002), and Jorgensen (2001). These authors have all determined the bonus strategy at outset. The bonuses may be stochastic, but they are a given function of other variables such as the asset value. Hence these authors are able to set charges that are fair in the sense that the expected value of the guarantees and charges are equal under the equivalent martingale measure. However these charges will be inequitable if the bonus algorithm is changed during the policy. Indeed Jorgensen (2001) shows the profits and losses to the insurer if the regulatory environment is changed during the policy term.

The flexibility of the method used in this thesis is particularly important given the discretion insurers have in the U.K. over setting bonuses. The insurer can set their bonuses with regard to their current solvency level and competitors rates. The insurer need not announce a bonus methodology in advance. Hence any charging strategy that is fixed at policy outset is unlikely to be equitable.

In fact the bonus flexibility of this method can be taken further. The policyholders could in theory even be allowed to choose their own bonuses, subject to the maximum. However in practice the policyholders may find this contract too complex. The record keeping required to keep track of the chosen bonuses and the resulting investments is also likely to be beyond insurers' existing systems.

3.4.6 Flexibility of Asset Allocation

So far we have assumed that the policyholder's assets are only invested in units of an equity index and put options written on this index. Such options are only likely to be available, even over-the-counter, for major quoted share indices. Hence if the insurer's share portfolio does not track a quoted index closely, the options will not provide 100% security and the cost of the options will not reflect the true cost of the shares.

It is common in the U.K. for with-profits funds to be invested in a variety of assets including overseas equities and property. It is unlikely that appropriate options will be available for such portfolios. It may be possible instead to follow a dynamic hedging strategy, although this will still be impossible for assets with low marketability such as property.

However, it is possible to apply the approach of Sections 3.2 and 3.3 to a portfolio of shares and zero coupon bonds. Zero coupon bonds with the same maturity date as the contract could be used to purchase part of the guarantee. The option pricing approach could be used as before on any remaining guarantee.

However a potential problem in the U.K. is that the insurer retains discretion over the choice of investments. There is considerably less chance of the guarantee biting if the assets are invested largely in zero coupon bonds. Indeed the last few years have seen many insurers switch a large proportion of with-profits assets from risky shares to safer gilts, 'with the average equity weighting of U.K. with-profits funds falling from 60 to 35 per cent in the past year' (Financial Times, 3 February 2003).

It is quite acceptable for insurers to switch assets if that is what the policyholder has been led to expect. The insurer could invest almost entirely in equities most of the time. When stock markets fall, the insurer could switch more into zero coupon bonds. The correct balance between shares and zero coupon bonds is actually given by the hedging portfolio of the option pricing strategy. There would be no need to charge the policyholder for the guarantees as any move from shares to bonds would make an implicit charge.

The problem occurs if the insurer has charged the policyholder as if they would invest entirely in equities to maturity. Any switch to bonds in the reference portfolio would then deprive the policyholder of part of the value of their option.

The option pricing approach provides a solution to the problem. Whenever a switch is made between shares and bonds in the reference portfolio, the policyholder should be credited with the value of the options that they (actually or notionally) hold at that date. New options should then be bought to match the guarantee in excess of the bond proceeds. Hence, in effect, the policyholder receives a rebate on their charges whenever the insurer switches from shares to bonds.

3.4.7 Asset Shares and Reserves for Conventional and Unitised With-Profits

In this thesis we are primarily concerned with charging the policyholder for the guarantees to determine an equitable maturity value. However, we will now briefly discuss the calculation of reserves.

The fair value of the policy is given by the values of the matching units of the equity index and put options. Hence the fair value of the policy is simply the value of the assets A_t. The prudential reserves required in excess of the fair value will depend on which of the three investment strategies described in Section 3.4.3 the insurer actually follows.

If the insurer actually buys the options then the payout is fully matched so that in theory no further reserves are required. However, the insurer is exposed to counterparty risk such that the writer of the option defaults in the event of a claim. Hence further reserves may be required.

Again, if the insurer continuously hedges the risk and there are no transactions costs, then no further reserves are required. However, in practice rebalancing can only take place at discrete time intervals and transactions costs are incurred. Hence the insurer must hold further reserves for hedging error and transactions costs, for example by holding either quantile or conditional tail expectation reserves derived from a number of simulations.

If the insurer does not hold options nor the hedge portfolio then it must hold mismatching reserves. These reserves could be given by the quantiles or conditional tail expectations of the loss from a number of simulations.

Boyle and Hardy (1996), Boyle and Hardy (1997), Hardy (2000), and Hardy (2002) compare the reserves required for the three investment strategies for unit-linked policies with guarantees. Tong (2004) extends this work to unitised with-profits policies.

3.4.8 Discussion of Further Work to Improve the Modelling of Unitised With-Profits

In the remainder of this thesis we will investigate further the option pricing approach to charging for UWP guarantees. There are a number of ways in which the simple model investigated in this chapter can be improved.

So far we have only observed the effects of the option pricing approach under a single investment period. In later chapters we will observe the method under a large number of stochastic simulations. We will consider two different investment models for shares: geometric Brownian motion in Chapter 4 and the Wilkie model in Chapter 5.

We discussed in Section 3.4.1 the possibility that in practice the risk-free interest rate could decrease so that the options became more expensive. In Chapter 6 the options are priced using a stochastic risk-free rate derived from the Wilkie model.

In this chapter the bonus mechanism has been a very simple one whereby the same bonus is declared each year unless it is unaffordable. In practice bonus rates are more dynamic. For example, the bonus rate should be related to investment returns, perhaps subject to smoothing. We consider alternative bonus strategies in Chapter 7.

Chapter 4

A Simple Investment Model — Geometric Brownian Motion for Equities and Deterministic Risk-Free Asset

In Section 3.3.1 we introduced an option pricing methodology for unitised with-profits policies. Then in Section 3.3.2 we presented the results of this strategy using the actual historic data from 1965 to 1985. Having seen how this approach behaves in the past we would now like to see how it might behave in the future. To do this we need an investment model.

In Section 3.3.2 we assumed that the options were priced using the Black-Scholes equation. Underlying this equation are the assumptions that shares follow geometric Brownian motion and that the return on the risk-free asset is constant. This is not to say that shares in the 'real world' will actually follow geometric Brownian motion, but only that the market believes that this is a good approximation to the behaviour of the share price.

In this chapter we assume that the 'real world' model for share prices is the same geometric Brownian motion used by the market when setting option prices. We will call this model GBM for short.

We begin with a description of the investment model in Section 4.1. We then illustrate the distribution of the investment returns generated by the model in Section 4.2. The model is then used to simulate the behaviour of the option pricing methodology in Section 4.3. In Section 4.4, we look at the sensitivity of the results to the parameters of the GBM model. Finally, in Section 4.5 we calculate the reduction in yield caused by the purchase of options to match the maturity guarantees.

4.1 Investment Model

In Section 3.2.1 we introduced the equity index S_t with reinvested dividends. The insurer holds a number of units of this equity index and options written upon this index.

The real world model is consistent with the Black-Scholes formula in that the equity index, S_t, follows a geometric Brownian motion as follows

$$dS_t = \mu\, S_t\, dt + \sigma\, S_t\, dz_t.$$

The parameter μ governs the expected rate of return on the index, while the parameter σ represents the volatility of the rate of return. z_t follows Brownian motion.

The level of the equity index at time T given the index at some earlier time t is then lognormally distributed as follows

$$\ln S_T \sim N\left[\ln S_t + \left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right]. \tag{4.13}$$

Hence we can generate the level of our equity index at yearly intervals as follows

$$S_t = S_{t-1} \cdot \exp\left(\mu - \frac{\sigma^2}{2} + \sigma \cdot sz_t\right)$$

where sz_t is a random number with unit normal distribution.

We now need to choose the parameters μ and σ. We have assumed that the real world investment model is the same as that underlying the Black-Scholes equation. Therefore we assume that the real world volatility σ is equal to 20% p.a. as in Section 3.3.2.

The Black-Scholes equation is independent of the expected return on shares and so we can choose any value for the parameter μ and still be consistent with the option pricing model. We will assume that the real world parameter μ equals 11.44947% p.a.. This choice of μ will allow comparisons to be made with an alternative real world investment model in Chapter 5.

Finally we must decide on a real world model for the risk-free asset. In Section 3.3.2 we priced the options using a constant risk-free rate of return of 7% p.a. (corresponding to a force of interest of 6.76586% p.a.). Therefore we will also use a constant risk-free rate of 7% p.a. in our real world investment model.

4.2 Investment Model Results

We generate 10,000 simulations of the investments using the GBM model and parameters described above. In this way we can illustrate the investment returns produced by the model. Figure 4.10 shows three sample paths for the equity index simulated by the GBM model.

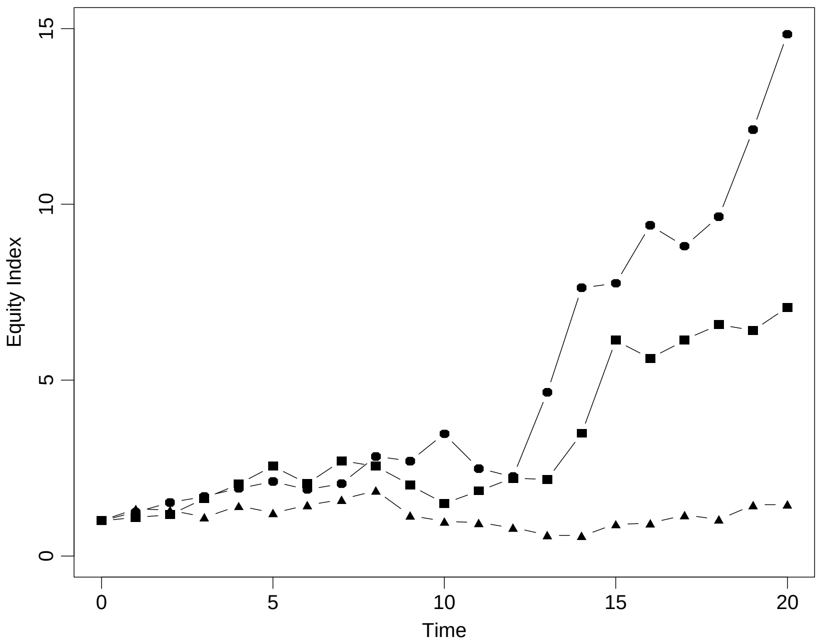

Figure 4.10: Sample Paths for the Equity Index, S_t, Simulated by the GBM Model

We estimate the mean M, standard deviation SD, skewness SK and excess kurtosis KU for the annualised returns, GS_t, on the equity index, S_t, where

$$GS_t = 100\left[\left(\frac{S_t}{S_0}\right)^{1/t} - 1\right].$$

Table 4.15 shows these summary statistics estimated from the simulations of the GBM model. Both the mean and standard deviation of the equity returns decrease with term. Both the skewness and excess kurtosis fall towards zero. The return on the risk-free asset is of course a constant 7% p.a..

Table 4.15: Simulated Summary Statistics from the GBM Investment Model

	Term					
	1	2	5	10	20	50
$M(GS)$	11.93	10.77	10.44	10.20	10.04	9.94
$SD(GS)$	22.79	15.67	9.88	6.98	4.89	3.08
$SK(GS)$	0.59	0.42	0.23	0.15	0.11	0.05
$KU(GS)$	0.51	0.34	0.04	0.05	−0.01	0.06

From Equation 4.13 we see that

$$\frac{1}{t}\ln\left(\frac{S_t}{S_0}\right) = \ln\left(\frac{S_t}{S_0}\right)^{1/t} \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right), \frac{\sigma^2}{t}\right].$$

Hence we can calculate the mean, standard deviation and skewness of GS_t analytically as follows:

$$\begin{aligned} M(GS_t) &= 100\left(E\left(\frac{S_t}{S_0}\right)^{1/t} - 1\right) \\ &= 100\left(\exp\left(\mu - \frac{\sigma^2}{2} + \frac{\sigma^2}{2t}\right) - 1\right) \end{aligned}$$

$$\begin{aligned} SD(GS_t) &= 100\,SD\left(\frac{S_t}{S_0}\right)^{1/t} \\ &= 100\left(e^{2\mu-\sigma^2+\sigma^2/t}\left(e^{\sigma^2/t} - 1\right)\right)^{1/2} \end{aligned}$$

$$
\begin{aligned}
SK(GS_t) &= SK\left(\frac{S_t}{S_0}\right)^{1/t} \\
&= \left(e^{\sigma^2/t}+2\right)\left(e^{\sigma^2/t}-1\right)^{1/2}.
\end{aligned}
$$

The analytical results in Table 4.16 and the simulated results in Table 4.15 are reassuringly in close agreement.

Table 4.16: Analytical Summary Statistics from the GBM Investment Model

	Term					
	1	2	5	10	20	50
$M(GS)$	12.13	11.01	10.35	10.13	10.02	9.95
$SD(GS)$	22.65	15.78	9.89	6.97	4.92	3.11
$SK(GS)$	0.61	0.43	0.27	0.19	0.13	0.08

Further we can use Jensen's inequality to show why the mean equity return falls with time. GS_t is a concave function of S_t for $t > 1$. Therefore:

$$
\begin{aligned}
E\{GS_t\} &= E\left\{100\left[\left(\frac{S_t}{S_0}\right)^{1/t}-1\right]\right\} \\
E\{GS_t\} &\leq 100\left[\left(E\left\{\frac{S_t}{S_0}\right\}\right)^{1/t}-1\right] \quad \text{for } t>1 \\
E\{GS_t\} &\leq 100\left[\left(e^{\mu t}\right)^{1/t}-1\right] \\
E\{GS_t\} &\leq 12.13\%.
\end{aligned}
$$

To give us some idea of the likelihood of poor returns we can calculate the probability that the annual rate of return on the equity index is less than some target value i, $P(GS_t < i)$. The results in Figure 4.11 are obtained over a 20 year term. For example, over this 20-year period, there is a 2% probability of a negative return, and a 27% probability of a return less than the risk-free rate of 7% p.a.. To compensate for the possibility of very low returns there is a probability of 15% that the return on the equity index will exceed 15% p.a..

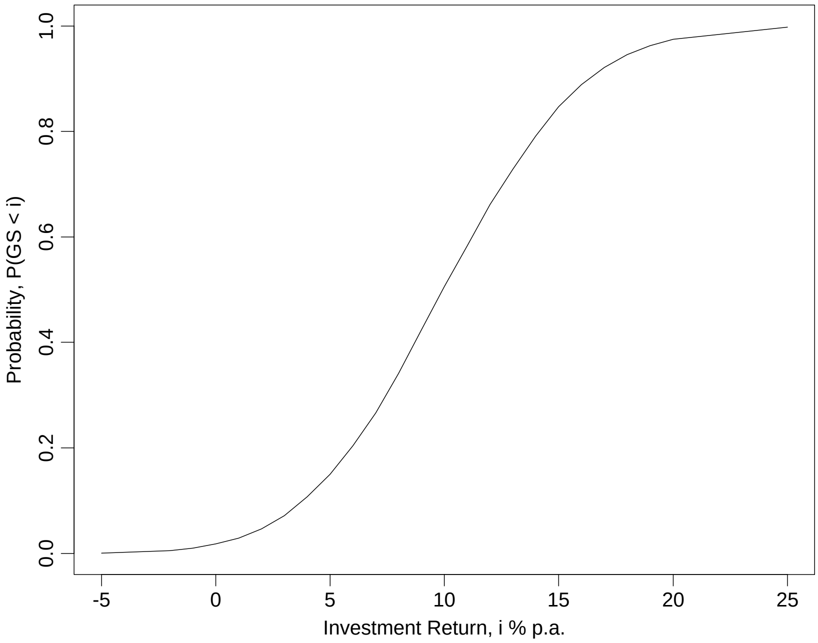

Figure 4.11: Cumulative Distribution Function for Equity Returns over a 20 Year Period under Geometric Brownian Motion, $P(GS_{20} < i)$

We can also calculate summary statistics of the annualised log returns, LGS, for our 10,000 simulations, where

$$LGS_t = 100 \cdot \ln\left(\frac{GS_t}{100} + 1\right).$$

We find that for any term, t, we approximately have mean of 9.45% p.a., volatility of $\frac{0.2}{\sqrt{t}}$, skewness of zero and excess kurtosis of zero. These results are in agreement with the true underlying values from the lognormal model.

4.3 Results for the Unitised With-Profits Charging Mechanism

The simulations from the GBM model are then used to calculate the payout on 20-year regular premium unitised with-profits (UWP) policies assuming guarantees are priced using Black-Scholes with risk-free return of 7% and volatility of shares of 20% as in Section 3.3.2. We consider UWP policies with a range of values for the minimum guaranteed growth rate y and the desired bonus rate z. In this chapter we take the bonus rates y and z as constants as in Section 3.3.2.

For comparison, we also simulate the payouts from a unit-linked (UL) policy and a risk-free (RF) policy which both have the same term and premiums as the UWP policy. The unit-linked policy is invested entirely in the equity index and offers no guarantees. The risk-free policy is invested in the risk-free asset, which under the GBM model has a constant return of 7% p.a..

Table 4.17 shows the results of the simulations for UWP policies with a range of different values for the minimum guaranteed growth rate y and the desired bonus rate z. The table also shows these results for the unit-linked and risk-free policies. The maximum guarantee is the guaranteed maturity payout if the desired bonus rate z is affordable every year. The table also shows the mean and standard deviation of the maturity payout and the achieved guarantee at maturity G_{19}.

Table 4.17: Mean and Standard Deviation of the Payout and Achieved Guarantee using the GBM Model

y	z	Maximum Guarantee	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		0.00	4086.15	3044.07	0.00	0.00
0.00	0.00	1000.00	4034.92	3005.97	1000.00	0.00
0.00	0.04	1488.90	3959.08	2987.87	1476.74	36.43
0.00	0.08	2288.10	3763.47	2922.99	2094.00	255.94
0.02	0.00	1239.17	3939.93	2920.78	1239.17	0.00
0.02	0.04	1892.28	3741.09	2839.26	1839.70	92.73
0.02	0.08	2974.63	3388.30	2573.33	2414.31	445.34
0.04	0.00	1548.46	3703.03	2680.44	1548.46	0.00
0.04	0.04	2422.22	3252.93	2353.87	2209.54	203.88
0.05	0.00	1735.96	3474.32	2422.42	1735.96	0.00
0.05	0.04	2746.95	2851.15	1779.25	2325.69	261.06
0.06	0.00	1949.64	3093.56	1939.29	1949.64	0.00
0.06	0.04	3119.56	2424.43	803.07	2308.69	223.82
RF		2193.26	2193.26	0.00	2193.26	0.00

Clearly the maximum possible guarantee increases as we increase either the compulsory bonus rate y or the desired bonus rate z. We also see above that the mean achieved guarantee normally increases with larger y or z.

However, we see that the mean guarantee achieved with compulsory bonuses at 5% and desired additional bonuses of 4% is £2,326; whereas if we increase the compulsory bonus to 6% and retain desired bonuses of 4% we achieve a mean guarantee of only £2,309. In fact this feature is very common if we look at individual simulations. We often find investment scenarios which produce a higher achieved guarantee with a *lower* value of y or z.

Clearly larger values of the bonus rates y and z have the potential to increase the guarantee achieved. However, declaring larger bonuses requires more options to be bought, which leaves less invested in shares. It is strong performance by shares which allows us to declare high bonuses later in the policy. Hence there comes a point where a high desired or compulsory bonus rate becomes counter productive.

The problem is more acute with a high compulsory bonus because it must apply to every future year. The desired bonus is only declared one year at a time.

We see that the maximum guarantee is always achieved when the desired guarantee is zero. This is because any compulsory guarantee up to 7% can be met by investing part of the premium at the risk-free return of 7%.

However there are always simulations where a positive desired guarantee cannot be declared even for the lowest bonus strategies. This is because these guarantees must be bought from assets initially invested in shares, which may be impossible if the share price has dropped, especially in the early years of the contract.

Hence we see that the mean achieved guarantee falls short of the maximum possible guarantee, especially when z is high.

Finally we see that the standard deviation of the achieved guarantee usually increases as the mean of the achieved guarantee increases. This is because as the guarantees become more onerous it becomes more likely that a desired bonus cannot be declared, not only in a single year, but on several occasions.

Now we turn to the actual payouts rather than the guarantees. The mean payout falls as the bonus rates rise. This is because higher guarantees force more money to be switched from equities into options.

We would hope that by increasing the bonuses we could trade some of the mean payout for a lower level of risk represented by the standard deviation. Increasing the compulsory bonus y has the larger effect on both the mean and standard deviation because the guarantee must be increased in all future years. Increasing the desired bonus z decreases the mean substantially with only a small reduction in standard deviation.

For example, the UWP policy with bonus rates y and z of zero has a mean maturity payout of £4,035 and a standard deviation of £3,006. A small increase in the compulsory bonus rate y to 2% decreases the standard deviation to £2,921 at the expense of a lower mean payout of £3,940. However, if we keep the compulsory bonus y at zero, we must substantially increase the desired bonus z to 8% to achieve a similar standard deviation of £2,923, but this comes at the expense of a much lower mean payout of £3,763.

It appears that a reduction in risk can only be obtained by paying a very high penalty on the expected payout. This is due in part to the nature of the bonus algorithm. The guarantees must be bought if they are affordable. This means that often, when share prices are low, additional guarantees are purchased at very high prices. More advanced bonus strategies will be considered in Chapter 7.

Table 4.18 shows the number of the 10,000 simulations which display certain features. The first column shows the number of simulations in which the maximum guarantee is achieved. The second column shows the number of simulations in which the option is exercised. The next three columns show the number of simulations for which the UWP payout, achieved guarantee, and maximum guarantee respectively are greater than the unit-linked payout. The final two columns show the number of simulations for which the risk-free policy payout is greater than the UWP payout and achieved guarantee respectively.

Higher bonus rates mean that the maximum guarantee can be achieved less often. However, even in the most onerous strategy with compulsory bonus y of 6% and desired bonus z of 4% a small proportion of the simulations, just under 3%, yield sufficient equity returns throughout the term of the policy to ensure the maximum guarantee is met.

If the equity price is above the exercise price of the options we are able to declare a terminal bonus and the option expires worthless. We see that the option is exercised

Table 4.18: The Number of the 10,000 Simulations where the Payouts and Guarantees Display Certain Features using the GBM Model

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	284	295	262	262	2731	10000
0.00	0.04	8774	1460	959	950	1036	3065	10000
0.00	0.08	5595	3845	1968	1948	2869	3917	4405
0.02	0.00	10000	684	602	576	576	2887	10000
0.02	0.04	7000	3071	1627	1620	1923	3943	10000
0.02	0.08	2927	5956	2520	2503	4456	3426	3735
0.04	0.00	10000	1562	1169	1156	1156	3308	10000
0.04	0.04	3831	5765	2363	2355	3182	4422	4752
0.05	0.00	10000	2479	1551	1545	1545	3821	10000
0.05	0.04	1871	7445	2630	2622	3961	3808	4047
0.06	0.00	10000	4100	2065	2062	2062	4830	10000
0.06	0.04	295	8901	2676	2673	4736	3338	3490

more often when either the guaranteed or desired bonuses are high.

We see that the UWP policy is more likely to pay more than the UL policy when the bonus rates are high. The reason becomes clear when we compare the achieved guarantee and the UL payout. When the achieved guarantee is greater than the UL payout then the option is almost certainly exercised and of course the UWP policy outperforms the UL policy. The higher the guarantee the more likely that the UL policy falls short.

There are a handful of occasions when the achieved guarantee does not exceed the unit-linked payout but the UWP policy still outperforms. Taking both bonus rates as zero we find 33 simulations in which the UWP payout is greater than the UL payout, which in turn is greater than the achieved guarantee. For example in simulation 5 the UWP payout is £1,414, while the UL payout is only £1,404. We can see the performance of the equity index in simulation 5 as the lower of the three sample paths in Figure 4.10. The equity index falls in value each year between times 8 and 14. This causes the value of the option to rise and so protects the UWP policy to some extent from the market falls. Each year a premium is received so that the insurer can rebalance its portfolio of shares and options by selling some of the more valuable put options and buying cheaper options with a lower exercise price. The market then recovers so that the final equity index is above the final exercise price and so the option is not exercised. However the index never recovers to its high of year 8 and the value of the UL policy remains below that of the UWP policy to the end of the term.

Also we see a number of simulations where the option is exercised but the payout is less than under the UL policy. For example, when the compulsory bonus is zero and the desired bonus is 4% we see that the option is exercised on 1460 occasions, but the UWP policy only outperforms the UL policy on 959 occasions. This happens if a guarantee is purchased when options are very expensive (*i.e.* the equity index is low). A large number of shares are sold to buy the option which has a very high exercise price. In effect the investment has been largely switched into the risk-free asset and so is unlikely to be able to outperform the UL policy.

In conclusion we see that the UWP policy is most likely to outperform the UL policy when the compulsory bonus rate is 6% and the desired bonus rate is 4%, although this still only occurs on 27% of the simulations.

Finally we compare the risk-free policy with the UWP policy. If we invest all the premiums at the risk-free rate the payout is £2,193.26. We can see that this risk-free payout is always higher than the achieved guarantee when the bonus rates are low. However, when the bonus rates are higher than the risk-free rate of 7% p.a. the achieved guarantee can be greater than the risk-free payout, although there are some simulations where the desired bonus rate is unaffordable in some years. For example, the maximum possible guarantee with a compulsory bonus of zero and a desired bonus of 8% is £2,288, which is larger than the risk-free payout. However, in 4,405 simulations the return on shares is insufficient to declare the 8% bonus every year and so the achieved guarantee falls short of the risk-free payout.

There are two factors affecting the link between the size of the bonuses and the likelihood that the risk-free payout exceeds the UWP payout. Firstly, higher bonus rates mean that more shares must be sold to buy options, and so the probability of the holding in shares outperforming the risk-free policy falls. This explains why the number of simulations in which the risk-free policy outperforms the UWP policy with zero compulsory bonus rises as the desired bonus rate is increased.

Secondly, higher bonus rates lead to the possibility of higher guarantees which will protect the UWP policy from falls in the share value late in the term of the policy. This explains why the UWP policy with a 6% compulsory bonus is more likely to outperform the risk-free policy with a high desired bonus rate.

4.4 Sensitivity of the Results to the Parameters of the GBM Model

In this section we consider the sensitivity of the results in Section 4.3 to changes in the parameters of the GBM model.

In Section 4.3 we assumed that the UWP policy was backed by shares which followed the GBM model with parameters μ of 11.44947% p.a. and σ of 20% p.a.. Correspondingly the guarantees were priced using Black-Scholes with risk-free return of 7% p.a. and volatility of shares of 20% p.a..

We will now investigate the effect of holding portfolios of shares with different levels of risk. The first portfolio we will consider will consist of more risky shares, with a correspondingly higher expected rate of return. The second portfolio will consist of less risky shares, with a correspondingly lower expected rate of return.

We will first assume that the UWP policy is backed by more risky shares which follow the GBM model with parameters μ of 13% p.a. and σ of 25% p.a.. The price of the options is adjusted to reflect the higher risk of this new portfolio. Hence, the guarantees are priced using Black-Scholes with volatility of shares of 25% p.a., but an unchanged risk-free return of 7% p.a..

Secondly we will assume that the UWP policy is backed by less risky shares which follow the GBM model with parameters μ of 10% p.a. and σ of 15% p.a.. Correspondingly the guarantees are priced using Black-Scholes with risk-free return of 7% p.a. and volatility of shares of 15% p.a..

We consider UWP policies with the same range of values as the previous section for the minimum guaranteed growth rate y and the desired bonus rate z.

For comparison, we also simulate the payouts from a unit-linked policy and a risk-free policy which both have the same term and premiums as the UWP policy. We assume that the unit-linked policy is invested in either the high risk or low risk portfolio of shares which corresponds to the investments of the UWP policy. The risk-free policy is invested in the risk-free asset, which under the GBM model has a constant return of 7% p.a. in all cases.

Tables 4.19 and 4.21 show the results for the high risk portfolio of shares with volatility of 25%. Tables 4.20 and 4.22 show the results for the low risk portfolio of shares with volatility of 15%. These tables can be compared with Tables 4.17 and 4.18 which show the results for the medium risk portfolio of shares with volatility of 20%.

Table 4.19: Mean and Standard Deviation of the Payout and Achieved Guarantee using the GBM Model for High Risk Shares

y	z	Maximum Guarantee	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		0.00	5071.85	5197.56	0.00	0.00
0.00	0.00	1000.00	4923.11	5043.40	1000.00	0.00
0.00	0.04	1488.90	4779.05	4982.07	1471.67	44.04
0.00	0.08	2288.10	4484.68	4843.96	2072.26	271.09
0.02	0.00	1239.17	4731.09	4814.11	1239.17	0.00
0.02	0.04	1892.28	4415.87	4631.63	1830.23	100.80
0.02	0.08	2974.63	3929.37	4217.24	2407.36	456.83
0.04	0.00	1548.46	4323.50	4285.86	1548.46	0.00
0.04	0.04	2422.22	3691.46	3741.16	2200.69	208.53
0.05	0.00	1735.96	3966.25	3784.41	1735.96	0.00
0.05	0.04	2746.95	3121.75	2810.61	2326.28	265.40
0.06	0.00	1949.64	3409.51	2927.24	1949.64	0.00
0.06	0.04	3119.56	2515.95	1304.94	2321.82	239.80
RF		2193.26	2193.26	0.00	2193.26	0.00

Firstly, we consider the mean and standard deviation of the payouts and achieved guarantees in Tables 4.19, 4.17, and 4.20, corresponding to the high, medium, and low volatility share portfolios respectively.

Table 4.20: Mean and Standard Deviation of the Payout and Achieved Guarantee using the GBM Model for Low Risk Shares

y	z	Maximum Guarantee	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		0.00	3352.63	1733.94	0.00	0.00
0.00	0.00	1000.00	3342.83	1729.29	1000.00	0.00
0.00	0.04	1488.90	3313.79	1728.25	1482.08	26.60
0.00	0.08	2288.10	3198.99	1702.19	2117.22	234.84
0.02	0.00	1239.17	3309.67	1707.66	1239.17	0.00
0.02	0.04	1892.28	3203.94	1680.57	1851.33	81.35
0.02	0.08	2974.63	2965.77	1509.00	2413.85	426.18
0.04	0.00	1548.46	3196.43	1617.72	1548.46	0.00
0.04	0.04	2422.22	2901.66	1431.77	2219.13	197.39
0.05	0.00	1735.96	3066.39	1499.62	1735.96	0.00
0.05	0.04	2746.95	2633.59	1079.60	2321.13	253.06
0.06	0.00	1949.64	2826.76	1244.90	1949.64	0.00
0.06	0.04	3119.56	2353.81	466.70	2291.96	201.10
RF		2193.26	2193.26	0.00	2193.26	0.00

Table 4.21: The Number of the 10,000 Simulations where the Payouts and Guarantees Display Certain Features using the GBM Model for High Risk Shares

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	532	512	476	476	2877	10000
0.00	0.04	8376	1961	1219	1212	1319	3375	10000
0.00	0.08	5359	4174	2107	2091	2897	4173	4641
0.02	0.00	10000	1044	920	896	896	3107	10000
0.02	0.04	6635	3514	1865	1858	2130	4228	10000
0.02	0.08	3012	6063	2591	2576	4183	3558	3863
0.04	0.00	10000	2083	1426	1415	1415	3603	10000
0.04	0.04	3749	5974	2511	2504	3160	4602	4927
0.05	0.00	10000	2983	1826	1821	1821	4143	10000
0.05	0.04	1943	7522	2708	2703	3769	3891	4101
0.06	0.00	10000	4551	2241	2241	2241	5178	10000
0.06	0.04	377	8932	2754	2753	4412	3305	3427

Table 4.22: The Number of the 10,000 Simulations where the Payouts and Guarantees Display Certain Features using the GBM Model for Low Risk Shares

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	89	113	86	86	2644	10000
0.00	0.04	9251	968	638	632	709	2810	10000
0.00	0.08	5799	3526	1821	1802	2914	3667	4201
0.02	0.00	10000	324	319	287	287	2727	10000
0.02	0.04	7503	2537	1373	1367	1675	3593	10000
0.02	0.08	2679	5868	2507	2480	5043	3247	3570
0.04	0.00	10000	1032	861	843	843	3054	10000
0.04	0.04	3876	5588	2288	2279	3341	4270	4597
0.05	0.00	10000	1848	1261	1249	1249	3482	10000
0.05	0.04	1700	7418	2591	2579	4386	3718	3988
0.06	0.00	10000	3477	1850	1846	1846	4487	10000
0.06	0.04	197	8822	2640	2636	5465	3386	3562

We see that increasing the volatility of the shares leads to an increase in the standard deviation of the maturity payout. However, the increased standard deviation is compensated for by a higher mean payout, as the expected return on shares is also increased.

The effect on the achieved guarantee depends on the size of the guaranteed growth rate and the desired bonus rate. For low bonuses we find that increasing the volatility of the shares increases the probability of very low returns and so there are more occasions when the desired bonuses are unaffordable. This leads to a lower mean achieved guarantee, but a higher standard deviation. For example, when the guaranteed growth rate is 2% and the desired bonus rate is 4% we find that the mean achieved guarantee falls from £1,851.33 to £1,830.23 as we increase the volatility of the shares from 15% to 25%. In the same example, the standard deviation of the achieved guarantee rises from 81.35 to 100.80.

However, for high bonuses the desired bonuses can only be afforded when investment returns are very high. Hence, we find that increasing the volatility of the shares increases the probability of very high returns and so there are fewer occasions when the desired bonuses are unaffordable. This leads to a higher mean achieved guarantee, although the standard deviation is also still higher. For example, when the guaranteed growth rate is 5% and the desired bonus rate is 4% we find that the mean achieved guarantee rises from £2,321.13 to £2,326.28 as we increase the volatility of the shares from 15% to 25%. In the same example, the standard deviation of the achieved guarantee rises from 253.06 to 265.40.

Secondly, we consider Tables 4.21, 4.18, and 4.22, corresponding to the high, medium, and low volatility share portfolios respectively.

As we saw above, the effect on the guarantees depends on the size of the bonuses. For lower bonuses an increase in share volatility leads to more very low returns and so the maximum guarantee is achieved less often. However, high bonuses can only be achieved when investment returns are high, and so the maximum guarantee is achieved more often in these cases when the share volatility is high.

Increased share volatility leads to the option being exercised more often and also an increase in the number of simulations in which the UWP policy outperforms the unit-linked policy. However, increased share volatility also leads to a greater number of simulations in which the risk-free policy outperforms the UWP policy.

4.5 Reduction in Yield

In Sections 4.3 and 4.4 we simulated the payouts on UWP and unit-linked policies. If the performance of the shares over the term of the policy were high enough, the UWP policy would receive a terminal bonus so that the payout exceeded the guarantee. In these cases the options held to match the guarantee would expire worthless. The cost of buying the options acts as a charge for the guarantees.

One way to measure the cost of the guarantees is to calculate the yield obtained by the policyholder on their UWP policy, compared to that which could have been obtained on a corresponding unit-linked policy. The yield obtained on the UWP policy i_{UWP} is given as follows:

$$\text{UWP Payout} = \sum_{t=0}^{T-1} P_t(1 + i_{UWP})^{T-t}.$$

Similarly, the yield on the unit-linked policy i_{UL} is given as:

$$\text{Unit-Linked Payout} = \sum_{t=0}^{T-1} P_t(1 + i_{UL})^{T-t}.$$

Hence we can calculate the reduction in yield caused by buying the options to match the guarantees as:

$$RIY = i_{UL} - i_{UWP}.$$

When shares have performed badly, it is possible that the payout from the UWP policy exceeds the payout on the unit-linked policy, and in these cases the reduction in yield is negative.

The reduction in yield has been calculated for each simulation. Table 4.23 shows the mean and standard deviation of the reduction in yield for each combination of guaranteed growth rate and desired bonus rate for the three different share portfolios.

Table 4.23: Mean and Standard Deviation of the Reduction in Yield

y	z	Low Risk ($\sigma = 15\%$)		Medium Risk ($\sigma = 20\%$)		High Risk ($\sigma = 25\%$)	
		Mean (%)	SD (%)	Mean (%)	SD (%)	Mean (%)	SD (%)
0.00	0.00	0.02	0.17	0.07	0.47	0.14	0.90
0.00	0.04	0.08	0.61	0.19	1.12	0.32	1.73
0.00	0.08	0.32	1.33	0.51	1.99	0.69	2.71
0.02	0.00	0.08	0.37	0.20	0.81	0.32	1.35
0.02	0.04	0.30	1.16	0.53	1.81	0.73	2.53
0.02	0.08	0.74	2.01	1.04	2.80	1.28	3.64
0.04	0.00	0.28	0.78	0.52	1.39	0.74	2.09
0.04	0.04	0.88	2.05	1.26	2.89	1.58	3.77
0.05	0.00	0.53	1.15	0.85	1.86	1.14	2.66
0.05	0.04	1.40	2.68	1.89	3.66	2.30	4.67
0.06	0.00	1.00	1.76	1.45	2.63	1.84	3.56
0.06	0.04	2.00	3.35	2.65	4.52	3.18	5.69

We see that the reduction in yield can be very small. For example, for the UWP policy with guaranteed growth rate of zero and no desired bonuses written on the low risk portfolio of shares, we see that the reduction in yield is only 0.02%.

However, an increase in either the guaranteed growth rate, or the desired bonus rate, will increase the reduction in yield, representing the greater cost of the larger guarantees. Similarly, increasing the volatility of the share portfolio leads the option pricers to increase the cost of the options, and so again the reduction in yield increases. For the UWP policy with guaranteed growth rate of 6% and desired bonuses of 4% written on the high risk portfolio of shares, we see that the reduction in yield is 3.18%, which represents a substantial reduction in the return which the policyholder will receive.

Similarly, we see that the standard deviation of the reduction in yield increases as we increase the guaranteed growth rate, desired bonus rate, or the share volatility. The standard deviations are large compared to the mean reduction in yields because the actual performance of the shares can have a large impact on the cost of the options.

Tillinghast (1997) stated that the most common way to charge for guarantees in the U.K. was to deduct a proportion of the asset share each year. This 'asset share charging approach' was one of the methods considered in Hare et al. (2000). For any given simulation the option pricing approach will lead to the same maturity payout as the asset share charging approach where the annual charge is equal to the reduction in yield.

Hare et al. (2000) considered an asset share charge of 0.15% p.a. and 0.25% p.a.. In both cases they found that the charges were too low compared to the put-spread strategy for a policy with no guaranteed growth rate and no reversionary bonus.

Standard Life (2004) recently announced that they would deduct a proportion of the asset share each year as a charge for guarantees. For example, UWP pensions policies with no guaranteed growth rate are currently liable for a charge of 0.50% p.a., while UWP pensions policies with guaranteed growth rate of 4% are currently liable for a charge of 0.75% p.a..

The results in this thesis are not directly comparable to the results discussed above for a number of reasons including differences in investment and bonus policies. However, the charges mentioned above are similar to the means of the reduction in yield calculated in this section. However, the large standard deviation in the reduction in yield indicates that insurers will need to review the charges they take on a regular basis.

Chapter 5

A More Realistic Investment Model — The Wilkie Model

5.1 Introduction

In Chapter 4 we considered the performance of the UWP policy under the GBM model of the 'real world'. In practice, insurers will model their business using a richer investment model which allows them to model the interaction of the equity returns with other asset classes.

In the option pricing approach suggested by Wilkie (1987) the office invests in a mixture of equities and put options. New bonuses are declared at the end of each year and the portfolio is rebalanced. There is no trading of assets during the year and dividends are reinvested. Hence we need an investment model that can produce equity values at yearly intervals. The Wilkie model (Wilkie (1986) and Wilkie (1995)) is the best known long term model of equity returns in the actuarial literature. We will use the Wilkie model as presented in Wilkie (1995).

The option pricing approach uses the Black-Scholes formula to value the options. Therefore in addition to the real world investment model we need to know the option pricer's model of the risk-free interest rate r_f and the return on the equity index. We assume that the option pricers use the same model as in Chapter 4 *i.e.* they assume that the risk-free rate is constant and that the equity index follows geometric Brownian motion with constant volatility σ.

In Chapter 4 we assumed that the real world model and the option pricing model were the same *i.e.* the equity index also follows geometric Brownian motion and the risk-free rate is constant. Hence we have implicitly assumed that the pricers of options have no model error *i.e.* the option pricers know the real world model and its parameters.

However in this Chapter we use two different investment models. In effect at any point in time during the contract we will use the real world model for all past calculations, and Black-Scholes for future calculations. This approach is justified as the two different models represent two different things — the real world model simulates the economy whose actual working is unknown, and the Black-Scholes model is the way the market believes the economy works for pricing purposes. If we are looking at the historic performance of the insurer, then we can use the actual investment data for that period as we did in Chapter 3. However, in this chapter we want to project what the actual performance might be and so use the Wilkie model to simulate the real world.

Therefore, by comparing the results of Chapter 4, where the real world and option pricing models were the same, with the results in this chapter, where the real world and option pricing models are different, we can see the effect of model error in the option pricing formula.

Brennan and Schwartz (1979) p.85 and Collins (1980) p.68 also consider the effects of model error — they both assume that equities follow geometric Brownian motion in both the real world and option pricing models, but that these models have different parameters. Hardy (1999) considers the error caused when hedging is performed under the assumption of geometric Brownian motion, but the real world model is RSLN. Hardy (2002) then goes on to look at the error caused from differences in the model and parameter uncertainty. Wilkie (1987) and Hare et al. (2000) use geometric Brownian motion in the option pricing model, but use the Wilkie model as their real world model without comparing the results with a geometric Brownian motion real world model. Collins (1982), Boyle and Hardy (1996), Boyle and Hardy (1997) and Hardy (2000) assume that the real world and option pricing models are the same. The remaining papers discussed in Chapter 2 use only one model, either because they do not consider options and so have no need for an option pricing model, or because they only price the option and do not consider the performance of the policy under the real world model.

We begin with a description of the Wilkie investment model in Section 5.2. We then illustrate the distribution of the investment returns generated by the model in Section 5.3. The Wilkie model and the GBM model are then compared to the assumptions underlying the Black-Scholes equation in Section 5.4. The Wilkie investment model is then used to simulate the behaviour of the option pricing methodology in Section 5.5.

5.2 The Wilkie Model

The Wilkie model was first presented in Wilkie (1986). We will use the updated form of the Wilkie model as given in Wilkie (1995).

The Wilkie model as described in Wilkie (1995) simulates the following at yearly intervals: inflation, wages, share dividend yields and amounts, long and short interest rates, returns on property, and currency exchange rates. However, in this thesis we will only use the part of the Wilkie model which models inflation, shares, and interest rates.

Inflation is needed due to the cascade nature of the model, as all other variables are derived from it. Inflation $I[t]$ and the corresponding price index $Q[t]$ are modelled as follows

$$\begin{aligned} QE[t] &= QSD \cdot QZ[t] \\ I[t] &= QMU + QA \cdot (I[t-1] - QMU) + QE[t] \\ Q[t] &= Q[t-1] \cdot \exp(I[t]) \end{aligned}$$

where $QZ[t]$ is a random variable with unit normal distribution and QSD, QMU and QA are parameters. We will use the form of the Wilkie model with a constant value for the parameter QSD. Therefore the standard deviation of the inflation innovation $QE[t]$ will be equal to the constant value of QSD.

Share dividend amounts and yields are needed to calculate the return on the equity index which is the asset underlying the with-profits fund. The dividend yield $Y[t]$ on the shares that make up this index is modelled as follows

$$
\begin{aligned}
YE[t] &= YSD \cdot YZ[t] \\
YN[t] &= YA \cdot YN[t-1] + YE[t] \\
Y[t] &= \exp(YW \cdot I[t] + \ln(YMU) + YN[t])
\end{aligned}
$$

where $YZ[t]$ is a random variable with unit normal distribution and YSD, YA, YW and YMU are parameters.

An index of the dividends $D[t]$ is modelled as follows

$$
\begin{aligned}
DE[t] &= DSD \cdot DZ[t] \\
DM[t] &= DD \cdot I[t] + (1 - DD) \cdot DM[t-1] \\
DI[t] &= DW \cdot DM[t] + DX \cdot I[t] \\
D[t] &= D[t-1] \cdot \exp(DI[t] + DMU + DY \cdot YE[t-1] + DB \cdot DE[t-1] + DE[t])
\end{aligned}
$$

where $DZ[t]$ is a random variable with unit normal distribution and DSD, DD, DW, DX, DMU, DY and DB are parameters.

Hence we can derive an index of prices $P[t]$ on the dividend paying shares as

$$
P[t] = D[t]/Y[t].
$$

Long and short interest rates are not used in this chapter but we will need them in all future chapters. We give the Wilkie model for interest rates at this stage for completeness. We model the long interest rate as the yield on a consol $C[t]$ (*i.e.* a perpetual coupon paying government stock) as follows

$$
\begin{aligned}
CE[t] &= CSD \cdot CZ[t] \\
CN[t] &= CA1 \cdot CN[t-1] + CA2 \cdot CN[t-2] + CA3 \cdot CN[t-3] + CY \cdot YE[t] + CE[t] \\
CM[t] &= CD \cdot I[t] + (1 - CD) \cdot CM[t-1] \\
C[t] &= CW \cdot CM[t] + CMU \cdot \exp(CN[t])
\end{aligned}
$$

where $CZ[t]$ is a random variable with unit normal distribution and $CSD, CA1, CA2, CA3, CY, CD, CW$ and CMU are parameters. In order to avoid the possibility of negative yields we set a minimum value of 0.5% for $C[t]$.

We model the short term interest rate as the base rate $B[t]$ as follows

$$
\begin{aligned}
BE[t] &= BSD \cdot BZ[t] \\
BD[t] &= BMU + BA \cdot (BD[t-1] - BMU) + BE[t] \\
B[t] &= C[t] \cdot \exp(-BD[t]).
\end{aligned}
$$

where $BZ[t]$ is a random variable with unit normal distribution and BSD, BMU and BA are parameters.

We now need to choose the parameters for the Wilkie model. In Sections 3.2.2 and 3.3.2 we considered the actual return on equities, during the period 1965 to 1985, where dividends had been taxed at 35%. We now want to model future investment returns allowing for the taxation regime which will apply in the future.

We will consider two different parameterisations of the Wilkie model. Firstly we will use the same parameters as Wilkie (1995), which were calculated from UK data from 1923 to 1994, and assume that all investment returns are untaxed.

However, in 1997 pension funds and life offices in the U.K. became unable to reclaim the tax credit on dividends. This resulted in a 20% reduction in the value of dividends. A discussion of the effects on insurance companies and pensions schemes of the treatment of U.K. dividends can be found in Masters et al. (1997).

So, secondly we will update the Wilkie model parameters to allow for taxed dividends. Our treatment of taxation is appropriate for UWP policies sold as pensions savings vehicles. Other UWP policies will be subject to further taxation of investment returns. We can achieve a 20% reduction in the dividend yield by multiplying the parameter YMU by 0.8. The parameterisations with both untaxed and taxed dividends are given in Table 5.24.

Table 5.24: Parameters used in the Wilkie Model

Parameter	Untaxed Dividends	Taxed Dividends
QA	0.58	0.58
QMU	0.047	0.047
QSD	0.0425	0.0425
YA	0.55	0.55
YMU	0.0375	0.03
YSD	0.155	0.155
YW	1.8	1.8
DB	0.57	0.57
DD	0.13	0.13
DMU	0.016	0.016
DSD	0.07	0.07
DW	0.58	0.58
DX	0.42	0.42
DY	-0.175	-0.175
$CA1$	0.9	0.9
$CA2$	0.0	0.0
$CA3$	0.0	0.0
CD	0.045	0.045
CMU	0.0305	0.0305
CSD	0.185	0.185
CW	1.0	1.0
CY	0.34	0.34
BA	0.74	0.74
BMU	0.23	0.23
BSD	0.18	0.18

We now want to choose the initial conditions for the Wilkie model, *i.e.* the values taken by the series $I[0], Y[0]$ *etc.* at the start of each simulation. We use what Wilkie

(1995) describes as 'neutral' initial conditions "in which the starting values are set at what their long-run means would be if all the standard deviations were zero".

Again we have two different sets of initial conditions corresponding to the untaxed and taxed parameterisations. The neutral initial conditions corresponding to the taxed parameters are the same as the untaxed conditions except for $Y[0]$ and $P[0]$. The initial conditions with both untaxed and taxed dividends are given in Table 5.25.

Table 5.25: Initial Conditions used in the Wilkie Model

Initial Condition	Untaxed Dividends	Taxed Dividends
$Q[0]$	100.0	100.0
$I[0]$	0.047	0.047
$Y[0]$	0.040811	0.032648
$YN[0]$	0.0	0.0
$D[0]$	100.0	100.0
$P[0]$	2450.346	3062.933
$DM[0]$	0.047	0.047
$C[0]$	0.0775	0.0775
$CM[0]$	0.047	0.047
$CN[0]$	0.0	0.0
$CN[-1]$	0.0	0.0
$CN[-2]$	0.0	0.0
$B[0]$	0.061576	0.061576
$BD[0]$	0.23	0.23

5.3 Wilkie Model Results

From the output of the Wilkie model we can calculate a total return index (with income reinvested) for prices Q, equities PR, consols CR and cash BR (see Wilkie (1995) Section 11.1.5). The notation used here is the same as used in Wilkie (1995). $PR[t]$ is the equity index simulated by the Wilkie model, which takes on the same role as the equity index simulated by the GBM model, S_t, in Chapter 4.

$$
\begin{aligned}
Q[t] &= Q[t-1]\exp(I[t]) \\
PR[t] &= PR[t-1]\frac{P[t]+D[t]}{P[t-1]} \\
CR[t] &= CR[t-1]\left(\frac{C[t-1]}{C[t]}+C[t-1]\right) \\
BR[t] &= BR[t-1](1+B[t-1]). \qquad (5.14)
\end{aligned}
$$

We can then estimate the means M, standard deviations SD, and correlation coefficients C, for the annualised returns, $GX[t]$, over any period of term t, for each of the total return indices, $X[t]$, above.

$$
GX[t] = 100\left[\left(\frac{X[t]}{X[0]}\right)^{1/t} - 1\right]
$$

The results obtained using 10,000 simulations of the untaxed parameterisation are given in Table 5.26.

Table 5.26: Summary Statistics from the Wilkie Investment Model with Untaxed Dividends

	Term					
	1	2	5	10	20	50
$M(GQ)$	4.91	4.93	4.90	4.85	4.85	4.84
$SD(GQ)$	4.45	4.14	3.60	2.96	2.25	1.48
$M(GPR)$	12.42	11.53	11.15	11.05	11.00	10.96
$SD(GPR)$	19.62	12.72	7.05	4.94	3.60	2.38
$M(GCR)$	7.63	7.61	7.73	7.85	7.95	8.01
$SD(GCR)$	7.96	5.34	2.94	1.62	1.02	1.05
$M(GBR)$	6.16	6.23	6.36	6.47	6.53	6.59
$SD(GBR)$	0.00	0.63	1.11	1.30	1.31	1.17
$C(GPR, GQ)$	-0.31	-0.13	0.17	0.39	0.54	0.64
$C(GCR, GQ)$	-0.32	-0.41	-0.54	-0.55	-0.16	0.48
$C(GCR, GPR)$	0.33	0.25	0.06	-0.06	0.04	0.36
$C(GBR, GQ)$	0.00	0.11	0.20	0.31	0.42	0.60
$C(GBR, GPR)$	0.00	-0.05	0.01	0.15	0.26	0.39
$C(GBR, GCR)$	0.00	-0.27	-0.33	-0.25	0.25	0.75

The results are sufficiently close to those shown in Wilkie (1995) Table 11.1 to give us confidence in the computer program used. Inevitably some differences will exist as we have used a different set of simulations. Now that we have established that the model is in good agreement with the results of Wilkie (1995) we will use the taxed parameterisation from now on.

The Wilkie model with taxed parameters only differs in the return on shares. We see from the Table 5.27 that a 20% tax on dividends reduces the mean return on equities by a little under 1% p.a.. There is little effect on the standard deviation of the return on equities, or its correlation with the return on other asset classes.

In addition Table 5.27 also shows the skewness (SK) and excess kurtosis (KU) for the equity returns. We see that both skewness and excess kurtosis are significantly positive if we look at returns over 1 year, but as we increase the term both skewness and excess kurtosis tend to zero.

Figure 5.12 shows three sample paths for the equity index simulated by the Wilkie model.

Table 5.27: Summary Statistics from the Wilkie Investment Model with Taxed Dividends

	Term					
	1	2	5	10	20	50
$M(GPR)$	11.55	10.65	10.26	10.17	10.12	10.07
$SD(GPR)$	19.60	12.71	7.03	4.90	3.56	2.35
$SK(GPR)$	0.52	0.33	0.20	0.12	0.09	0.05
$KU(GPR)$	0.54	0.25	0.11	0.07	-0.02	-0.05
$C(GPR, GQ)$	-0.31	-0.14	0.16	0.38	0.53	0.64
$C(GCR, GPR)$	0.33	0.25	0.06	-0.05	0.04	0.36
$C(GBR, GPR)$	0.00	-0.05	0.01	0.15	0.26	0.39

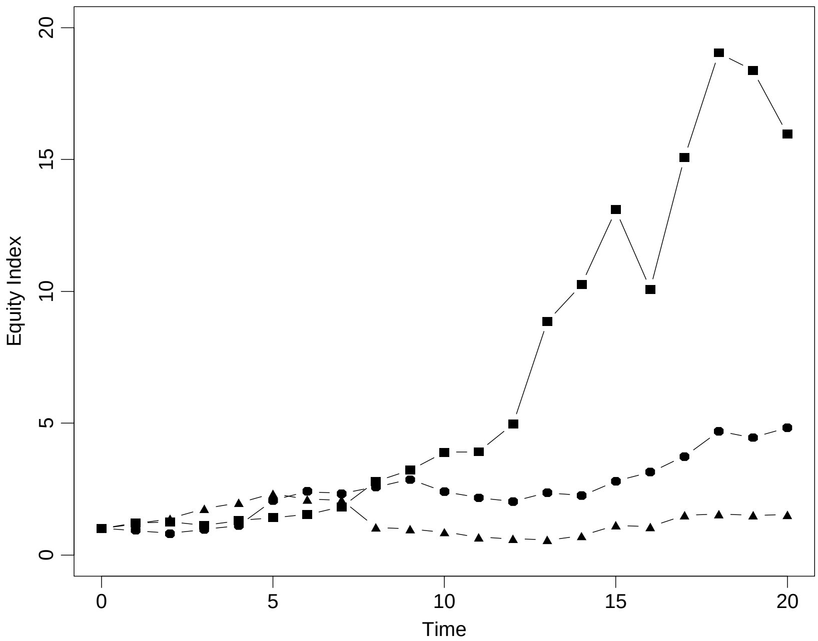

Figure 5.12: Sample Paths for the Equity Index, $PR[t]$, Simulated by the Wilkie Model

We can see in Figure 5.13 the probability that the annual rate of return on the equity index over a 20-year period is less than some target value i, $P(GPR[20] < i)$. For example, under the Wilkie model, there is a 0.1% probability of a negative return, and a 19% probability of a return less than the risk-free rate of 7% p.a.. The GBM model is much more likely to produce low returns with a probability of 1.8% and 27% of returns lower than zero and the risk-free rate respectively. Looking at this another way, we can be 99% confident of a return in excess of 2% p.a. under the Wilkie model, but are only 99% confident of a return in excess of -1% p.a. under the GBM model. However, high returns are much more likely under the GBM model. The probability of a return in excess of 15% p.a. is only 9% under the Wilkie model, but is 15% under the GBM model. The median return is around 10% p.a. under both models.

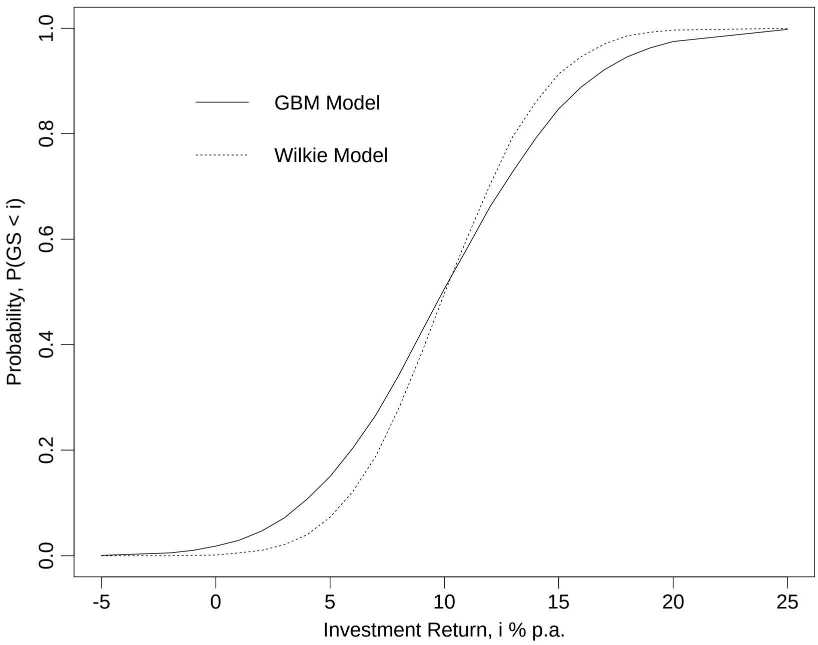

Figure 5.13: A Comparison of the Cumulative Distribution Functions for Equity Returns over a 20 Year Period under Geometric Brownian Motion ($P(GS_{20} < i)$) and the Wilkie Model ($P(GPR[20] < i)$)

5.4 Comparison of the Wilkie and Geometric Brownian Motion Investment Models with the Black-Scholes Option Pricing Model

In this section we will compare the Wilkie model described in Section 5.2, and the GBM model introduced in Chapter 4, with the model underlying the Black-Scholes option pricing formula. We start in Section 5.4.1 by comparing the equity returns generated by the Wilkie and GBM models with the Black-Scholes assumptions. Then, in Section 5.4.2, we compare how the two investment models simulate interest rates with the Black-Scholes assumption of risk-free returns.

5.4.1 Comparison of Equity Returns

We would like the equity returns under the Wilkie model to be similar to those under the GBM model, so that we can compare the two sets of results. What do we mean by similar? We assume that the market prices options using the Black-Scholes model. So, we want the investment model to be as consistent as possible with pricing options using Black-Scholes. Black-Scholes assumes that the log returns are normally distributed. Therefore we set our investment model parameters so that the log returns are consistent.

Further, we use an equity index with re-invested dividends as the underlying asset for the options. Hence it is the log returns of this index that we require.

Of course the Wilkie model does not generate lognormal equity returns. However the pricers of options cannot know the exact dynamics of the real market. Therefore, given our choice of the Wilkie model as the real world model, we have assumed that the pricers of options have error in their choice of model.

We will first consider the volatility of the log returns of the equity index generated by the Wilkie model and compare it with the GBM model. We will then do the same with the expected log return.

The Wilkie model generates annual returns on the equity index. We can calculate $LPR[t]$, the log return over one year for successive time steps as follows

$$LPR[t] = 100 \cdot \ln\left(\frac{PR[t]}{PR[t-1]}\right) = 100 \cdot \ln\left(\frac{P[t]+D[t]}{P[t-1]}\right) \tag{5.15}$$

The option pricers can find an estimate of the volatility of the equity returns by observing a large number of price movements (see for example Hull (1997) p. 232). In practice the option pricers will want to consider a long enough time period to provide enough data to get a reliable estimate. However, the underlying volatility of the market does change through time and so the option pricers will not want to use too long a time period.

We can therefore estimate the volatility of the logged price process by considering the annual returns on the equity index generated by the Wilkie model over a single simulation of many years. There is no reason to use a shorter time period as the same Wilkie model is used throughout. So we consider a long time period (10,000 years) in order to gain as accurate an estimate as possible. Hence we calculate the mean and standard deviation of LPR as follows

$$M(LPR) = \sum_{\tau=1}^{10{,}000} \frac{LPR[\tau]}{10,000} \tag{5.16}$$

$$SD(LPR) = \sum_{\tau=1}^{10{,}000} \frac{(LPR[\tau] - M(LPR))^2}{9,999}. \tag{5.17}$$

Using the Wilkie model with taxed dividends we estimate the volatility of the equity returns as 19.8367% p.a.. Hence we will get similar results from the GBM model where the volatility was 20% p.a.. Therefore under either the Wilkie or GBM model of the real world the option pricer would observe the historic log returns and estimate the volatility as 20% to use in the Black-Scholes pricing formula.

In the same way we estimate the mean log return under the Wilkie model as 9.44947% p.a.. From Equation 4.13 the mean log return under the GBM model is given by

$$100\,E\left[\ln\left(\frac{S_t}{S_{t-1}}\right)\right] = 100\left[\mu - \frac{\sigma^2}{2}\right] = 100\left[0.1144947 - \frac{0.2^2}{2}\right] = 9.44947\%.$$

Hence we have chosen our parameters so that we obtain the same mean log return under both the Wilkie model and the GBM model. The Black-Scholes equation is independent of the expected return on shares, but does depend on the risk-free return. We will compare the interest rates generated by the investment models with the risk-free rate assumed by the option pricers in Section 5.4.2.

We have chosen our investment model parameters so that the Wilkie and GBM models have the same mean and standard deviation for the log equity return over a one year period. We will now see to what extent the two models have similar distributions over longer time periods.

We can calculate the annualised log return under the Wilkie model, LGPR[t], over any term t, where

$$LGPR[t] = 100 \cdot \ln\left(\frac{GPR[t]}{100} + 1\right).$$

Table 5.28 shows the mean and standard deviation of LGPR[t] estimated from the simulations of the Wilkie model. Another useful measure, also shown in Table 5.28, is the standard deviation multiplied by the square root of the term, $SD(LGPR[t]) \cdot \sqrt{t}$.

Table 5.28: The Mean and Standard Deviation of the Annualised Log Return from the Wilkie Investment Model

	Term					
	1	2	5	10	20	50
$M(LGPR[t])$	9.41	9.46	9.57	9.58	9.58	9.57
$SD(LGPR[t])$	17.46	11.47	6.36	4.45	3.23	2.14
$SD(LGPR[t])\ \sqrt{t}$	17.46	16.22	14.23	14.07	14.46	15.10

We can calculate the annualised log returns under the GBM model directly using Equation 4.14 as follows

$$LGS_t = 100 \cdot \ln\left(\left(\frac{S_t}{S_0}\right)^{1/t}\right) \sim N\left[100\left(\mu - \frac{\sigma^2}{2}\right), \frac{100^2\,\sigma^2}{t}\right]$$

Hence under the GBM model we see that $SD(LGS_t) \cdot \sqrt{t}$ is constant and equal to the volatility of 20% used in the Black-Scholes equation.

However, for the Wilkie model, the pattern of $SD(LGPR[t]) \cdot \sqrt{t}$ is very strange. Initially it falls in value up to term 10, but then rises a little. We would expect the Wilkie model to become less volatile compared to the GBM model as we increase the term because of the mean reverting nature of the Wilkie model. The rate of inflation $I[t]$ will tend to revert to its average value of QMU over time. Hence the dividend yield $Y[t]$, which is derived from the rate of inflation, will vary around its neutral initial condition. Therefore periods of low dividend yields, and hence a high level of the equity index, are likely to be followed by higher yields, and hence a fall in the equity index.

Note that even for terms of one year we do not get the volatility of 20% which we obtained looking at the annual returns over 10,000 years of a single simulation. When we considered a one year time period above, we started each simulation from the same neutral initial starting conditions. However, when we considered 10,000 consecutive one year periods, each year would start with a different set of conditions. Volatility is added by changing the conditions as we move through time. The model will take a number of time steps to burn-in *i.e.* to obtain a good spread of conditions at the start of the year. To demonstrate this we calculate $LPR[t]$, the log returns over one year, for successive time steps as in Equation 5.15. However, when we calculated the mean and standard deviation of LPR as 9.44947% p.a. and 19.8367% p.a. respectively, we used only a single simulation of 10,000 years (see Equations 5.16 and 5.17). But now in Table 5.29 we calculate the mean and standard deviation of $LPR[t]$ for a single time t over 10,000 simulations *i.e.*

$$
\begin{aligned}
M(LPR[t]) &= \sum_{\text{sims}=1}^{10{,}000} \frac{LPR[t]}{10,000} \\
SD(LPR[t]) &= \sum_{\text{sims}=1}^{10{,}000} \frac{(LPR[t] - M(LPR[t]))^2}{9,999}.
\end{aligned}
$$

We see in Table 5.29 that by the fourth time step the model appears to have reached its long term volatility of 20%.

Table 5.29: The Mean and Standard Deviation of the Log Return over a Single Year from the Wilkie Investment Model

	t						
	1	2	3	4	5	10	20
$M(LPR[t])$	9.41	9.52	9.44	9.61	9.87	9.63	9.74
$SD(LPR[t])$	17.46	18.93	19.42	19.99	19.94	20.12	20.18

5.4.2 Comparison of Interest Rates

We would also like the Wilkie model and GBM model to generate similar interest rates to the risk-free rate assumed in the Black-Scholes equation.

The Wilkie model simulates both a short term interest rate $B[t]$ and a long term interest rate $C[t]$. These two interest rates are stochastic. They also give us some idea of how the yield curve changes through time, although we only have two points on the yield curve. We see in Table 5.26 that the mean return on cash is around 6.4% p.a. and on consols is around 7.8% p.a. depending on the duration of the projection (these figures are unaffected by the tax on dividends).

The GBM model considered in Chapter 4 assumes that the interest rate is a constant 7% p.a.. So the GBM model is much simpler than the Wilkie model. The GBM model is deterministic and has no term structure for interest rates.

The assumption underlying the Black-Scholes equation is also that the risk-free rate is a constant 7% p.a. regardless of the term of the option. Therefore the GBM 'real world' model is consistent with the option pricer's model of interest rates.

In order to compare the Wilkie and GBM models we will, in Section 5.5, ignore the stochastic interest rates produced by the Wilkie model. Instead we will assume that the Wilkie model produces a constant interest rate of 7% p.a.. This will allow us to concentrate on the differences produced by the two models for equity returns. However, in Chapter 6 we will consider the effects of using the full stochastic Wilkie model for interest rates.

5.5 Results for the Unitised With-Profits Charging Mechanism

We now use 10,000 simulations of the Wilkie investment model with taxed dividends to calculate the return on the equity index. However, for comparison with the GBM model, we will use a constant risk-free rate of 7% p.a. in the real world model at this stage. Using this real world model for equities and the risk-free asset we will calculate the payout on 20-year regular premium unitised with-profits (UWP) policies, the unit-linked (UL) policy, and the risk-free (RF) account. We assume that UWP guarantees are priced using Black-Scholes with a constant risk-free rate of 7% p.a. and a constant volatility of equity returns of 20% p.a..

Table 5.30 shows the mean and standard deviation of the payout and achieved guarantee for the UWP, UL, and RF policies. Table 5.31 shows the number of the 10,000 simulations which display certain features.

Table 5.30: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities and a Constant Risk-Free Rate

y	z	Maximum Guarantee	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		0.00	3761.91	2096.38	0.00	0.00
0.00	0.00	1000.00	3709.79	2071.64	1000.00	0.00
0.00	0.04	1488.90	3627.17	2064.63	1483.38	22.99
0.00	0.08	2288.10	3397.22	1999.11	2133.08	216.55
0.02	0.00	1239.17	3612.36	2014.62	1239.17	0.00
0.02	0.04	1892.28	3390.70	1949.89	1858.27	70.84
0.02	0.08	2974.63	2997.69	1628.65	2425.37	400.65
0.04	0.00	1548.46	3372.59	1846.51	1548.46	0.00
0.04	0.04	2422.22	2888.23	1505.87	2224.41	182.83
0.05	0.00	1735.96	3147.31	1657.88	1735.96	0.0
0.05	0.04	2746.95	2553.85	984.63	2311.10	228.04
0.06	0.00	1949.64	2794.04	1288.75	1949.64	0.00
0.06	0.04	3119.56	2306.40	319.55	2270.34	147.21
RF		2193.26	2193.26	0.00	2193.26	0.00

We can compare the results in Tables 5.30 and 5.31 which use the Wilkie model for

Table 5.31: The Number of the 10,000 Simulations where the Payouts and Guarantees Display Certain Features using the Wilkie Model for Equities and a Constant Risk-Free Rate

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	81	81	72	72	2227	10000
0.00	0.04	9341	880	499	492	551	2581	10000
0.00	0.08	5857	3649	1494	1480	2405	3580	4143
0.02	0.00	10000	313	244	235	235	2433	10000
0.02	0.04	7645	2597	1109	1102	1343	3544	10000
0.02	0.08	2477	6357	2077	2064	4268	2975	3259
0.04	0.00	10000	1034	644	641	641	2951	10000
0.04	0.04	3552	6082	1845	1840	2780	4287	4606
0.05	0.00	10000	1954	980	976	976	3563	10000
0.05	0.04	1252	8028	2090	2087	3669	3622	3842
0.06	0.00	10000	3923	1483	1481	1481	4869	10000
0.06	0.04	59	9241	2135	2133	4656	3078	3211

equity returns with the results in Tables 4.17 and 4.18 which use the same option pricing approach (with $r_f = 0.0677$ and $\sigma = 0.20$), and use the same real world constant risk-free return, but use the GBM investment model for equity returns.

The basic form of the results is very similar. Higher bonuses, either guaranteed or desired, reduce the variability of the final payout, but the cost of the guarantees also reduces the expected payout.

The most striking difference between the results of the two investment models is that the mean payouts, regardless of bonus strategy, are around 10% higher under the GBM model (*cf.* Table 4.17 with Table 5.30). The standard deviation of the payout is around 50% higher under the GBM model. This is surprising given that the mean and standard deviation of the lognormal returns on the equity index were set to be equal under the two investment models. This shows that the Wilkie model has behaviour very far from lognormal. In fact the share price $P[t]$ under the Wilkie model is lognormal given the information at time $t-1$, but the total return index $PR[t]$ is not.

Figure 5.13 shows us that the annualised return on the equity index over the 20 year term of the policy has higher standard deviation under the GBM model than the Wilkie model, but similar mean. The effects of compounding these annualised returns over the 20 years will result in both a higher mean and standard deviation of payout from the UL policy invested entirely in equities. Hence the equity investments within the UWP policies will also have higher expected return and greater variability leading to the higher mean and standard deviation of UWP payouts under the GBM model.

We can also see in Figure 5.13 that relatively low equity returns are more common under the GBM model. When asset values are low during a policy's term the guarantees are more expensive to buy. If the desired bonus rate is too high it may be unaffordable. Hence we often find that under the GBM model: the mean achieved guarantee is lower, the maximum guarantee is achieved less often, the options are more likely to be exercised, and the UWP payout is more likely to exceed the unit-linked payout (*cf.* Table 4.18 with Table 5.31). However, this pattern is not observed for all UWP policies. The higher mean value of the equity index after 20 years under the GBM model, described in the previous paragraph, operates in the opposite direction to the effects of the more common low returns. This is especially true for the higher bonus rates because the desired guarantees are only affordable if equity returns have been very high.

The risk-free rate is the same (7%) under both investment models, so the return if all premiums are invested at the risk-free rate (RF) is the same. However, again due to the higher probability of low equity returns under the GBM model, we see that the risk-free account is more likely to outperform the unitised with-profits policies under the GBM model than the Wilkie model. There is one exception where the risk-free payout is more likely to exceed the UWP payout under the Wilkie model. In this case the compulsory bonus of 6% and desired bonus of 0% means that the UWP policy needs high equity returns to outperform the RF policy, and these are more likely under the GBM model.

As in Section 4.3 it appears that a reduction in risk can only be obtained at the expense of a very high penalty on the expected payout. The mean reverting nature of the Wilkie model means that we tend to buy the options when they are most expensive. Also, the volatility over the 20-year period assumed by the Black-Scholes formula is higher than that actually generated by the Wilkie model. Therefore option prices are higher than if we had used an option pricing formulae which more closely fitted our real world model.

Chapter 6

Consistent Risk-Free Rate

6.1 Introduction

The Black-Scholes equation assumes that the risk-free rate of interest is constant. Over a short period of time this is a reasonable approximation. However, the risk-free asset for an option expiring in n years time is a zero coupon bond with remaining term of n years. Hence, in practice the risk-free rate changes through time, and also depends on the remaining term of the option.

In Chapters 4 and 5 we have used a constant risk-free force of interest ($r_f = .0677$) regardless of the time or term of the option. However, the Wilkie model can simulate changes in the risk-free return in a way which is consistent with changes in the share price.

In this chapter we will price the options using a risk-free rate derived from the Wilkie model as described in Section 6.2 below. The options will continue to be priced using the standard Black-Scholes formula which assumes that in the future the risk-free rate will be constant. However, whenever we require the value of an option we assume that the market will update its risk-free assumption to use the current risk-free rate at that time.

For example, the Wilkie model simulates a scenario where the risk-free rate is 5%, 6% and 7% at times 0, 1, and 2 respectively. We initially price the options assuming that the risk-free rate will remain a constant 5% until the option expires. A year later we decide to sell the options and buy new options with a different exercise price. The market now observes the higher risk-free rate of 6% and prices the options assuming that this rate will remain constant in the future. At time 2 the options are revalued using a risk-free rate of 7% and so on.

Therefore the option pricing formulae is assuming a constant risk-free rate of interest (and shares following geometric Brownian motion), while the 'real world' investment model has a stochastic risk-free rate following the Wilkie model (and shares following the Wilkie model).

6.2 Deriving the Risk-Free Rate from the Wilkie Model

The Wilkie model generates at each time t and for each simulation j a base rate $B[t,j]$ and a consols yield $C[t,j]$, but does not generate a full yield curve. Yang (2001) p. 30 and Wilkie et al. (2003) Appendix B show how a simple par bond yield curve can be fitted to the base rates and consol yields as follows:

$$parbond[n,t,j] = C[t,j] + (B[t,j] - C[t,j]) \cdot e^{-\beta n}$$

where $parbond[n,t,j]$ is the par bond yield for a bond of remaining term n, at time t, for simulation j. β is a parameter determining the shape of the yield curve.

We require the zero coupon yield for our risk-free rate. We know that the coupon on a bond priced at par (with redemption proceeds of 1 payable at time $n+t$) equals $parbond[n,t,j]$. Hence the price of the par bond is equal to the discounted value of the coupons (assumed paid annually in arrear) and redemption proceeds at the zero-coupon bond yield $zcby[m,t,j]$ of the appropriate remaining term m. Hence, omitting time t and simulation j from our notation for simplicity, we have:

$$1 = \frac{parbond[n]}{(1+zcby[1])} + \frac{parbond[n]}{(1+zcby[2])^2} + \cdots + \frac{parbond[n]}{(1+zcby[n])^n} + \frac{1}{(1+zcby[n])^n}.$$

Hence, we first calculate the par bond yield for each term n, and then use the above equation iteratively to derive the zero coupon bond yields. We then convert the zero coupon bond yield to a force of interest

$$r[n,t,j] = \ln(1 + zcby[n,t,j]).$$

$r[n,t,j]$ will then take the place of r_f in the Black-Scholes equation whenever we value options with remaining term n, at time t, within simulation j.

There are a number of things to note about the yield curve derived above. Firstly, with only two points ($B[t]$ and $C[t]$) to fit, we cannot model changes in the shape of the yield curve.

From Figure 6.14 we see that for higher values of β the yield curve becomes flatter. However, in practice the yield curve would not be so flat. This suggests that a low value for β may be most appropriate.

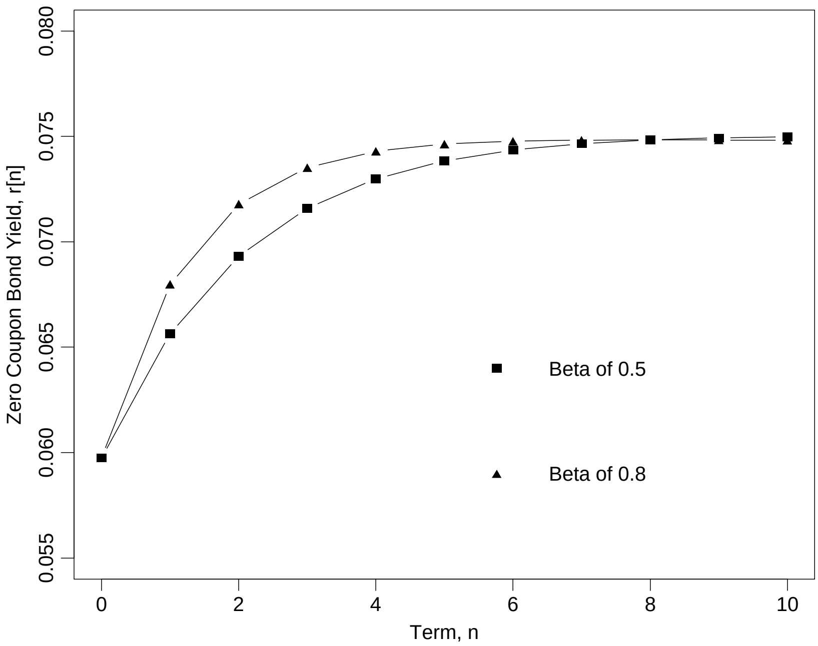

Figure 6.14: Risk-free Force of Interest $r[n]$

Wilkie et al. (2003) found that for low values of β (for their choice of parameters this was 0.38 and below) they would occasionally find simulations where

$$\frac{1}{(1 + zcby[n])^n} < 0$$

for the higher terms, and so the zero coupon bond yield was negative or complex. In this thesis we use $\beta = 0.5$ (Yang (2001) also uses 0.5, Wilkie et al. (2003) use 0.39).

The par bond curve derived above is either monotone increasing or decreasing. However the zero coupon yield curve we obtain is often slightly humped. For example, when the consol yield is higher than the base rate, the yield rises with term until it is a little above the consol yield and then falls back towards the consol yield. The hump is very much more pronounced if we consider forward rates.

We calculate the mean and standard deviation over 10,000 simulations for the force of interest at given times t and terms n as follows:

$$E_j[n,t] = \sum_{j=1}^{10000} \frac{r[n,t,j]}{10000}$$

$$SD_j[n,t] = \left(\sum_{j=1}^{10000} \frac{(r[n,t,j] - E_j[n,t])^2}{9999} \right)^{\frac{1}{2}}.$$

We see in Tables 6.32 and 6.33 the mean and standard deviation of the risk-free force of interest $r[n,t,j]$ produced using the above method with $\beta = 0.5$. The mean of the risk-free force of interest is also shown in Figure 6.15. (I have also investigated the shape of the yield curve and find no examples of negative forward rates using $\beta = 0.5$.)

Table 6.32: Mean of the Risk-free Force of Interest $E_j[n,t]$ with $\beta = 0.5$

Time		Term n							
t	Base Rate	1	2	3	4	5	10	20	Consols
0	5.98	6.56	6.93	7.16	7.30	7.38	7.50	7.49	7.46
5	6.30	6.83	7.15	7.36	7.48	7.56	7.66	7.65	7.63
10	6.39	6.90	7.22	7.42	7.54	7.62	7.72	7.71	7.68
20	6.40	6.92	7.24	7.44	7.56	7.64	7.74	7.73	7.70

From Table 6.32 we see that the mean force of interest $E_j[n,t]$ rises with term n. This is in keeping with Table 5.26 where we see that the mean base rate is lower than the mean return on consols. (Note, Tables 5.26 and 6.32 are not directly comparable because Table 6.32 looks at the *force* of interest earned over the *term* of a zero coupon bond held to redemption, while Table 5.26 looks at an *effective rate* of interest earned over a *single* year including changes in market value.)

Table 6.33: Standard Deviation of the Risk-free Force of Interest $SD_j[n,t]$ with $\beta = 0.5$

Time t	Base Rate	Term n 1	2	3	4	5	10	20	Consols
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
5	1.95	1.51	1.34	1.28	1.27	1.28	1.29	1.29	1.28
10	2.21	1.81	1.66	1.62	1.62	1.62	1.64	1.64	1.63
20	2.32	1.96	1.84	1.82	1.82	1.83	1.85	1.85	1.83

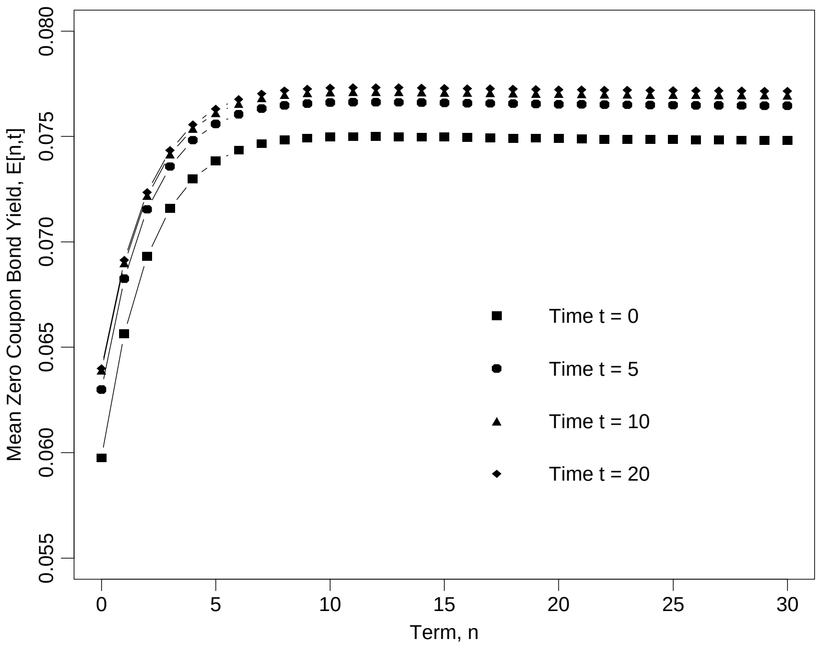

Figure 6.15: Mean of the Risk-free Force of Interest $E_j[n,t]$ with $\beta = 0.5$

The mean force of interest $E_j[n,t]$ is also changing through time t as can be seen in Figure 6.15. The mean base rate and consols yield increase from their neutral initial values, which correspond to the median of their distribution, towards their long run mean values. Hence, the mean risk-free return increases with time as the mean base rate and consols yield, from which it is derived, are also increasing.

From Table 6.33 we see that at time zero the standard deviation is zero because both the base rate and consols yield are fixed in the initial parameters. At later times we see that the yields are most variable at the short end. The standard deviation of the yield also increases with time.

6.3 Results for the Unitised With-Profits Charging Mechanism

We will not produce results for the full range of bonus rates considered in Sections 4.3 and 5.5. In Chapters 4 and 5 the risk-free rate was fixed throughout the contract and we only investigated compulsory bonus rates y which were less than the risk-free rate. Therefore in the past we have always known that at least the compulsory bonus rate would be affordable regardless of the return on equities. However, we now have a variable risk-free return. This means that future premiums may be invested at a time when the risk-free rate of return is lower than the compulsory guarantee.

Of course, when the compulsory guarantee is zero, we have no problems. We also find no problems when $y = 2\%$ and $z = 0\%$ — the risk-free rate does occasionally fall below 2%, but the return on past premiums is sufficient to purchase the new compulsory guarantee in all simulations. (Note that the consols yield is limited to a minimum of 0.5% in this implementation of the Wilkie model.) When $y = 2\%$ and $z = 4\%$ we find no problems except for one simulation where the compulsory guarantee is not quite affordable in the final year — in this case we assume that the fund switches entirely into cash and the payout at maturity is very slightly more than the value of the assets. For higher rates of bonus the compulsory guarantee becomes unaffordable far more frequently, so we will not consider these cases further.

Many UWP contracts allow the compulsory bonus to change for future premiums (future bonuses on past premiums must of course still be declared at the compulsory bonus rate in-force at the time the premium was paid). We could consider policies where the compulsory guarantee for premiums invested in a particular year is set with reference to the risk-free return at that time, but will not consider this further in this thesis.

Tables 6.34 and 6.36 show the results for the simulations using the stochastic risk-free rate. They show the same patterns as we have seen in previous tables. For example, higher bonuses lead to a higher mean achieved guarantee, a lower mean payout, and to the option being exercised more often.

It is particularly interesting to compare these results with Tables 5.30 and 5.31 which used the same Wilkie model for equities as this chapter, but used a constant risk-free rate of 6.77%. We repeat the relevant parts of Tables 5.30 and 5.31 here as Tables 6.35 and 6.37 for ease of reference. The results are different in two ways.

Firstly, Tables 6.34 and 6.36 show greater variability because the stochastic risk-free rate can be very high or very low, whereas in Tables 5.30 and 5.31 it is fixed.

Secondly, the fixed risk-free rate of 6.77% was chosen to be a typical rate lying between the base rate and consols yield. We can see from Table 6.32 that 6.77% is indeed approximately half way between the averages of the base rate and consols yield. However, we see that 6.77% is in fact *always* lower than the mean risk-free return used in the option pricing of a policy sold at time zero (*i.e.* at time 0, 5, 10 and 19 we have mean risk-free rates, with corresponding durations of 20, 15, 10 and 1, of 7.5%, 7.7%, 7.7%, and 6.9% respectively. Only at time 20 and duration zero is the mean stochastic risk-free return less than 6.77%, but we do not buy options with a term of less than 1 year.) Hence the average cost of options is cheaper in the stochastic risk-free model.

Comparing Tables 6.34 and 6.35 we see that using a stochastic risk-free rate increases both the mean and standard deviation of UWP payout for all levels of bonus. The stochastic risk-free rate model gives higher payouts because the mean risk-free rate is higher than the fixed rate of 6.77%. On average this leads to cheaper options and a larger proportion of assets retained in equities. The increased standard deviation is caused by the variability in the risk-free rate and in the larger equity

Table 6.34: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities and a Stochastic Risk-Free Rate

y	z	Maximum Guarantee	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		0.00	3761.91	2096.38	0.00	0.00
0.00	0.00	1000.00	3724.61	2084.38	1000.00	0.00
0.00	0.04	1488.90	3655.80	2089.22	1482.57	26.05
0.00	0.08	2288.10	3479.34	2051.43	2138.16	223.34
0.02	0.00	1239.17	3655.65	2051.46	1239.17	0.00
0.02	0.04	1892.28	3485.10	2022.95	1857.00	77.35
R-F		NA	2473.26	268.23	2473.26	268.23

Table 6.35: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities and a Constant Risk-Free Rate

y	z	Maximum Guarantee	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		0.00	3761.91	2096.38	0.00	0.00
0.00	0.00	1000.00	3709.79	2071.64	1000.00	0.00
0.00	0.04	1488.90	3627.17	2064.63	1483.38	22.99
0.00	0.08	2288.10	3397.22	1999.11	2133.08	216.55
0.02	0.00	1239.17	3612.36	2014.62	1239.17	0.00
0.02	0.04	1892.28	3390.70	1949.89	1858.27	70.84
RF		2193.26	2193.26	0.00	2193.26	0.00

holding.

We see that the mean achieved guarantee is little changed, but the standard deviation increases, again because of the variability of the risk-free rate.

Table 6.34 also shows the figures for the unit-linked and risk-free policies. The unit-linked policy is entirely invested in shares and so is unaffected by the stochastic risk-free rate.

The risk-free account invests each premium at the current risk-free rate (*i.e.* zero coupon bonds maturing at time 20). Table 6.34 shows a higher mean payout than Table 6.35 because the mean stochastic risk-free rate is greater than the fixed rate of 6.77%. However, Table 6.35 shows that the standard deviation of the payout is zero because the reinvestment rate for future premiums is known and fixed at outset. However, Table 6.33 shows some variability in the risk-free account payout because the risk-free rate at the time future premiums are paid is unknown at outset. The mean and standard deviation of payout on the risk-free policy are much lower than the UWP policies.

Table 6.36: The Number of the 10,000 Simulations where the Payouts and Guarantees Display Certain Features using the Wilkie Model for Equities and a Stochastic Risk-Free Rate

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	81	80	72	72	2878	10000
0.00	0.04	9306	905	492	488	551	3200	10000
0.00	0.08	6124	3474	1518	1502	2405	4249	8978
0.02	0.00	10000	315	244	235	235	3048	10000
0.02	0.04	7760	2493	1104	1097	1343	3956	9999

Comparing Tables 6.36 and 6.37 we see that using a stochastic risk-free rate slightly reduces the number of simulations where the maximum guarantee is achieved for low bonuses ($y = 0\%, z = 4\%$), but otherwise increases the number of simulations where the guarantee is achieved. The pattern is reversed for the number of simulations in which the option is exercised *i.e.* whenever the stochastic risk-free

Table 6.37: The Number of the 10,000 Simulations where the Payouts and Guarantees Display Certain Features using the Wilkie Model for Equities and a Constant Risk-Free Rate

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	81	81	72	72	2227	10000
0.00	0.04	9341	880	499	492	551	2581	10000
0.00	0.08	5857	3649	1494	1480	2405	3580	4143
0.02	0.00	10000	313	244	235	235	2433	10000
0.02	0.04	7645	2597	1109	1102	1343	3544	10000

rate has a higher number of simulations where the maximum guarantee is achieved, it also has a lower number of simulations where the guarantee is exercised, and vice versa. The different characteristics of the low and high bonuses arise because the low bonus strategies are more sensitive to the variability of the risk-free rate, whereas the higher bonuses are more sensitive to the mean of the risk-free rate.

The results under the two different risk-free assumptions are very similar when we compare the UWP policies to the unit-linked policy. Even though the mean UWP payout is higher under the stochastic risk-free rate, the UWP policy does not outperform the unit-linked policy more often because of the added variability.

We saw in Tables 6.34 and 6.35 that the mean payout from the risk-free account increased by £280 when we switched to the stochastic risk-free rate. This increase is much larger than the increase in UWP payouts. Therefore comparing Tables 6.36 and 6.37 we see that the risk-free account outperforms the UWP policies more often under a stochastic risk-free rate.

In conclusion, we have seen that using a constant risk-free rate of interest in our projections over-simplifies the model and ignores some of the variability in the results. The constant risk-free rate of 6.77% chosen in Chapter 5 is generally too low for the investment model we have chosen and so understates the payouts which can be achieved by the UWP policy. In the following chapter we will use the more realistic stochastic risk-free rate introduced in this chapter.

Chapter 7

Dynamic Bonuses

7.1 Introduction

In Chapters 3, 4, 5 and 6 we have considered a very simple bonus strategy. The compulsory and desired bonus rates y and z are fixed in advance. However, the bonus strategy is still dynamic to the extent that the desired bonus is not declared in any year in which the matching options are unaffordable.

In this chapter we will consider more advanced bonus mechanisms. Ideally, the bonuses declared should:

- reflect the performance of the underlying assets,
- change in a smooth way from year to year,
- be competitive with other contracts, and
- maintain adequate solvency.

We discuss these features in more detail below.

Firstly, bonuses should follow the investment returns. Importantly, on average, bonus rates should be lower than the investment return in order to build up a terminal bonus cushion and allow future investment freedom. Traditionally when investment returns are low, bonus rates are reduced in order to reduce the probability of the guarantees being uncovered and to maintain solvency. The option pricing approach also shows us that if the bonus rates do not reduce then they may become unaffordable. When investment returns are high there are marketing pressures to increase bonus rates. Hence traditional insurers, who do not use the option pricing methodology, will have larger terminal bonus cushions, and better solvency and so will feel more able to grant higher bonuses. Using the option pricing approach, higher past investment returns mean that options with a given exercise price become cheaper and so guarantees can be increased at reasonable cost.

Secondly, bonuses should be declared in a smooth manner from year to year. With-profits policies are marketed as lower risk investments which protect the policyholder to some extent from the fluctuations of the stock market. The requirement to smooth bonuses conflicts with the requirement for bonuses to follow investment returns. Smoothing works very well whenever the market is temporarily overpriced or underpriced — the value of the policyholder's policy increases steadily, whilst the terminal bonus cushion fluctuates within acceptable limits. However smoothing becomes more problematic whenever the market performs particularly well or badly over long periods. The more important problem occurs when the stock market is depressed over a long period. Smoothing can stop the bonus rates from being cut rapidly enough, which for traditional insurers reduces the terminal bonus cushion and threatens solvency. Even if options are bought to match all bonuses, a long period of low stock market values can lead to a portfolio that is effectively entirely invested in cash unless bonus rates are reduced. Long periods of rapidly rising markets can still cause problems with smoothing. Bonus rates will lag the increasing market returns making the with-profits policy look uncompetitive.

Thirdly, the bonus rates must be competitive, both with other insurers and comparable alternative investments. Policyholders will want higher bonuses and may (unfairly) compare bonuses with returns on cash and unit-linked policies. However, the option pricing technique shows that higher bonuses come at a cost. Higher guarantees must be paid for by selling shares to buy options. Bonus rates will need to be set high enough to make them competitive, but not so high that the policy cannot invest substantially in shares.

Finally, bonus rates should be set with reference to the solvency of the insurer. A healthily solvent insurer can declare the bonuses that it thinks best fits the three criteria above. However, as the solvency of an insurer falls, its bonus policy becomes constrained, because higher bonuses lead to higher liabilities and hence yet lower solvency. Ultimately, an insurer in severe financial distress must abandon the first three criteria and declare only such small bonuses as its solvency allows. If the insurer actually buys options to match the guarantees or performs the corresponding hedging, then the solvency of the insurer is ensured. However, the bonus policy may still become constrained if poor investment returns mean that the options to back the desired new bonuses are unaffordable.

7.2 Literature Review

Many authors have considered stochastic models for UK-style with-profits policies.

Conventional with-profits policies have been considered by Limb et al. (1986), Wilkie (1987), Forfar et al. (1989), Ross (1991), Ross and McWhirter (1991), Macdonald (1995), Thomson et al. (1995), Yap (1999), Hare et al. (2000), Kouloumbos (2000), Miranda (2001), and Hibbert and Turnbull (2003).

Unitised with-profits policies have been considered by Thomson et al. (1995), Chadburn (1997), Chadburn and Wright (1999), Yap (1999), Hare et al. (2000), Kouloumbos (2000), Bruskova (2001), Hairs et al. (2002), Abbey (2003), and Lal (2003).

Hare et al. (2000) do not consider a dynamic bonus strategy. They consider a policy, either conventional or unitised with-profits, which has been in-force for t years. Up to time t the investment return and bonuses are deterministic. From time t, investments are modelled stochastically, but no more reversionary bonuses are declared.

All other authors mentioned above have used a bonus strategy that is dynamic to some extent (sometimes in addition to considering deterministic bonuses). The dynamic bonus mechanisms can be split into four main groups: 'fixed' bonuses, bonuses based on net premium reserves, bonus earning power, and bonuses directly linked to investment return.

7.2.1 'Fixed' bonuses

Wilkie (1987) considers two mechanisms for declaring bonuses on conventional with-profits policies. The first mechanism is the one we considered in Section 3.2. It is a simple method for single premium policies. When the policy is sold the sum assured and desired bonus rate are fixed. The same rate of bonus is declared each year if the matching options are affordable. If the options are unaffordable then no bonus is declared that year. Hence this method is still dynamic in that the bonus is cut to zero in some circumstances, but when a bonus is declared it is always the same. Therefore this bonus mechanism does not change smoothly in response to investment performance.

This mechanism is slightly more complicated in the regular premium case because the premiums in the early years cannot typically buy the options to match the sum assured. In this case it is assumed that part of the sum assured and bonuses will be purchased by the current assets, and the remainder will be bought by the future premiums. The proportion bought by current assets is equal to the value at maturity of the current assets, divided by the value at maturity of all assets including future premiums (see equation 3.6), assuming in both cases that investment is in the risk-free asset. This method has the disadvantage that if the risk-free rate decreases in the future, then the future premiums may be less than the cost of the matching options, so that the guarantee may become uncovered if shares perform badly. My MSc students Yap (1999), Kouloumbos (2000) and Miranda (2001) all use the bonus mechanism of Wilkie described above.

We commented that Wilkie's bonus mechanism does not reflect investment performance. In fact, when investment returns are poor for a number of years, the method will continue to declare bonuses at the desired rate until they are no longer affordable, at which point the bonus rate is cut suddenly to zero. This has the effect that more and more units of equity are sold to buy the increasingly expensive options, until the entire fund is effectively invested at the risk-free rate. It is desirable to maintain some equity investment in order to hopefully benefit from a recovery of share prices, and so Wilkie (1987) suggests that a bonus is only declared if the ratio of the resultant guarantee over the maximum possible guarantee (if invested entirely in the risk-free asset) is less than some critical amount. Wilkie does not provide any results for this 'critical ratio' approach, but it is considered by Yap (1999) and Kouloumbos (2000).

In the preceding chapters we have taken an approach similar to that of Wilkie above, but instead applied to unitised with-profits and without using a 'critical ratio'. We have only declared a bonus if the options to match the desired bonus rate are affordable. This approach was first used by my MSc student Yap (1999), and has since been used by my MSc students Kouloumbos (2000), Bruskova (2001) (with the modification used by Hare et al. (2000) that options were bought so that they gave a probability of shortfall of 1%), Abbey (2003), and Lal (2003). This bonus mechanism is very simplistic. The desired bonus rate is fixed at the start of the policy and does not change to reflect investment performance. The bonus rates do not change smoothly, being either zero or declared at the same rate z each year.

7.2.2 Bonuses based on net premium reserves

Limb et al. (1986) consider two different bonus mechanisms for conventional with-profits. Under the first method the retrospective accumulation of premiums less claims is rolled up with investment income, but not changes in capital, and compared with a net premium valuation. A bonus is then declared such that the cost of bonus equals the excess of the retrospective accumulation over the net premium valuation. By ignoring capital appreciation, an implicit allowance is made to build up a terminal bonus.

7.2.3 Bonus earning power

All the following bonus mechanisms are variations on the bonus earning power concept.

Limb et al. (1986) considered their second method to be more realistic. A smoothed discounted future income value of the assets is compared with a bonus reserve for the liabilities. A bonus is declared equal to the bonus earning power *i.e.* the rate of assumed future bonus which equates the value of the assets and liabilities. Forfar et al. (1989) also use a similar bonus earning power approach.

Ross (1991) and Ross and McWhirter (1991) also set conventional with-profits bonuses using bonus earning power. In both cases the asset values are smoothed rather than using market values. A terminal bonus target of 30% is allowed for. The reversionary bonus is set so that the asset share and present value of future premiums for the whole portfolio equals the present value of future benefits. However, the process is repeated for all business within 5 years of maturity assuming zero terminal bonus, and if this results in a lower bonus earning power than the portfolio calculation then the lower rate is used for all business. Ross (1991) uses only compound bonuses, whilst Ross and McWhirter (1991) use super compound bonuses. Ross and McWhirter (1991) also smooth the reversionary bonuses by limiting the change in bonuses from year to year.

Macdonald (1995) uses a bonus earning power approach for conventional with-profits. For each policy the bonus rate is calculated for which the smoothed asset share, projected using a geometric average of gilt yields, equals the maturity payout with allowance for terminal bonus. The bonus actually declared is a weighted average of the bonus earning power of each policy. Bonus rates cannot change by more than a given amount from year to year in order to smooth the build up of guarantees. Thomson et al. (1995) use a bonus earning power approach for both conventional and unitised with-profits based on Macdonald (1995).

The bonus earning power methods meet the criteria for bonuses which smoothly reflect investment performance, but to varying extents.

7.2.4 Bonuses directly linked to investment return

The bonus mechanisms described above are all affected by the investment return indirectly. However, the methods described here directly link the bonus declared to the investment return.

The second mechanism considered by Wilkie (1987) is more dynamic than his fixed strategy, the change in the bonus rate being linked to the change in the equity dividends. The rate of increase of dividends, perhaps negative, is closely related to the investment performance. However, dividend changes tend to be smoother than changes in the share price. Therefore this bonus mechanism fits our requirements of a smoothed link to investment performance. However, since the changes to the taxation of dividends announced in 1997 (see Masters et al. (1997)), companies are using a greater variety of ways to return profits to shareholders, and so changes in dividends may no longer be a reliable method.

Chadburn (1997) and Chadburn and Wright (1999) consider the same unitised with-profits bonus mechanism. The aim is to declare bonuses which reflect the return on consols. However, the bonuses are restricted so that the guarantees do not exceed the 'reduced asset share' (*i.e.* the asset share based on 75% of the actual asset return) — this implicitly builds up a terminal bonus cushion. The bonuses are then restricted to change by no more than a given percentage each year so that the guarantees build up smoothly. Notice that this is not a bonus earning power method and indeed no maturity date is assumed in the bonus mechanism.

Hibbert and Turnbull (2003) set bonuses on conventional with-profits policies equal to the current yield on long dated gilts. These bonuses are smoothed so that they may increase or decrease by no more than 1% each year. A further check is made similar to a bonus earning power calculation. If the asset share accumulated to maturity at the long gilt yield is smaller than the projected guarantees assuming continuation of the current bonus rate, then the bonus rate is reduced by 1%.

Hairs et al. (2002) declare a bonus of 60% of the investment return, but give no further details of their unitised with-profits bonus mechanism.

In the following sections we will investigate a number of different bonus mechanisms within the option pricing framework.

7.3 Alternative Bonus Mechanism Results

7.3.1 Bonuses directly linked to investment return

In this section we will follow the approach of Hairs et al. (2002) and declare a bonus rate each year as a proportion bp of the return on shares in that year, subject to a minimum of zero.

$$
\begin{aligned}
z[t] &= bp\left(\frac{PR[t]}{PR[t-1]} - 1\right) && \text{if} \quad PR[t] \geq PR[t-1] \\
z[t] &= 0 && \text{if} \quad PR[t] < PR[t-1].
\end{aligned} \tag{7.18}
$$

This bonus mechanism has the advantage of following the actual investment returns, but will not be smooth. We will consider smoothing these bonuses in Section 7.3.2.

We need to check that each bonus under this mechanism is affordable. Surprisingly, it is not the low rates of return which cause the problems, at least not immediately. When returns are negative, we declare no bonus, and maintain the same investment in shares and options as the previous years.

In fact, the problems potentially come when the stock market rises again after a fall. In effect, as the stock market falls we invest more and more in the risk-free asset. Hence the portfolio is only partially invested in shares and so obtains a lower return than the return on shares. Therefore the actual return earned may be less than the bonus we are trying to declare and so may in some circumstances be unaffordable.

We could solve this problem by basing the bonuses on the return on the hedging portfolio. However, we want to use a bonus strategy which is as close as possible to that which may be used by an office which does not use option pricing, to allow comparisons to be made.

If the desired guarantee is unaffordable we will set the bonus to zero.

Tables 7.38 and 7.40 show the results for the unsmoothed investment-linked bonus mechanism.

We can compare Tables 7.38 and 7.40 with Tables 6.34 and 6.36. The only difference between these two sets of tables is in the bonus mechanism used. Tables 6.34 and 6.36 use the fixed bonus approach of Yap, whereas Tables 7.38 and 7.40 link the bonus directly to the investment return. We repeat Tables 6.34 and 6.36 here as Tables 7.39 and 7.41 for ease of reference.

Comparing Tables 7.38 and 7.39, it appears that the investment-linked bonus mechanism is superior producing a lower standard deviation of payout for a given mean payout. For example, with $y = 0\%, bp = 60\%$ we have both a higher mean

Table 7.38: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Unsmoothed Investment-linked Bonus Strategy

y	bp	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		3761.91	2096.38	0.00	0.00
0.00	0.00	3724.61	2084.38	1000.00	0.00
0.00	0.50	3594.97	2040.35	2226.45	490.31
0.00	0.60	3521.48	1978.41	2573.37	725.71
0.00	0.70	3430.42	1853.34	2874.77	1019.01
R-F		2473.26	268.23	2473.26	268.23

Table 7.39: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Fixed Bonus Strategy

y	z	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		3761.91	2096.38	0.00	0.00
0.00	0.00	3724.61	2084.38	1000.00	0.00
0.00	0.04	3655.80	2089.22	1482.57	26.05
0.00	0.08	3479.34	2051.43	2138.16	223.34
0.02	0.00	3655.65	2051.46	1239.17	0.00
0.02	0.04	3485.10	2022.95	1857.00	77.35
R-F		2473.26	268.23	2473.26	268.23

Table 7.40: The Number of the 10,000 Simulations where the Payouts and Guarantees Display Certain Features using the Unsmoothed Investment-linked Bonus Strategy

y	bp	Desired Bonuses Always Affordable	Option Exercised	UWP > UL	Ach Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	81	80	72	2878	10000
0.00	0.50	8925	2053	1165	1151	3428	7246
0.00	0.60	7489	3462	1791	1767	3388	4957
0.00	0.70	5133	5022	2477	2449	3260	3962

Table 7.41: The Number of the 10,000 Simulations where the Payouts and Guarantees Display Certain Features using the Fixed Bonus Strategy

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	81	80	72	2878	10000
0.00	0.04	9306	905	492	488	3200	10000
0.00	0.08	6124	3474	1518	1502	4249	8978
0.02	0.00	10000	315	244	235	3048	10000
0.02	0.04	7760	2493	1104	1097	3956	9999

and lower standard deviation of payout than with either $y = 0\%, z = 8\%$ or $y = 2\%, z = 4\%$.

Under the fixed bonus approach the bonuses must be declared as long as the matching options are affordable even if the options are very expensive and require the sale of a large proportion of the equity investment. However, under the investment-linked bonus mechanism, bonuses are low in years of poor investment returns when the cost of options is increasing. Conversely under the investment-linked bonus mechanism, bonuses are high in years of high investment returns when the cost of options is decreasing. Therefore the investment-linked bonus mechanism is superior because guarantees are purchased for lower cost.

Hence it appears that flexibility to adjust bonuses in line with investment returns is of benefit to the policyholder.

We also see that investment-linked bonuses produce a higher mean and standard deviation for the achieved guarantee than the fixed bonus method. Fixed bonuses have an upper limit to their size, whereas very high bonuses can be declared under the investment-linked method whenever investment returns are high.

Comparing Tables 7.40 and 7.41, we see that the higher average bonuses under the investment-linked mechanism leads to the options being exercised more frequently and UWP outperforming unit-linked more frequently. However, whenever the option is exercised, no terminal bonus is paid. The insurer would wish to add some terminal bonus in all but the most adverse circumstances. This suggests that bonuses of 70% of the investment return are too high, and that to retain a high probability of paying a terminal bonus the regular bonuses should perhaps be no more than 50% of the investment return.

In Table 7.41 the maximum guarantee is achieved whenever the fixed desired bonus (*e.g.* $z = 8\%$) can be afforded every year. Under the investment-linked bonus mechanism there is no upper limit to the size of the bonus, but again we can count the number of simulations where the desired guarantee can be afforded each year. For example, with fixed bonuses of $y = 0\%, z = 8\%$ the mean guarantee is £2138, and the desired guarantee can only be afforded every year in 6124 simulations. However, with investment-linked bonuses of $y = 0\%, bp = 50\%$ the average guarantee is a similar £2226, and the desired guarantee can be afforded in 8925 simulations, because the rate of desired bonus is cut back in years of poor investment returns and increased in good years to compensate.

7.3.2 Bonuses directly linked to investment return with smoothing

In the previous section, bonuses were directly linked to the return on equities. However, we commented that these bonuses were not at all smooth, reflecting the volatility of equities.

In this section we will add smoothing to the investment-linked mechanism described in the previous section. Again we will declare bonuses equal to some proportion *bp* of the total return on equities as in Equation 7.18. As before we will add the constraints that the bonuses cannot be negative. We now add the additional constraint that the bonus rate cannot change by more than a given percentage each year. In the results below, bonus rates are not permitted to increase by more than 20% or decrease by more than 16.67%. Hence a decrease and increase in bonus rates at the maximum permitted amounts will cancel out.

$$1.2\,z[t-1] \geq z[t] \geq \frac{z[t-1]}{1.2}.$$

So, for example, if last year's bonus rate is 5%, then this year's bonus rate must lie between 4.1667% and 6% inclusive.

This bonus smoothing mechanism has been employed previously by Macdonald (1995). Ross and McWhirter (1991) and Thomson et al. (1995) limit the change in bonus rate by a fixed amount, *e.g.* 1%, rather than by a proportion of the previous bonus. The approach of Macdonald appears better as a cut in bonus of 2% is more likely to be acceptable to policyholders if the current bonus rate is 10% than if it were 5%. Chadburn (1997) and Chadburn and Wright (1999) also employ smoothing of bonus rates, but in a more complex manner.

However, we still retain the constraint that bonuses are only declared if matching options are unaffordable.

If the desired guarantee is unaffordable then we set the desired bonus to zero that year. However, we carry over the desired bonus rate to the following year as a starting point for the next bonus. For example, if in year 15 we wish to declare a bonus of $z = 10\%$, but cannot afford to buy the corresponding options, then we will declare a bonus of $z = 0\%$. Then at time 16 we use 10% as a base for our smoothing and can declare any bonus between 8.33% and 12%. We may find at time 16 that we wish to declare a bonus of 8.5%, but cannot afford it. We then declare no bonus at time 16 and carry forward 8.5% as the new base value for smoothing. Notice that the smoothing mechanism operates as if the desired bonus had been declared. In practice this often means that once the desired guarantee is unaffordable in one year, it remains unaffordable in each future year. An alternative would be to rebase the bonuses at a very low level, say 0.5%, and start a new smoothed series from this point.

Declaring no bonuses in a year when the desired bonus is unaffordable is not smooth, but the desired bonus could only be matched if external capital was added to the policy.

However, the method described in this section is much smoother than the method in Section 7.3.1 at the expense of bonuses that are less closely matched to the investment return. The smoothing mechanism is quite realistic in that insurers will not want to change their bonuses too much because of the bad publicity that a cut in bonus rates brings.

This method will be particularly effective if investment returns are slowly increasing or decreasing, because bonus rates will be able to change to reflect this. This is quite likely to happen if the Wilkie model produces a number of years of high or low inflation respectively.

The method is much less effective when returns are very volatile from one year to the next. The number of times the bonus rate increases and decreases will follow the number of years of high and low returns. However, the method lacks any mechanism to look at the size of these returns, and in particular the average return. For example the same bonuses will be declared if returns in consecutive years are 10% and -20% or 20% and -10%.

In the same way that we use neutral initial conditions for the variables in the Wilkie investment model, we should also start each of our simulations with the same neutral value for the bonus rate. An alternative approach would be to use current market values for the parameters in the Wilkie model and a current typical bonus rate. The important thing is that we should have consistent starting values for both the investment model and the bonus rate.

The bonus rate $z[t]$ declared at time t, before smoothing, is given by

$$
\begin{aligned}
z[t] &= bp\left(\frac{PR[t]}{PR[t-1]} - 1\right) \\
z[t] &= bp\left(\frac{P[t] + D[t]}{P[t-1]} - 1\right)
\end{aligned}
$$

where $PR[t]$ is the value of the total equity return index, $P[t]$ is the price of one share and $D[t]$ is the dividend from one share. Given that $D[t] = P[t] \cdot Y[t]$ we get:

$$
z[t] = bp\left(\frac{Y[t-1] \cdot D[t]}{Y[t] \cdot D[t-1]}(1 + Y[t]) - 1\right).
$$

We can now find the bonus rate under neutral initial conditions by setting the innovations to zero. We find from the equations in Section 5.2 that both the yield and the ratio of consecutive dividends are constants as follows:

$$
Y[t] = YMUe^{YW \cdot QMU}
$$

$$
\frac{D[t]}{D[t-1]} = e^{QMU+DMU}.
$$

Hence the neutral intial bonus rate given the parametrisation we are using from Wilkie (1995) is:

$$
\begin{aligned}
z[0] &= bp\left(e^{QMU+DMU}(1 + YMUe^{YW \cdot QMU}) - 1\right) \\
z[0] &= bp\left(e^{0.047+0.016}(1 + 0.03e^{1.8*0.047}) - 1\right) \\
z[0] &= 0.099798314\, bp. \qquad (7.19)
\end{aligned}
$$

The results of the smoothed bonus mechanism are given in Tables 7.42 and 7.43.

Table 7.42: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Smoothed Investment-linked Bonus Strategy

y	bp	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		3761.91	2096.38	0.00	0.00
0.00	0.00	3724.61	2084.38	1000.00	0.00
0.00	0.50	3636.43	2080.23	1711.42	284.68
0.00	0.60	3597.35	2070.16	1911.52	392.58
0.00	0.70	3545.17	2048.37	2122.18	520.03
R-F		2473.26	268.23	2473.26	268.23

Table 7.43: The Number of the 10,000 Simulations where the Payouts and Guarantees Display Certain Features using the Smoothed Investment-linked Bonus Strategy

y	bp	Desired Bonuses Always Affordable	Option Exercised	UWP > UL	Ach Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	81	80	72	2878	10000
0.00	0.50	9217	1144	643	637	3280	9792
0.00	0.60	8501	1761	928	919	3487	9099
0.00	0.70	7550	2606	1251	1239	3624	7958

Firstly we compare Tables 7.38 and 7.40 which have investment-linked bonuses without smoothing with Tables 7.42 and 7.43 which have investment-linked bonuses with smoothing.

We see that with smoothing we get a lower mean and standard deviation of achieved guarantee for a given value of bp. When returns are very high, without smoothing we declare very high bonuses, and so the mean and standard deviation of the guarantee is larger than with smoothing where bonuses increase only slowly. When returns are very low, without smoothing we declare very low bonuses, and so the mean is lower, but the standard deviation again is larger than with smoothed bonuses. Therefore it is clear that both high and low returns lead to greater volatility without smoothing. However, when considering the mean we appear to have two opposing effects. The bonus is restricted to a minimum of zero in bad years, whereas the bonus is unrestricted in good years. Hence the high return effect dominates and the mean guarantee of the unsmoothed method is greater than the smoothed method.

As smoothing leads to lower guarantees on average, then the mean and standard deviation of payout are increased due to the lower cost of the options. The lower guarantees also mean that the desired bonus is more likely to be affordable, and the guarantee will be exercised less often.

However, when we compare policies with a similar standard deviation of payout, we see that the unsmoothed bonus strategy is superior. The unsmoothed policy with $bp = 0.5$ has both higher expected payout and lower standard deviation of payout compared to the smoothed policy with $bp = 0.7$. Hence it appears that smoothing regular bonuses may actually disadvantage policyholders if they are charged for the guarantees using the option pricing approach.

Secondly we can compare Tables 7.39 and 7.41 which have fixed bonuses with Tables 7.42 and 7.43 which have investment-linked bonuses with smoothing. The Tables are otherwise identical. The two methods are initially quite similar. Under the fixed method a bonus of z is declared every year if it is affordable. Under the investment model parameters used here, Equation 7.19 shows us that the smoothed investment-linked method will declare a bonus close to $0.1bp$ in the early years, although the bonus rate may drift away from this as time progresses. The more we smooth the bonus rate the closer the two methods become. For example, a smoothed investment-linked bonus with $bp = 0.6$ should be very similar to the fixed strategy with $z = 6\%$.

Looking at the mean achieved guarantee we find that the results for the smoothed investment-linked bonus rates with $bp = 0.5, 0.6, 0.7$ do indeed lie between the results for the fixed bonuses with $z = 4\%$ and 8%. The standard deviation of the guarantees is higher for the smoothed investment-linked method because the bonuses can drift upwards or downwards from their starting values.

Given the very similar mean achieved guarantee under the fixed and smoothed investment-linked methods, it is not surprising to see similar results for the payouts and number of simulations the guarantee is exercised *etc.*

7.4 Conclusion

There has been pressure on insurers to be more specific in the way they calculate bonuses. The reason for this has been to remove the conflict of interest between policyholders (who want high guarantees) and shareholders (who want low guarantees). However, this conflict only exists because the charge for guarantees is not accurately calculated and adjusted for the guarantees given.

No such conflict exists if we follow the methodology in this thesis. The policyholder is charged the fair value for any guarantees they have. If the policyholder wants higher guarantees then they will pay a higher charge via the increased cost of the matching options.

We see that the policyholder benefits from a more flexible bonus strategy if they are correctly charged for their guarantees. Firstly in Section 7.3.1, we see that the investment-linked bonus strategy is superior to the fixed bonus strategy, which implies that retaining flexibility to change bonuses with investment returns is of benefit to the policyholder. Secondly in Section 7.3.2, we see that the unsmoothed strategy is superior to the smoothed strategy, which implies that retaining flexibility to change bonuses quickly without smoothing is of benefit to the policyholder.

Chapter 8

Multiple Generations

8.1 Introducing the Multiple Generation Model

In the previous chapters we have considered a single UWP policy with regular premiums. In this chapter we will extend this work to an insurer with a portfolio of policies with multiple generations. In the previous chapters we saw that the guarantees could be matched by holding a mixture of shares and put options. In this chapter we will investigate whether we can improve on this approach when we consider multiple generations.

We begin in Section 8.2 with a description of the multiple generation model. We present the results if we use the maturity guarantee charges to buy matching options for a portfolio of policies with different commencement dates. Simulations are performed in the same way as Chapter 5 using the Wilkie model, with options priced using a constant risk-free rate of interest, and bonuses declared according to the simple bonus strategy.

From Section 8.3 onwards we consider a major change in approach. We will investigate the extent to which the insurer can diversify the investment risk through time by selling multiple generations of policies. We will still charge the policyholder in the same way according to the cost of matching options. However, we will examine the effect of investing these charges in the risk-free asset rather than options. In Section 8.3 we investigate the potential profits and losses to the insurer of this approach, where simulations are again performed in the same way as Chapter 5 using the Wilkie model, with options priced using a constant risk-free rate of interest, and bonuses declared according to the simple bonus strategy. In Section 8.4 we calculate the free asset ratio as the ratio of the accumulated profits or losses to the asset shares of the in-force policies. In Section 8.5 we improve the model in the same way as Chapter 6 by using a risk-free interest rate derived from the Wilkie model. In Section 8.6 we consider the dynamic bonus mechanism used in Chapter 7. Finally, Section 8.7 gives a summary of the results in this chapter.

8.2 Matching the Guarantees of Multiple Generations using Options

So far we have considered regular premium policies. However in this chapter we will consider single premium policies for the following reason. Most UWP business is written as recurrent single premiums. We have no need to split premiums in each cohort between genuine new business and additions to existing policies. Initially we will assume that the office receives premiums of £50 each year, although in Section 9.1 we will allow for growth in new business.

Also, in previous chapters we have considered policies with a 20-year term. In this chapter we will consider policies with a 10-year term because this speeds up the computer program and reduces the number of intermediate steps that must be stored in memory.

Initially the office has no business on the books. Each year it issues a new cohort of policies with term of 10 years and total premiums of £50. The office is projected for 50 years. Hence, at the end of the projection we have had 40 maturing cohorts of policies, and still have 10 cohorts in-force.

Each policy is individually charged for its guarantees in the same way as was described in Section 3.3.1. However, as these policies are single premium, a modification is required to deal with cases when the actual guarantee gets very close to the maximum possible guarantee. This was not a problem in the regular premium cases because the new money always allowed an increase in the guarantee and the guarantee chosen was always restricted to be less than the maximum. However, in many simulations of the single premium case the share price might drop so low that the hedging portfolio is almost entirely invested in cash *i.e.* the guarantee is very close to the maximum. The computer would often record that the guarantee was not only close to, but at its maximum. This meant that the program was unable to solve for the correct exercise price. Hence, in cases where it is impossible to increase the guarantee, we simply maintain the same investment of shares and options as last year. These investments will of course meet the guarantee if held to maturity.

8.2.1 Results

We now look at the results of using options to match the maturity guarantees in the same way as Chapter 5. We use the Wilkie investment model with taxed dividends and option pricing using a constant risk-free rate of 7% (to be precise, a force of interest of 0.0676586) and volatility of 20% in the Black-Scholes equation. The simple bonus mechanism is used such that the desired bonus rate z is fixed in advance, and is declared every year that it is affordable.

First of all we consider the first cohort of policies starting at time zero and maturing at time 10. The results are shown in Tables 8.44 and 8.45. The pattern of results is very similar to the previous investigations.

Table 8.44 shows that the highest mean payout is obtained from the unit-linked policy which has no guarantees at all and hence also has the highest standard deviation of payouts. Increasing either of the bonus rates y or z leads to lower mean and standard deviation of payouts. Increasing the bonus rates not only increases the mean achieved guarantee, but also its standard deviation because there are more occasions when the desired guarantee will be unaffordable and hence set to zero.

In Table 8.45 we see that as the bonus rate y or z increases the maximum guarantee of course increases but can be achieved less often. Higher guarantees mean that it is worthwhile exercising the option on more occasions, firstly because the guarantee is higher, but secondly the high cost of the guarantee means that we hold fewer shares.

It is interesting to compare the results in Tables 8.44 and 8.45 with Tables 5.30 and 5.31, which are repeated here as Tables 8.46 and 8.47 for ease of reference.

Table 8.44: Mean and Standard Deviation of the Payout and Achieved Guarantee for a 10-year Single Premium Policy using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy
Start Year 0

y	z	Maximum Guarantee	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		0.00	143.91	67.18	0.00	0.00
0.00	0.00	50.00	139.04	64.62	50.00	0.00
0.00	0.04	71.17	133.85	63.80	70.59	1.77
0.00	0.08	99.95	125.06	58.97	91.79	9.89
0.02	0.00	60.95	133.86	61.35	60.95	0.00
0.02	0.04	86.75	123.67	56.90	84.23	4.14
0.02	0.08	121.84	113.74	44.71	100.16	14.61
0.04	0.00	74.01	124.32	53.73	74.01	0.00
0.04	0.04	105.34	109.54	38.90	95.79	7.54
0.06	0.00	89.54	107.80	33.15	89.54	0.00
R-F		98.36	98.36	0.00	98.36	0.00

Table 8.45: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy
Start Year 0

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	196	162	162	162	2865	10000
0.00	0.04	8764	1612	742	742	861	3501	10000
0.00	0.08	5013	4544	1631	1631	2718	4385	4987
0.02	0.00	10000	634	432	432	432	3219	10000
0.02	0.04	6554	3802	1390	1390	1789	4745	10000
0.02	0.08	2021	7078	2208	2208	4371	5046	5495
0.04	0.00	10000	1849	1004	1004	1004	3990	10000
0.04	0.04	2634	7159	2111	2111	3139	5918	6308
0.06	0.00	10000	5482	1996	1996	1996	6270	10000

Both sets of results use the same parameters in the Wilkie model and Black-Scholes pricing formula. However Tables 8.46 and 8.47 consider a regular premium 20-year contract, while Tables 8.44 and 8.45 consider a single premium 10-year contract.

Table 8.46: Mean and Standard Deviation of the Payout and Achieved Guarantee for a 20-year Regular Premium Policy using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Maximum Guarantee	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		0.00	3761.91	2096.38	0.00	0.00
0.00	0.00	1000.00	3709.79	2071.64	1000.00	0.00
0.00	0.04	1488.90	3627.17	2064.63	1483.38	22.99
0.00	0.08	2288.10	3397.22	1999.11	2133.08	216.55
0.02	0.00	1239.17	3612.36	2014.62	1239.17	0.00
0.02	0.04	1892.28	3390.70	1949.89	1858.27	70.84
0.02	0.08	2974.63	2997.69	1628.65	2425.37	400.65
0.04	0.00	1548.46	3372.59	1846.51	1548.46	0.00
0.04	0.04	2422.22	2888.23	1505.87	2224.41	182.83
0.06	0.00	1949.64	2794.04	1288.75	1949.64	0.00
RF		2193.26	2193.26	0.00	2193.26	0.00

Comparing Tables 8.46 and 8.44 we see a larger difference between the mean payout and the mean achieved guarantee for the 20-year policy than for the 10-year policy. This is because the longer term allows any investment return in excess of bonuses to build up a larger terminal bonus cushion. Therefore, the standard deviation of the payout on the 10-year policy falls more rapidly as the bonus rates y and z are increased, because a greater part of that payout is guaranteed.

Comparing Tables 8.47 and 8.45 we see that the maximum guarantee is achieved less often for the 10-year single premium policy. The 20-year regular premium policy is more able to afford the desired bonuses each year for two reasons. Firstly, the regular premiums can be used to buy additional guarantees for existing units. Secondly, it is less sensitive to falls in equity values in the early years because it has a longer investment horizon. For example, if after 5 years the value of the assets falls by 50%, it takes an annual return of 14.87% to make good the deficit for the

Table 8.47: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 20-year Regular Premium Policy Display Certain Features using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	81	81	72	72	2227	10000
0.00	0.04	9341	880	499	492	551	2581	10000
0.00	0.08	5857	3649	1494	1480	2405	3580	4143
0.02	0.00	10000	313	244	235	235	2433	10000
0.02	0.04	7645	2597	1109	1102	1343	3544	10000
0.02	0.08	2477	6357	2077	2064	4268	2975	3259
0.04	0.00	10000	1034	644	641	641	2951	10000
0.04	0.04	3552	6082	1845	1840	2780	4287	4606
0.06	0.00	10000	3923	1483	1481	1481	4869	10000

10-year policy, but only 4.73% for the 20-year policy. Hence a one-off increase of $z\%$ is more likely to be affordable for the longer term policy because the options will be cheaper.

Also we see that the option is exercised more often for the 10-year single premium contract. This is because shares are more likely to outperform cash in the long term and hence the assets of the 20-year contract are more likely to be higher than the guarantee.

We now consider a later cohort of policies starting at time 20 and maturing at time 30. The results are shown in Tables 8.48 and 8.49. The pattern of results is very similar to Tables 8.44 and 8.45, but there are some interesting differences.

The differences stem from the initial conditions used in the Wilkie model. All the policies commencing at time zero have the same starting conditions, in this case the neutral initial conditions. However the initial conditions for the policies starting at time 20 will be very different because the simulations have already been running for 20 years.

Firstly, comparing Tables 8.44 and 8.48 we see that policies starting at time 20 have a higher mean and standard deviation of payout. This is consistent with Table

5.29 where we saw that the mean and standard deviation of equity returns were greater for the 20th year than the first year.

Table 8.48: Mean and Standard Deviation of the Payout and Achieved Guarantee for a 10-year Single Premium Policy using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy
Start Year 20

y	z	Maximum Guarantee	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		0.00	148.78	83.07	0.00	0.00
0.00	0.00	50.00	143.91	79.78	50.00	0.00
0.00	0.04	71.17	139.05	79.50	70.07	2.67
0.00	0.08	99.95	132.13	76.38	90.21	11.70
0.02	0.00	60.95	138.85	75.62	60.95	0.00
0.02	0.04	86.75	130.34	72.63	83.30	5.24
0.02	0.08	121.84	121.95	64.21	99.91	16.75
0.04	0.00	74.01	129.61	66.34	74.01	0.00
0.04	0.04	105.34	117.22	56.05	95.36	8.66
0.06	0.00	89.54	112.49	42.96	89.54	0.00
R-F		98.36	98.36	0.00	98.36	0.00

Secondly, we see in Tables 8.45 and 8.49 that the number of times the maximum guarantee is achieved is very different. In the cases with relatively low and affordable guarantees then the maximum guarantee is more likely to be achieved for the cohort starting at time 0. However, the position is reversed for higher desired bonuses *c.f.* $y = 0\%, z = 4\%$ with $y = 2\%, z = 8\%$. The reason for this is the greater variability of equity returns for the later cohort. When the bonuses are low, then the desired guarantee is affordable in all but the most extreme cases, which are of course more common for the cohort starting at time 20. However, when the bonus rates are high, the desired bonus can only be afforded when equity returns are unusually high, which is also more common for the cohort starting at time 20.

Table 8.49: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy
Start Year 20

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	358	308	308	308	3194	10000
0.00	0.04	8139	2159	1043	1043	1244	3804	10000
0.00	0.08	4975	4605	1845	1845	3072	4563	5025
0.02	0.00	10000	952	733	733	733	3493	10000
0.02	0.04	6184	4062	1730	1730	2180	4833	10000
0.02	0.08	2604	6643	2407	2407	4488	5093	5400
0.04	0.00	10000	2231	1409	1409	1409	4164	10000
0.04	0.04	3166	6740	2425	2425	3418	5719	6001
0.06	0.00	10000	5371	2355	2355	2355	6108	10000

Finally, we consider the cohort of policies starting at time 40 and maturing at time 50. The results are shown in Tables 8.50 and 8.51. The pattern of results is basically the same as Tables 8.48 and 8.49. This shows that at time 40 we have a similar spread of starting conditions as we had at time 20.

8.3 Mismatching Assets

So far we have used options to price the guarantee. The asset share is assumed to be invested in a mixture of shares and put options. The cost of buying the options implicitly charges for the guarantee. This strategy could in theory be adopted by an individual policyholder with access to the derivatives market.

In practice the policyholder will take out a unitised with-profits policy with an insurer. It is the insurer who will be liable to meet the guarantees. It is also the insurer who decides how the assets of the with-profits fund are invested. If the insurer decides to match the guarantees with options then the position is exactly as above.

Table 8.50: Mean and Standard Deviation of the Payout and Achieved Guarantee for a 10-year Single Premium Policy using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Start Year 40

y	z	Maximum Guarantee	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		0.00	148.87	83.17	0.00	0.00
0.00	0.00	50.00	144.03	79.83	50.00	0.00
0.00	0.04	71.17	139.23	79.51	70.08	2.65
0.00	0.08	99.95	132.43	76.43	90.26	11.70
0.02	0.00	60.95	138.98	75.65	60.95	0.00
0.02	0.04	86.75	130.58	72.69	83.30	5.20
0.02	0.08	121.84	122.41	64.36	99.95	16.78
0.04	0.00	74.01	129.73	66.36	74.01	0.00
0.04	0.04	105.34	117.61	56.10	95.40	8.66
0.06	0.00	89.54	112.60	42.94	89.54	0.00
R-F		98.36	98.36	0.00	98.36	0.00

Table 8.51: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Start Year 40

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	386	333	333	333	3209	10000
0.00	0.04	8168	2139	1057	1057	1255	3770	10000
0.00	0.08	5033	4591	1861	1861	3056	4530	4967
0.02	0.00	10000	948	720	720	720	3472	10000
0.02	0.04	6139	4022	1727	1727	2202	4851	10000
0.02	0.08	2612	6681	2401	2401	4495	4996	5310
0.04	0.00	10000	2255	1422	1422	1422	4150	10000
0.04	0.04	3157	6723	2400	2400	3417	5672	5941
0.06	0.00	10000	5436	2378	2378	2378	6148	10000

In the past, insurers have not attempted either to match the guarantees with options, or use the hedging strategy this implies. Typically insurers have tried to protect their solvency by increasing their investment in gilts at the expense of a decreased exposure to equities. However, a financially strong insurer would maintain a high equity backing ratio regardless of the level of the assets. In fact a strong insurer may wish to switch to equities as the market falls in order to benefit from the expected upswing — this is the exact opposite of the hedging portfolio which will gradually switch into the risk-free asset as shares fall and the option becomes more valuable.

Therefore in this section we will assume that the insurer does not invest in options. Indeed, it is questionable whether the derivatives market is large enough to allow all insurers to buy the put options they might need. Worse than this, it is possible, given the large proportion of the stock market held by with-profits funds, that if active hedging of the guarantee risk were used, the selling of equities required when shares fall would make any stock market falls considerably worse.

We assume that the insurer deducts charges from the asset share equal to the value of the matching options, and hence still pays out the same maturity benefit as under the option pricing approach described previously. To pay out anything different would admit a theoretical arbitrage (assuming the policyholder had access to the derivatives market to take advantage of it).

However, we will now assume that the insurer does not use these charges to buy the matching options. Instead they invest the charges in a guarantee account invested in the risk-free asset, and use this account to pay out the excess of any guarantees above the asset share. Clay et al. (2001) discuss such a guarantee account, but do not provide numerical results. We give a precise description of how the guarantee account operates in our case in Section 8.3.1, and provide the numerical results in Section 8.3.2.

Recall that in Section 3.4.3 we showed that we could hedge the put options by holding a positive amount of the risk-free asset and a negative amount of the equity index. Hence, by investing the charges entirely in the risk-free asset we have actually increased the insurer's exposure to the equity market. This mismatching strategy is in agreement with the belief that investment risk can be pooled over a number of years *i.e.* over short periods losses may be made, but profits in other years will more than compensate over the long run. We will investigate to what extent this belief is true.

We ignore here any regulatory requirement to hold further assets to ensure the guarantees are met with sufficiently high probability (for example a mismatching reserve) and any charges for use of this capital the insurer may wish to make. We concentrate solely on the accumulated profits and losses from writing the guarantees.

8.3.1 Annual Charges for Guarantees

In this section we will consider the charging mechanism and the build up of the guarantee account in detail.

The guarantees are not fixed in advance, but will increase with bonuses at the discretion of the directors of the insurer. Hence, we cannot accurately charge for the guarantees at the beginning of the policy because we do not know how the insurer will apply its discretion in the future.

However, once a bonus is declared we know the cost of the options which will match it. Bonuses are declared at the beginning of each year. Hence we will deduct charges annually. These charges are not fixed at the start of the contract, but their method of calculation is. The charges are varied to reflect the cost of the options which would match the guarantee.

We have seen in Section 3.3.1 how to match the guarantee with an equal number of shares and put options. For a single premium policy, at the beginning of the contract the premium is received and is guaranteed to grow at rate y. In later years we add bonuses z. Consider a policy at time t, which has been in-force for d years, and hence was sold at time $t-d$. The number of options required after declaration of bonuses at time t, $N_{t,t-d}$, and their exercise price, $E_{t,t-d}$, for the policy sold at time $t-d$ is calculated to match the guarantee, $G_{t,t-d}$, as follows

$$G_{t,t-d} = N_{t,t-d} \cdot E_{t,t-d} \tag{8.20}$$

such that the value of the portfolio of equities and put options is given by

$$A_{t,t-d} = N_{t-1,t-d} \cdot (S_t + O^-_{t,t-d}) = N_{t,t-d} \cdot (S_t + O^+_{t,t-d}) \tag{8.21}$$

where

- S_t is the value of a single unit of the equity index at time t,
- $O^-_{t,t-d}$ is the value at time t of a single option purchased at time $t-1$ (with exercise price $E_{t-1,t-d}$) to match the guarantees of the policy sold at time $t-d$,
- $O^+_{t,t-d}$ is the value at time t of a single option bought at time t (with exercise price $E_{t,t-d}$) to match the guarantees of the policy sold at time $t-d$.

The cost of the initial guarantee, for a policy sold at time t, is equal to the cost of the options that match it:

$$\text{Cost of Initial Guarantee} = N_{t,t} \cdot E_{t,t}. \tag{8.22}$$

The cost of the bonuses declared at time t, for a policy sold d years previously, is equal to the increase in the value of the put options after rebalancing (or equivalently the value of the equities sold to purchase the options):

$$\begin{aligned} \text{Cost of Bonus at time } t &= N_{t,t-d} \cdot O^+_{t,t-d} - N_{t-1,t-d} \cdot O^-_{t,t-d} \\ &= (N_{t-1,t-d} - N_{t,t-d}) \cdot S_t. \end{aligned} \tag{8.23}$$

Recall that in this chapter the policyholder does not buy the options directly, but purchases a unitised with-profits contract from an insurer. If the insurer deducts charges equal to the cost of guarantees given in Equations 8.22 and 8.23, then the policyholder is in exactly the same position as if they had bought the options directly (although the counterparty risk has changed). The insurer can then use the charge to buy matching options or to set up the hedge. If we ignore counterparty risk, transaction costs *etc.*, as we have done previously, then the insurer is exposed to no risk. However, traditionally insurers have not purchased options or used hedging, and as described previously, the with-profits market has probably been too large to make this practicable.

Hence, in the remainder of this section we will assume that the insurer correctly charges for the guarantees each year using the option pricing approach, but they hold these charges in a guarantee account. We will assume that the guarantee account is invested in the risk-free asset. In order to be able to calculate the charges in Equations 8.22 and 8.23, the insurer will have to solve Equations 8.20 and 8.21 each year for each cohort in order to find the number of options and their exercise price which would have been held under the matching options approach.

As the insurer does not hold the matching options, it must make good any losses at maturity if the value of the policyholder's equities are lower than the value of the guarantees. Therefore the loss to the insurer on a policy maturing at the end of its n-year term is:

$$\begin{aligned} \text{Loss on Policy Maturing at time } t &= \max([G_{t,t-n} - N_{t-1,t-n} \cdot S_t], 0) \qquad (8.24) \\ &= \max([N_{t-1,t-n} \cdot E_{t-1,t-n} - N_{t-1,t-n} \cdot S_t], 0). \end{aligned}$$

Hence, the loss to the insurer at maturity is the excess of the value of the options it would have held in the matched position over and above the value of the equities.

The guarantee account builds up as follows. For a particular cohort, a positive charge is received at the beginning of each year (the charge will be zero if the guarantees are not increased) equal to the cost of the additional guarantees. These charges roll up with interest to maturity. If, at maturity, the guarantee is greater than the asset share (reduced for guarantee charges), then the guarantee fund must pay out the difference. However, if the guarantee is less than the asset share at maturity, then no transfers are made to or from the guarantee account, but clearly the account has profited from all the previous charges.

The policyholder is charged annually. If this is reported explicitly to the policyholder as a reduced asset share, then it is clear that a charge has been made and the cost has been fixed.

The insurer holds a single guarantee account in respect of their with-profits business. Therefore, the value of the guarantee account, GA_t, is the value of the guarantee account brought forward from last year with interest at the risk-free rate, plus the cost of the initial guarantees on new business, plus the cost of bonuses declared on existing business, less the cost of any maturity guarantees. The guarantee account is calculated as follows:

$$\begin{aligned} GA_t &= GA_{t-1} \cdot e^{r_f} \\ &\quad + N_{t,t} \cdot E_{t,t} \\ &\quad + \sum_{d=1}^{n-1} N_{t,t-d} \cdot O^{+}_{t,t-d} - N_{t-1,t-d} \cdot O^{-}_{t,t-d} \\ &\quad - \max([N_{t-1,t-n} \cdot E_{t-1,t-n} - N_{t-1,t-n} \cdot S_t], 0). \end{aligned} \tag{8.25}$$

8.3.2 Results

We now look at the results of passing the charges to the guarantee account invested in the risk-free asset. We use the same model as in Section 8.2.1. We use the Wilkie investment model with taxed dividends and option pricing using a constant risk-free rate of 7% and volatility of 20% in the Black-Scholes equation. The simple bonus mechanism is used such that the desired bonus rate z is fixed in advance, and is declared every year that it is affordable.

Each year the insurer sells a 10-year unitised with-profits endowment assurance with single premium of £50. We consider the effect of using a range of different values for the compulsory bonus rate y and the desired bonus rate z. Our aim is to investigate the distribution of the mismatching profits and losses.

Figure 8.16 shows sample paths of the guarantee account for five different simulations. Each simulation has a guaranteed growth rate of zero and a desired bonus rate of 4%.

For the first 9 years there are no maturing policies. Each year the guarantee account will receive charges for any increases in guarantees. We see that the sample path is increasing in each simulation shown in Figure 8.16.

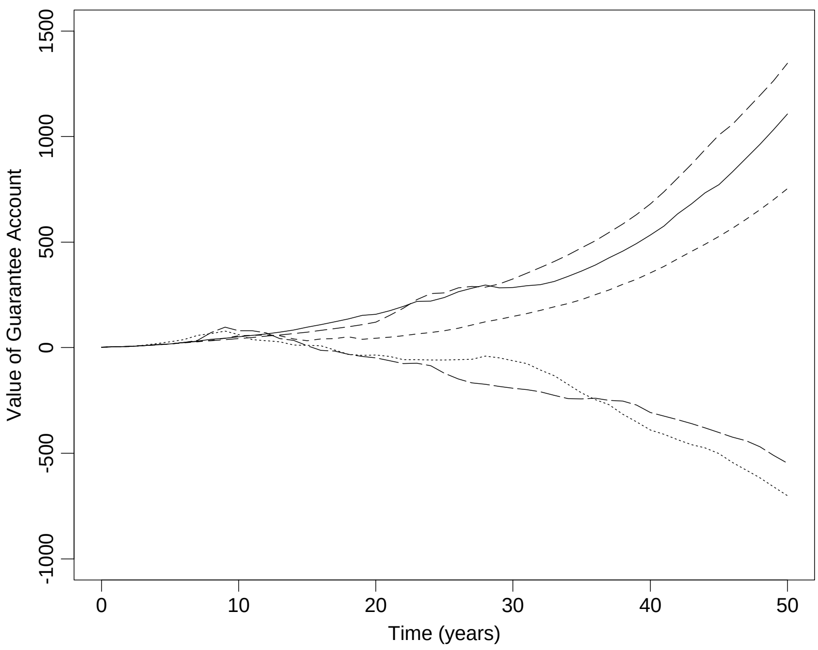

Figure 8.16: Sample Paths of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

However, at time 10 the first cohort of policies matures, and hence the first payment from the guarantee account is made if the value of the maturing policyholder's equities is less than the guarantee. Therefore, the guarantee account includes both the mismatching profits and losses from the first maturing policy and the charges from the unexpired policies. Indeed we see in Figure 8.16 that the guarantee account falls at time 10 in two of the simulations.

In each future year, the guarantee account rolls up with interest, charges deducted from the in-force policies are added to the guarantee account, and the guarantee account makes good any deficit on the maturing cohort.

By time 50 the guarantee account includes the accumulated mismatching profits and losses from the first 41 maturing cohorts, and charges from 10 unexpired cohorts.

The results of 10,000 simulations of the guarantee account GA_t are given below in Tables 8.52 to 8.54.

The first thing we notice about the mean of the guarantee account in Table 8.52 is that it is always positive. This is as expected because the guarantee account, invested in the risk-free asset, has a higher expected return than the hedging portfolio which is invested short in equities and long in the risk-free asset.

Table 8.52: Mean of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time 10	Time 30	Time 50
0.00	0.00	27.34	164.24	693.21
0.00	0.04	54.31	322.35	1355.76
0.00	0.08	103.13	554.89	2301.95
0.02	0.00	57.98	331.12	1386.52
0.02	0.04	116.78	601.29	2472.49
0.02	0.08	188.17	866.14	3489.31
0.04	0.00	119.76	625.04	2576.82
0.04	0.04	232.00	975.86	3852.91
0.06	0.00	261.11	1093.19	4308.50

Secondly, we see that the mean of the guarantee account is larger whenever the bonus rates y and z are larger. This is because if we matched the guarantees, then larger guarantees would require that more equities be sold with a correspondingly greater amount invested in the risk-free asset. Hence, if we actually mismatch, we make greater profits on average.

The mean guarantee account grows through time for two reasons. Firstly, on average the transfer to the account is positive over the lifetime of a contract, and secondly, these mismatching profits roll up at the risk-free rate.

The large positive mean of the guarantee account appears to vindicate the belief that we can pool the investment risk through time. By mismatching, the insurer expects in the long run to have a large guarantee account. This allows the insurer to build up an estate, or to pay dividends to shareholders, or to increase payouts to policyholders. However, so far we have ignored the downside risk that mismatching brings.

The standard deviation of the guarantee account in Table 8.53 rises with the size of the guarantees. In order to hedge the larger guarantees the guarantee account would have to hold a more negative position in equities. Hence, by investing the guarantee account entirely in the risk-free asset the extent of the mismatch increases as the guarantee increases.

Table 8.53: Standard Deviation of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time 10	Time 30	Time 50
0.00	0.00	1.35	42.00	170.97
0.00	0.04	16.62	115.66	468.29
0.00	0.08	30.38	240.45	974.70
0.02	0.00	3.21	97.81	396.70
0.02	0.04	24.43	238.26	969.09
0.02	0.08	34.04	415.55	1684.08
0.04	0.00	7.48	220.22	893.29
0.04	0.04	29.44	455.42	1851.76
0.06	0.00	17.36	494.08	2007.72

The standard deviation also rises through time. As the number of maturing policies increases the possible profits that can be made if the guarantees do not bite also increases. However the potential number of cohorts on which a loss can be made also increases.

Table 8.54 shows the number of simulations where the guarantee account is negative at times 10, 20, 30, 40 and 50. Note that the number of simulations which had a negative guarantee account for at least one year over the 50 year period would be much higher.

There are no examples of negative guarantee accounts at time 10 from the 10,000 simulations. Such examples should indeed be very rare because the loss at maturity on the first cohort would not only have to be larger than the charges from this cohort, but also the charges taken so far from the unexpired contracts.

Initially we see a rising pattern through time of the number of simulations with a negative guarantee account. At time 20 we have had 11 maturing cohorts, but by time 30 we have had 21 maturing cohorts and so there is a higher probability that the guarantee bites on several occasions. In the early years it is likely that two or three losses at maturity will exceed the accumulated charges so far.

However, at later durations the number of negative guarantee accounts falls. We saw in Table 8.52 that on average the charges exceed the costs of the guarantees.

Table 8.54: Number of the 10,000 Simulations which show Negative Guarantee Accounts using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time 10	Time 20	Time 30	Time 40	Time 50
0.00	0.00	0	153	171	155	143
0.00	0.04	0	230	274	249	228
0.00	0.08	0	293	345	330	306
0.02	0.00	0	197	217	196	185
0.02	0.04	0	252	304	285	269
0.02	0.08	0	245	348	351	323
0.04	0.00	0	248	258	241	230
0.04	0.04	0	169	299	319	319
0.06	0.00	0	172	292	314	320

As these excess charges build up through time it requires a larger single loss, or more smaller losses, to produce a negative guarantee account. We see that for larger guarantees, the rising pattern continues for longer as it takes more time to build up a guarantee account which can absorb the potentially larger losses.

At time 50 we see that lower guarantees lead to a lower proportion of negative guarantee accounts. However, even after 50 years and using zero bonus rates, we still have a probability of 1.4% of the account being negative. This shows that even over long periods there is still an appreciable probability of a loss. The next question to be asked is how significant can this loss be?

To gain an idea of the distribution of profits and losses, we will consider the quantiles of the guarantee account. At each time we will order the simulated guarantee accounts from the smallest to the largest. We will calculate $QGA_t(q)$, the qth quantile of the guarantee account at time t such that

$$\text{Number of simulations where } [GA_t \leq QGA_t(q)] = \frac{q}{10,000} \cdot 100.$$

Hence for example, we can say that 10% of the simulated guarantee accounts at time 20 are no greater than the 10th quantile, $QGA_{20}(10)$. Similarly, 1% of the simulated guarantee accounts at time 40 are at least as large as the 99th quantile,

$QGA_{40}(99.01)$.

A range of quantiles at certain times for selected value of y and z are given in Tables 8.55 to 8.59, and the corresponding Figures 8.17 to 8.21. Tables 8.55, 8.56, and 8.57 show the effect of increasing the desired bonus rate while keeping the guaranteed growth rate fixed at zero. Tables 8.56, 8.58, and 8.59 show the effect of increasing the guaranteed growth rate while keeping the desired bonus rate fixed at 4%. Figures 8.17 to 8.21 all use the same scale to aid comparison between the different bonus rates.

Table 8.55: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.00	0	2	2	2	2	2	2	2	2	2
0.00	0.00	1	4	4	4	4	4	4	4	4	4
0.00	0.00	2	6	6	6	6	6	6	6	6	6
0.00	0.00	3	8	8	8	8	8	8	8	8	8
0.00	0.00	4	10	10	10	10	10	10	10	10	10
0.00	0.00	5	12	12	12	12	12	12	12	12	12
0.00	0.00	10	1	8	21	27	27	27	27	27	27
0.00	0.00	15	-128	-66	1	49	49	49	49	49	49
0.00	0.00	20	-221	-123	-20	69	78	78	78	78	78
0.00	0.00	25	-300	-164	-35	98	120	120	120	120	120
0.00	0.00	30	-411	-243	-42	140	178	178	178	178	178
0.00	0.00	35	-571	-331	-51	198	259	259	259	259	259
0.00	0.00	40	-791	-462	-63	281	374	374	374	374	374
0.00	0.00	45	-1103	-637	-86	401	534	534	534	534	534
0.00	0.00	50	-1537	-884	-115	568	759	759	759	759	759

Guaranteed Growth Rate 0%, Desired Bonus Rate 0%

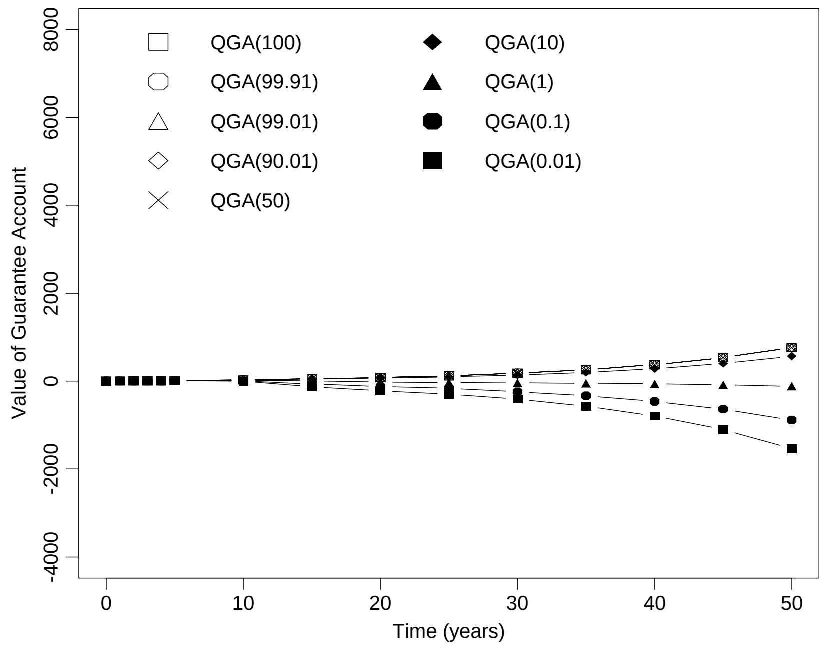

Figure 8.17: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Note that each table of quantiles shows the evolution through time of a single bonus strategy. The quantile at time t will not necessarily be derived from the same simulation as that quantile at any other time. It is possible for a particular simulation to have a number of years of good investment returns, placing it in the upper quantiles, but then to experience an investment crash and so move into the lower quantiles. Figure 8.16 shows a number of examples where the sample paths of the guarantee account cross.

The tables of quantiles show very clearly the trends found in Tables 8.52 to 8.54. Higher values for the bonuses y and z lead to a higher value of the median guarantee account, represented by the 50th quantile. Higher bonuses also lead to a greater spread of values of the guarantee account, producing both larger potential profits, and larger potential losses.

We see for the first 10 years that all the guarantee accounts are positive. However, at later durations the 1st quantile and below becomes more negative through time, whilst the 10th quantile and above becomes more positive through time.

Table 8.55 shows an unusual pattern. At any given time, many of the quantiles

Table 8.56: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
			Quantiles of the Guarantee Account, QGA_t								
0.00	0.00	0	2	2	2	2	2	2	2	2	2
0.00	0.04	1	4	4	4	4	4	4	4	5	5
0.00	0.04	2	6	6	6	6	6	7	8	8	10
0.00	0.04	3	8	8	8	9	9	11	13	14	20
0.00	0.04	4	11	11	11	12	13	16	20	25	38
0.00	0.04	5	14	14	14	15	17	22	31	40	46
0.00	0.04	10	32	33	35	38	49	79	104	122	135
0.00	0.04	15	-129	-69	3	69	92	123	151	173	197
0.00	0.04	20	-270	-170	-52	106	150	194	239	280	332
0.00	0.04	25	-359	-266	-96	142	229	294	357	417	463
0.00	0.04	30	-507	-379	-131	196	340	433	522	594	676
0.00	0.04	35	-767	-511	-172	280	492	625	748	851	963
0.00	0.04	40	-1060	-720	-228	402	707	896	1066	1235	1376
0.00	0.04	45	-1471	-986	-305	574	1009	1275	1517	1739	1971
0.00	0.04	50	-1998	-1357	-417	820	1430	1808	2142	2460	2787

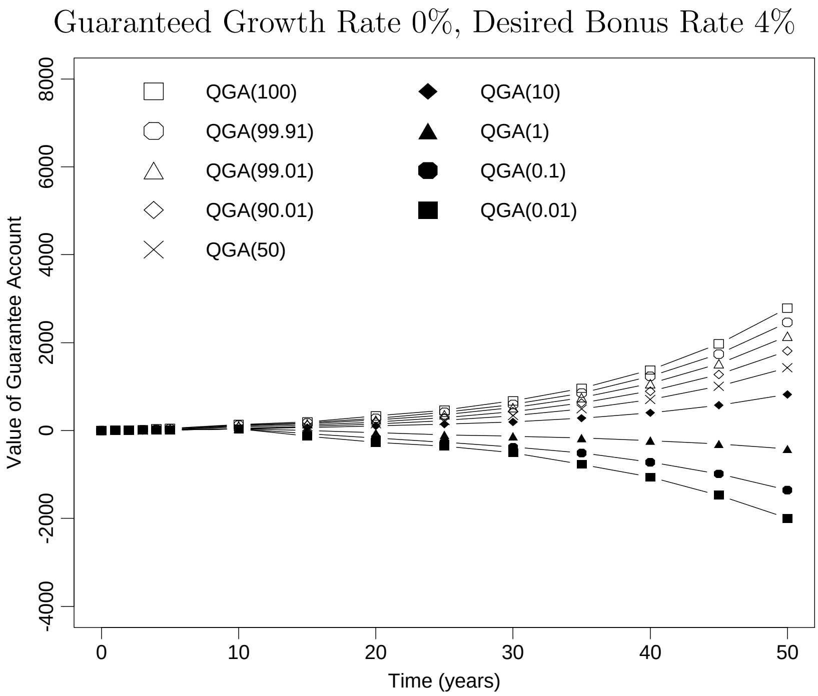

Figure 8.18: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Table 8.57: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.00	0	2	2	2	2	2	2	2	2	2
0.00	0.08	1	4	4	4	4	4	5	5	6	7
0.00	0.08	2	6	6	6	7	7	9	11	13	17
0.00	0.08	3	9	9	9	10	12	15	20	28	32
0.00	0.08	4	12	12	13	14	18	26	36	45	49
0.00	0.08	5	15	16	17	20	26	41	55	66	75
0.00	0.08	10	26	41	47	61	104	142	171	191	211
0.00	0.08	15	-164	-79	-8	103	176	225	260	283	307
0.00	0.08	20	-357	-203	-84	117	275	366	434	487	528
0.00	0.08	25	-430	-326	-140	155	413	551	641	695	773
0.00	0.08	30	-758	-479	-198	224	603	810	941	1029	1083
0.00	0.08	35	-1123	-648	-276	323	867	1170	1360	1489	1550
0.00	0.08	40	-1545	-930	-378	470	1237	1676	1950	2118	2291
0.00	0.08	45	-2131	-1284	-472	680	1757	2379	2772	3034	3207
0.00	0.08	50	-2917	-1751	-642	968	2484	3366	3901	4286	4592

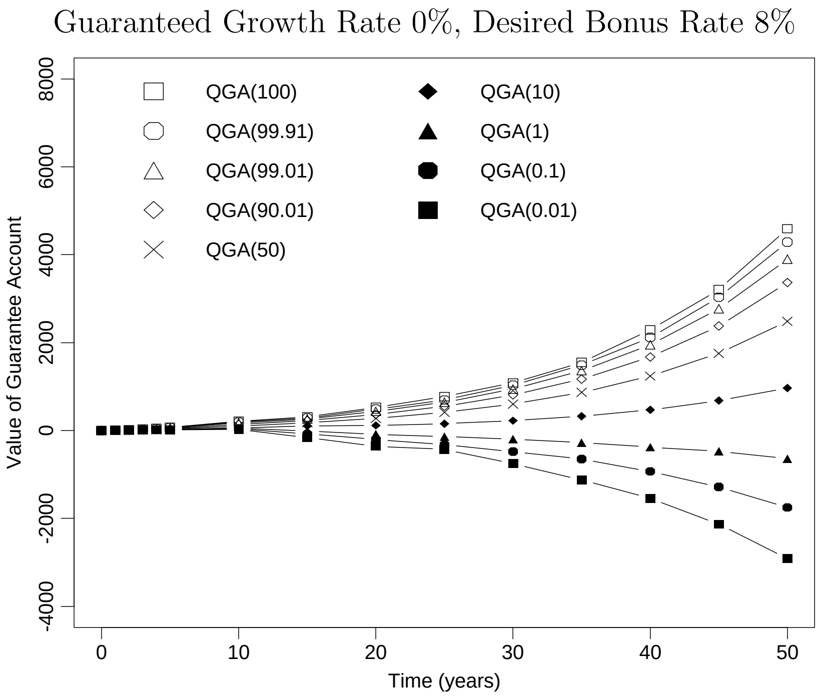

Figure 8.19: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Table 8.58: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.00	0	4	4	4	4	4	4	4	4	4
0.02	0.04	1	8	8	8	8	8	9	9	10	11
0.02	0.04	2	12	13	13	13	14	15	17	19	23
0.02	0.04	3	18	18	18	19	20	23	28	34	39
0.02	0.04	4	23	24	24	26	29	35	44	53	67
0.02	0.04	5	30	30	31	33	38	50	64	73	86
0.02	0.04	10	54	71	77	86	115	150	176	194	227
0.02	0.04	15	-153	-68	7	130	194	236	268	292	314
0.02	0.04	20	-357	-218	-69	144	305	377	438	479	553
0.02	0.04	25	-448	-352	-138	193	457	568	645	706	728
0.02	0.04	30	-755	-487	-192	273	663	835	942	1025	1070
0.02	0.04	35	-1141	-667	-268	387	952	1204	1363	1486	1541
0.02	0.04	40	-1573	-1023	-365	548	1356	1723	1952	2121	2226
0.02	0.04	45	-2167	-1349	-475	792	1924	2443	2768	3019	3172
0.02	0.04	50	-2966	-1878	-647	1123	2711	3457	3903	4266	4492

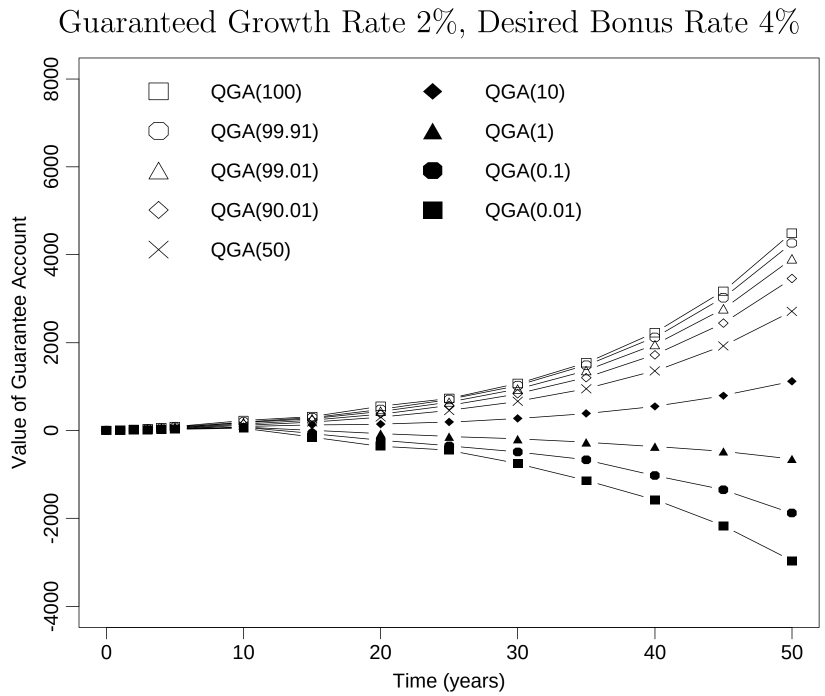

Figure 8.20: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Table 8.59: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
			Quantiles of the Guarantee Account, QGA_t								
0.04	0.04	0	8	8	8	8	8	8	8	8	8
0.04	0.04	1	16	16	17	17	17	18	19	21	23
0.04	0.04	2	26	26	27	28	29	32	37	43	46
0.04	0.04	3	37	38	39	41	44	51	60	67	70
0.04	0.04	4	50	51	52	56	63	75	86	94	110
0.04	0.04	5	64	65	67	73	87	103	116	126	133
0.04	0.04	10	129	147	164	193	232	269	299	325	343
0.04	0.04	15	-128	-16	59	192	363	441	485	517	530
0.04	0.04	20	-290	-200	-40	198	522	703	786	840	886
0.04	0.04	25	-427	-346	-135	249	737	1047	1168	1241	1324
0.04	0.04	30	-837	-529	-219	339	1042	1520	1721	1832	1902
0.04	0.04	35	-1272	-828	-348	453	1469	2170	2474	2627	2726
0.04	0.04	40	-1765	-1187	-487	634	2070	3069	3520	3759	3900
0.04	0.04	45	-2380	-1627	-684	897	2913	4325	4969	5309	5547
0.04	0.04	50	-3397	-2251	-951	1270	4098	6080	7001	7527	7791

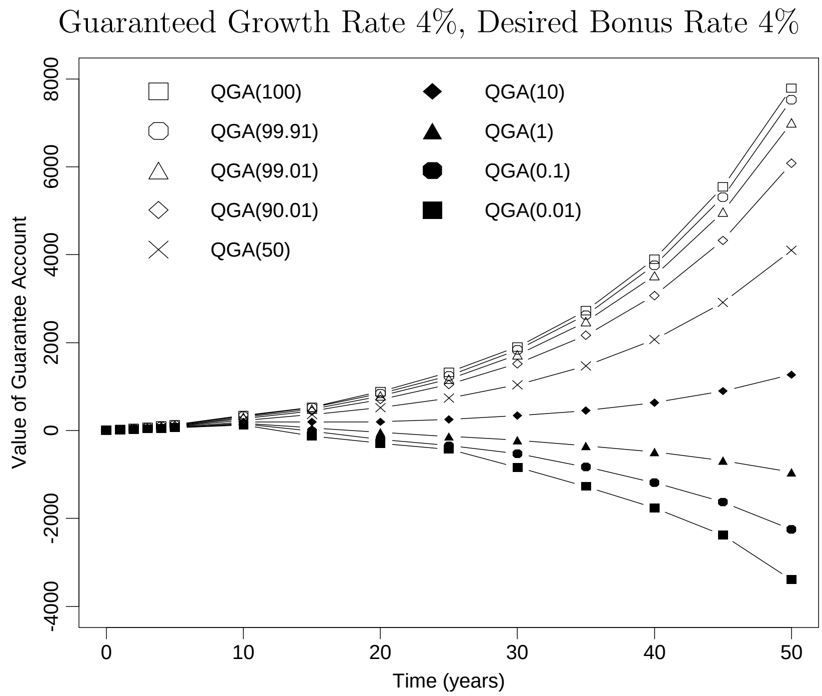

Figure 8.21: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

are identical. For example, at time 50 every simulation from the median onwards has a guarantee account of £759. In fact, at times zero to 9 inclusive, all the simulations have identical accounts. This is because the two bonus rates are zero. Hence options need only be bought at the outset of the contract and held to maturity. The cost of the option is always the same at outset because the ratio of the guarantee to the assets is always the same. (If bonuses are declared later in the term of the policy then the option cost will be different. Firstly, the size of the bonus declared may be different. Secondly, the increase in the equity price from outset will change the ratio of the guarantee to the value of the assets.) Hence, the guarantee accounts only differ due to the different number of payouts where the guarantee bites and the size of the corresponding losses. As no policies mature before time 10 there are no differences before then. The guarantee is relatively small and bites rarely, hence many simulations give the maximum size of account.

Of particular interest are the quantiles that show negative guarantee accounts. We would hope that by running the portfolio through time we would eventually benefit from the higher average return on equities. Indeed, in all the cases we have considered, the 10th quantile increases through time, showing that the insurer has a probability of over 90% that the guarantee account is positive at any given point in time.

However, the insurer cannot be 99% sure of a positive account even after 50 years in the cases shown. In fact the 1st quantile becomes more negative with time.

We can see two problems with this guarantee account approach. Most worrying is the occurrence of large negative accounts even after 50 years. For example, the 1st quantile of the accumulated losses after 50 years using a guaranteed growth rate of 4% and desired bonus rate of 4% is £951, which is 19 times the premium of a single cohort of business.

The second problem is the potential for very large guarantee accounts. Even in the case when no bonuses are declared at all, the median guarantee account grows to £759 after 50 years. The regulators will require the insurer to hold additional capital as a mismatching reserve to reduce the probability of negative accounts. If this additional capital is being provided by shareholders, then the positive accounts are simply the profits for the risk of potentially losing the capital in the mismatching reserve. If the capital is provided from a free estate within the with-profits fund then our second problem becomes apparent — how do we return the free estate to the policyholders if it becomes unnecessarily large? This is precisely the orphan estate problem. To provide guarantees (and also to smooth payouts) the with-profits fund must have a free estate. However, what constitutes a reasonable size for the estate? Which policyholders should benefit from the free estate and how? We will look at the effects of distributing the estate in Section 9.2.

Despite these two problems, the high probability of making mismatching profits still makes this approach of great interest to insurers.

However, Tables 8.55 to 8.59 must be read with caution. It is not sufficient to have a positive guarantee account. Recall that the account includes charges from unexpired business. Hence, in fact we need a sufficiently positive account to cover potential losses on the existing business. We now look at the free estate after deduction of the cost of options to match the guarantees on unexpired business.

8.4 Free Assets

So far we have considered only the absolute size of the guarantee accounts. It would be more useful to express the accounts in some way in relation to the amount of existing business. This will be particularly useful when we introduce new business growth in Section 9.1.

One such useful measure is the free asset ratio. The free assets are the excess of the assets over the liabilities, and the free asset ratio is the ratio of the free assets to the liabilities. There is no single method or basis for the calculation of the assets or liabilities to use in the free asset ratio (although its calculation in the statutory returns is currently tightly defined). We will consider a market value for assets and liabilities with no additional reserves for mismatching, resilience tests or solvency margins. We will consider in Section 9.2 how large the free asset ratio needs to be now to give an acceptably low probability of negative free assets in the future.

If the insurer actually buys the options which match the guarantees then the assets and liabilities are exactly equal and the free assets are zero.

From now on we will consider the case where the charges are invested in the risk-free asset within the guarantee account.

In our model office the insurer has two assets: the unit fund invested in equities, and the guarantee account invested in the risk-free rate.

The insurer's liabilities are unitised with-profits policies which pay out the greater of the guarantee, or the asset share where charges have been deducted equal to the cost of options which match the guarantee.

The guarantee account consists of the accumulated profits from matured policies plus the charges to date on the existing business. The unit fund belongs entirely to the policyholders. Hence the free assets are smaller than the guarantee account, because in addition to the portfolio of units, the liability includes the value of the option to receive the guarantee. If we set up a reserve for the guarantees exactly equal to the current cost of the options and deduct this from the guarantee account then we have a measure of the free assets as follows:

$$FAR_t = \frac{GA_t - \sum_{d=0}^{n-1}(N_{t,t-d} \cdot O^{+}_{t,t-d})}{\sum_{d=0}^{n-1} A_{t,t-d}} \tag{8.26}$$

where

- n is the term of the policies (in this case 10 years),
- $N_{t,t-d}$ is the number of options required to match the guarantees of the policies with duration d and sold in year $t-d$, which can be derived from Equations 8.20 and 8.21,
- $O^{+}_{t,t-d}$ is their price at time t,
- $A_{t,t-d}$ is the value of the asset share (*i.e.* the value of the portfolio of equities and put options which would match the liability) for policies with duration d and sold in year $t-d$, given by Equation 8.21.

Note that the asset share shown in the denominator is the value of the assets which would have been held if the insurer had matched with options. It is not the value of the actual assets held.

In effect the free asset ratio calculates the mismatching profits the insurer has made as a proportion of the liabilities, if the insurer were to immediately match the remaining guarantees with options.

8.4.1 Results

We now look at the results of simulating the free asset ratios. We use the same model as in the previous sections of this chapter. We use the Wilkie investment model with taxed dividends and option pricing using a constant risk-free rate of 7% and volatility of 20% in the Black-Scholes equation. The simple bonus mechanism is used such that the desired bonus rate z is fixed in advance, and is declared in every year in which it is affordable.

Figure 8.22 shows sample paths of the free asset ratios for the same five simulations as were shown in Figure 8.16.

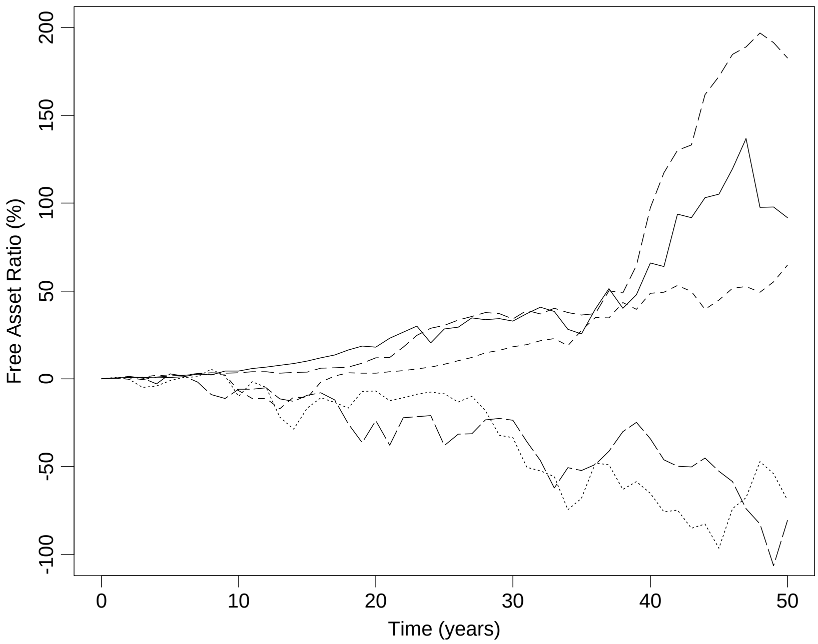

Figure 8.22: Sample Paths of the Free Asset Ratios using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

We can see that the sample paths of the free asset ratio are much more volatile than the sample paths of the guarantee account. The free asset ratio is the ratio of the free assets to the total asset shares. Hence, the free asset ratio has both a stochastic denominator and numerator which causes its greater volatility.

Again we can get an idea of the distribution of the free asset ratios by considering the quantiles. At each time we will order the simulated free asset ratios from the smallest to the largest. We will calculate $QFAR_t(q)$, the q th quantile of the free asset ratio at time t such that

$$\text{Number of simulations where } [FAR_t \leq QFAR_t(q)] = \frac{q}{10,000} \cdot 100.$$

A range of quantiles at certain times for selected value of y and z are given in Tables 8.60 to 8.64, and the corresponding Figures 8.23 to 8.27. Tables 8.60, 8.61, and 8.62 show the effect of increasing the desired bonus rate while keeping the guaranteed growth rate fixed at zero. Tables 8.61, 8.63, and 8.64 show the effect of increasing the guaranteed growth rate while keeping the desired bonus rate fixed at 4%. Figures 8.23 to 8.27 all use the same scale to aid comparison between the different bonus rates.

Table 8.60: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	Quantiles of the Free Asset Ratio, FAR_t (%) 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.00	0	0	0	0	0	0	0	0	0	0
0.00	0.00	1	-7	-4	-2	-1	0	1	1	1	1
0.00	0.00	2	-8	-6	-4	-1	1	1	1	1	1
0.00	0.00	3	-10	-8	-4	-1	1	2	2	2	2
0.00	0.00	4	-14	-10	-5	-1	1	2	2	2	2
0.00	0.00	5	-19	-11	-6	-1	1	2	2	2	2
0.00	0.00	10	-28	-23	-8	1	2	3	3	3	3
0.00	0.00	15	-55	-26	-8	3	5	6	6	6	6
0.00	0.00	20	-54	-29	-8	5	8	10	11	11	11
0.00	0.00	25	-51	-33	-7	8	13	17	18	19	19
0.00	0.00	30	-90	-38	-8	12	20	27	30	30	31
0.00	0.00	35	-125	-48	-8	17	30	42	46	47	48
0.00	0.00	40	-132	-67	-12	26	43	62	70	73	73
0.00	0.00	45	-176	-90	-12	37	62	90	105	109	111
0.00	0.00	50	-241	-131	-18	52	88	130	153	161	165

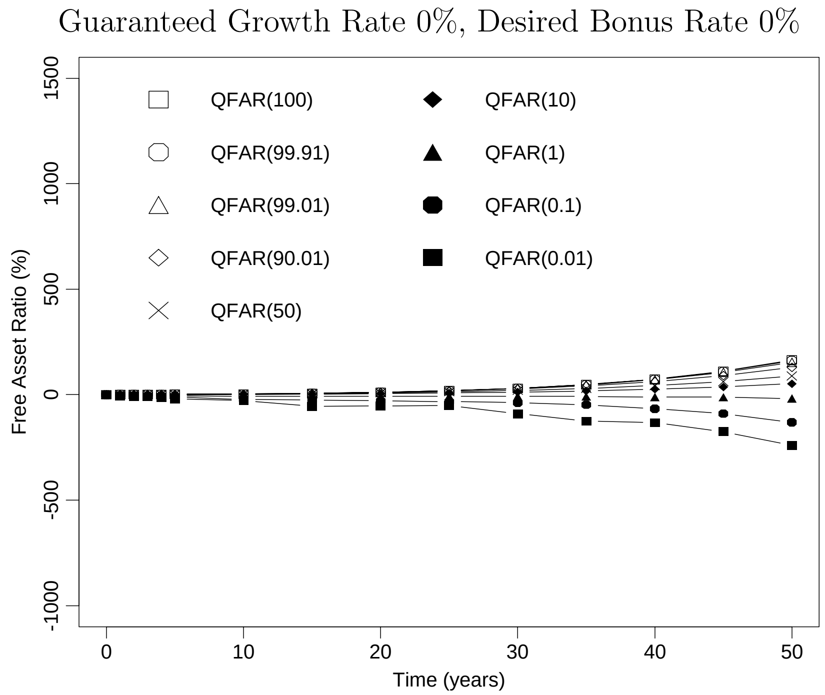

Figure 8.23: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Naturally we see that the free asset ratios in Tables 8.60 to 8.64 follow a similar pattern to the guarantee accounts in Tables 8.55 to 8.59. For example, the upper (lower) quantiles increase (decrease) with time and the size of bonuses.

However, recall that the free asset ratio has both a stochastic denominator and numerator. Hence the qth quantile of the free asset ratio may refer to a different simulation than the qth quantile of the guarantee account. For example, the largest free asset ratio may not occur when the guarantee account is largest, but may occur when the guarantee account is quite large in a simulation with a very low asset share. A comparison of Figures 8.22 and 8.16 show a number of occasions where one simulation has a greater guarantee account than another simulation, but a smaller free asset ratio.

There are some differences in the pattern of quantiles for the free asset ratio compared to the pattern of quantiles for the guarantee account. The free asset ratio is zero at time zero, but the guarantee account is positive. This is because at time zero an amount exactly equal to the cost of the guarantees is transfered to the guarantee account. If the insurer actually bought the options then the free asset

Table 8.61: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	Quantiles of the Free Asset Ratio, FAR_t (%) 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.04	0	0	0	0	0	0	0	0	0	0
0.00	0.04	1	-7	-4	-2	-1	0	1	1	1	1
0.00	0.04	2	-8	-6	-4	-1	1	1	2	2	2
0.00	0.04	3	-11	-9	-5	-1	1	2	2	2	2
0.00	0.04	4	-16	-12	-6	-1	1	2	2	3	3
0.00	0.04	5	-21	-13	-7	-1	2	2	3	3	4
0.00	0.04	10	-34	-29	-15	0	3	6	8	10	11
0.00	0.04	15	-59	-37	-19	3	8	12	15	17	19
0.00	0.04	20	-70	-41	-21	7	15	22	30	36	43
0.00	0.04	25	-89	-51	-20	12	25	37	47	55	67
0.00	0.04	30	-124	-65	-20	18	39	57	72	87	106
0.00	0.04	35	-141	-79	-26	27	58	86	110	131	142
0.00	0.04	40	-179	-104	-33	41	84	128	163	188	224
0.00	0.04	45	-236	-150	-43	59	122	186	238	279	307
0.00	0.04	50	-438	-206	-56	86	173	266	344	406	472

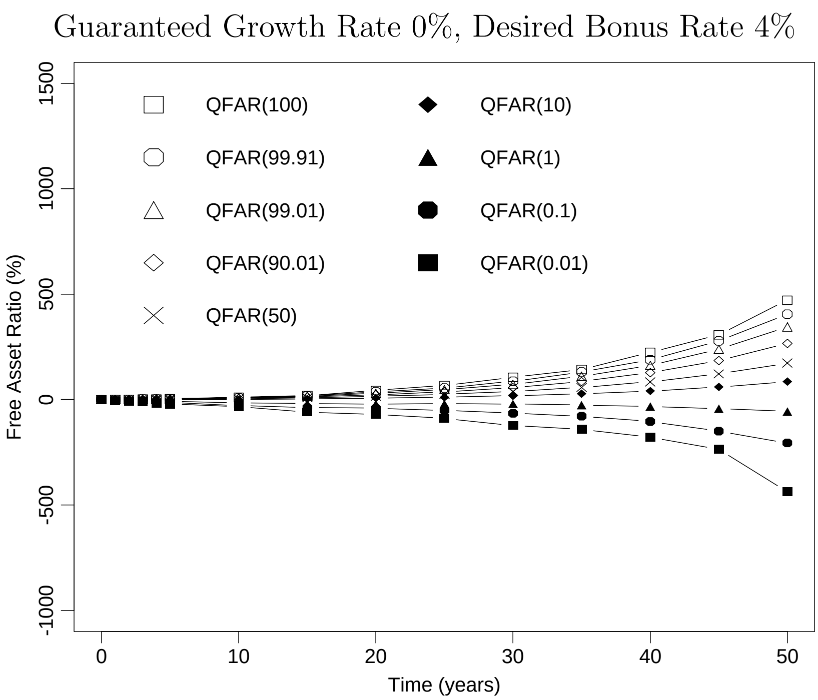

Figure 8.24: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Table 8.62: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	Quantiles of the Free Asset Ratio, FAR_t (%) 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.08	0	0	0	0	0	0	0	0	0	0
0.00	0.08	1	-7	-4	-2	-1	0	1	1	1	1
0.00	0.08	2	-9	-6	-4	-1	1	2	2	2	2
0.00	0.08	3	-12	-10	-5	-1	1	2	2	3	3
0.00	0.08	4	-17	-13	-7	-2	2	3	3	4	5
0.00	0.08	5	-22	-15	-9	-2	2	3	4	5	6
0.00	0.08	10	-38	-32	-20	-4	5	10	13	15	17
0.00	0.08	15	-64	-48	-28	-1	14	21	26	28	31
0.00	0.08	20	-79	-58	-33	4	26	40	50	57	62
0.00	0.08	25	-127	-75	-35	10	43	66	80	89	105
0.00	0.08	30	-179	-98	-38	19	66	102	127	143	159
0.00	0.08	35	-188	-117	-47	31	99	154	192	217	247
0.00	0.08	40	-237	-148	-60	49	144	227	285	326	352
0.00	0.08	45	-316	-202	-73	73	208	333	415	472	516
0.00	0.08	50	-574	-269	-97	108	296	477	603	685	784

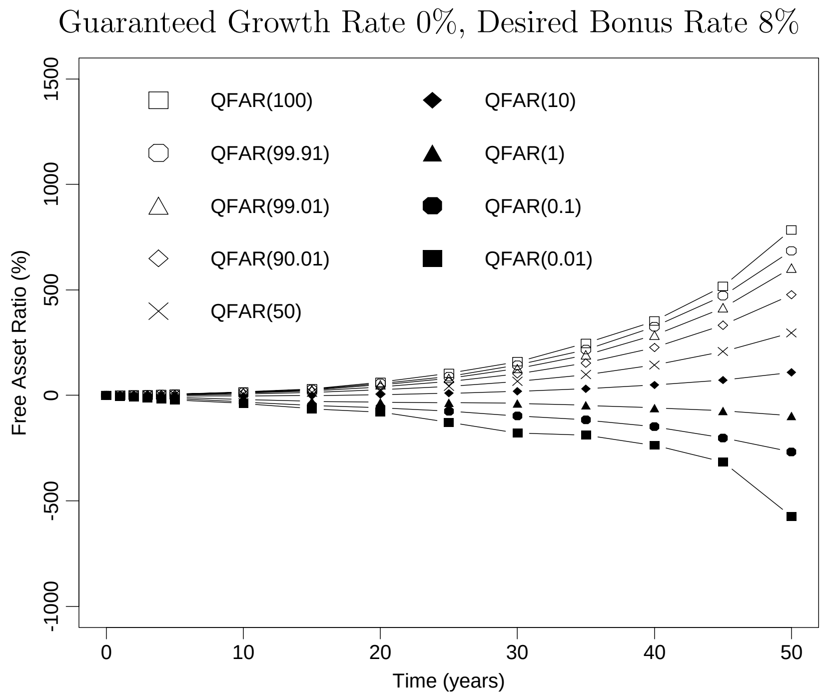

Figure 8.25: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Table 8.63: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
			Quantiles of the Free Asset Ratio, FAR_t (%)								
0.02	0.04	0	0	0	0	0	0	0	0	0	0
0.02	0.04	1	-10	-6	-4	-2	1	2	2	2	2
0.02	0.04	2	-12	-9	-6	-2	1	3	3	3	3
0.02	0.04	3	-16	-13	-8	-2	2	3	4	4	4
0.02	0.04	4	-20	-16	-10	-3	2	4	5	5	6
0.02	0.04	5	-26	-18	-11	-3	3	5	6	6	7
0.02	0.04	10	-38	-33	-21	-3	7	11	14	16	17
0.02	0.04	15	-67	-47	-28	0	16	23	26	29	32
0.02	0.04	20	-87	-56	-33	6	29	42	51	57	64
0.02	0.04	25	-121	-71	-36	14	48	68	82	89	97
0.02	0.04	30	-174	-103	-38	24	74	107	128	143	158
0.02	0.04	35	-188	-113	-47	39	109	160	193	216	233
0.02	0.04	40	-240	-157	-57	59	158	236	288	321	341
0.02	0.04	45	-332	-214	-73	87	228	345	417	467	508
0.02	0.04	50	-572	-259	-98	128	324	494	599	692	731

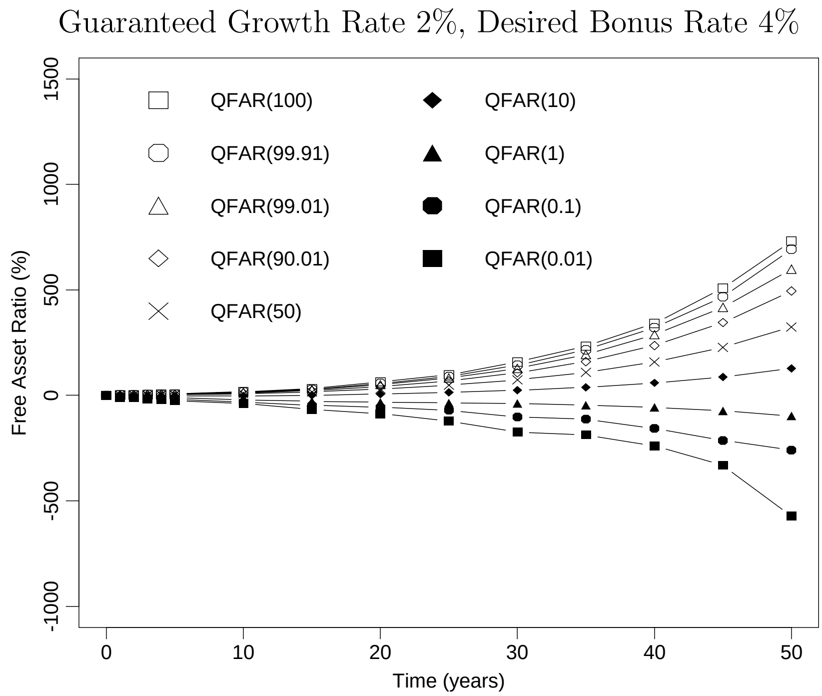

Figure 8.26: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

Table 8.64: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
			Quantiles of the Free Asset Ratio, FAR_t (%)								
0.04	0.04	0	0	0	0	0	0	0	0	0	0
0.04	0.04	1	-13	-8	-6	-3	1	3	4	5	5
0.04	0.04	2	-15	-12	-9	-3	2	5	6	7	7
0.04	0.04	3	-19	-16	-11	-4	3	6	8	8	9
0.04	0.04	4	-23	-20	-13	-5	4	8	9	10	11
0.04	0.04	5	-27	-21	-15	-5	5	9	11	12	13
0.04	0.04	10	-39	-34	-23	-7	12	19	23	25	27
0.04	0.04	15	-67	-50	-34	-5	27	40	45	48	50
0.04	0.04	20	-97	-66	-43	1	47	74	86	93	97
0.04	0.04	25	-131	-91	-49	10	75	120	139	151	159
0.04	0.04	30	-191	-123	-60	22	115	185	220	241	250
0.04	0.04	35	-218	-155	-74	38	170	276	329	354	374
0.04	0.04	40	-284	-182	-97	62	249	403	483	527	554
0.04	0.04	45	-433	-276	-120	96	360	581	703	769	800
0.04	0.04	50	-608	-379	-159	141	514	831	1010	1114	1201

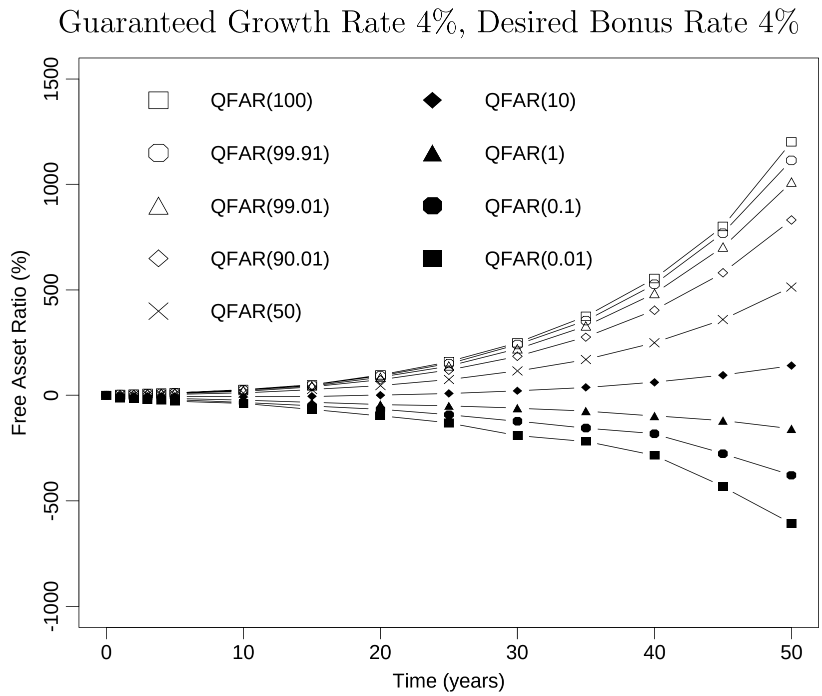

Figure 8.27: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Constant Risk-Free Rate, and the Fixed Bonus Strategy

ratio would remain at zero for ever. It is the mismatch of assets to liabilities which causes the later non-zero free asset ratios.

We also see that the 10th quantiles are positive for the guarantee account, but are negative in the early years for the free asset ratio because we have deducted the cost of the options to back existing business. For example, up to time 10 the guarantee account is always positive because no policies have yet matured and so no transfers out of the guarantee fund have yet been required. However, the value of the guarantees on existing business may exceed the charges to date and so the free assets will be negative in some cases.

We can see that regardless of the size of the bonuses, the free assets can grow to be many times larger than the asset share. For example, even in Table 8.60 which has no guaranteed growth and no reversionary bonus, we see that the median free asset ratio is 88% by time 50 *i.e.* the insurer's assets are almost twice their liabilities (represented by the asset share). As all the free assets are invested in the risk-free asset within the guarantee account, the insurer will be able to meet its liabilities in the future unless shares become totally worthless. This means of course that the policyholders have very high security. This gives the insurer the confidence to continue to mismatch in pursuit of higher profits.

We see that in the higher quantiles very high free asset ratios are possible. For example, in Table 8.62 which has no guaranteed growth and a desired reversionary bonus rate of 8%, the 99th quantile is 603% at time 50, meaning that the total assets are seven times larger than the liabilities. Such high free assets are excessive and the insurer should either return part of the free assets to the owners (either shareholders or policyholders depending on the type of company) or consider increasing the business they write (we consider new business growth in Section 9.1).

Even if free assets are positive they should be required to be greater than some level to provide a cushion against possible adverse events. We will discuss what a sensible cushion might be in Section 9.2.

The most worrying feature of mismatching the guarantees is not the possibility of excessive free assets, but the possibility of insolvency. We see in all our examples that the free assets are positive by time 20 in over 90% of simulations. However, for example, in Table 8.63 which has guaranteed growth of 2% and a desired reversionary bonus rate of 4%, the 1st quantile is -98% at time 50, meaning that the accumulated mismatching losses are almost as great as the liabilities.

Recall that in our model the insurer begins at time zero with no free assets. In practice the insurer would only sell UWP business if it had some additional capital outside the guarantee account. Hence, the insurer should be able to run a small negative guarantee account for some time in the hope that future profits would make good the losses. This is precisely what we see in the 10th quantile. However, in the worst scenarios it is clear that even an initially well capitalised insurer would be forced at some point to realise its losses and buy matching put options to avoid insolvency.

8.5 Risk-Free Rate from the Wilkie Model

In Sections 8.2, 8.3, and 8.4 we have considered a portfolio of UWP policies and calculated the policyholder payouts, the guarantee account, and the free assets, assuming that the risk-free rate was constant. In this section we will recalculate these results using a more realistic risk-free rate derived from the Wilkie model in the same way as in Chapter 6.

8.5.1 Results

First we look at the policyholder payouts in the same way as in Section 8.2. We obtain the following results using the Wilkie investment model with taxed dividends and option pricing using a risk-free rate derived from a yield curve fitted to the Wilkie model and volatility of 20% in the Black-Scholes equation. The simple bonus mechanism is used such that the desired bonus rate z is fixed in advance, and is declared in every year in which it is affordable.

Summary statistics are shown in Tables 8.65 to 8.70.

As in Section 6.3, I have considered only a guaranteed growth rate y of 0% and 2% because it is possible that the stochastic risk-free rate of return is below the guaranteed growth rate. In fact we find that the guaranteed growth rate is

Table 8.65: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

Start Year 0

y	z	Maximum Guarantee	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		0.00	143.91	67.18	0.00	0.00
0.00	0.00	50.00	140.29	65.22	50.00	0.00
0.00	0.04	71.17	136.30	64.69	70.67	1.61
0.00	0.08	99.95	129.30	60.91	92.84	9.33
0.02	0.00	60.95	136.39	62.63	60.95	0.00
0.02	0.04	86.75	128.65	59.39	84.70	3.70
R-F		105.83	105.83	0.00	105.83	0.00

Table 8.66: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

Start Year 0

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	187	162	162	162	3420	10000
0.00	0.04	8917	1511	763	763	861	3852	10000
0.00	0.08	5483	4247	1732	1732	2718	5145	10000
0.02	0.00	10000	578	432	432	432	3686	10000
0.02	0.04	7029	3379	1449	1449	1789	4791	10000

Table 8.67: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy
Start Year 20

y	z	Maximum Guarantee	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		0.00	148.78	83.07	0.00	0.00
0.00	0.00	50.00	145.00	81.16	50.00	0.00
0.00	0.04	71.17	140.94	81.32	70.01	2.86
0.00	0.08	99.95	135.55	79.11	90.61	11.91
0.02	0.00	60.95	141.12	78.60	60.95	0.06
0.02	0.04	86.75	134.55	76.91	83.28	5.58
R-F		NA	110.33	21.77	110.33	21.77

Table 8.68: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy
Start Year 20

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	364	308	308	308	3821	10000
0.00	0.04	8167	2126	1022	1022	1244	4363	9984
0.00	0.08	5274	4398	1880	1880	3072	5072	8489
0.02	0.00	9999	994	733	733	733	4112	10000
0.02	0.04	6397	3872	1709	1709	2180	5129	9627

Table 8.69: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

Start Year 40

y	z	Maximum Guarantee	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		0.00	148.87	83.17	0.00	0.00
0.00	0.00	50.00	145.07	81.14	50.00	0.00
0.00	0.04	71.17	141.05	81.22	70.03	2.83
0.00	0.08	99.95	135.82	79.05	90.62	11.89
0.02	0.00	60.95	141.15	78.50	60.95	0.10
0.02	0.04	86.75	134.60	76.80	83.26	5.59
R-F		NA	111.20	23.63	111.20	23.63

Table 8.70: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

Start Year 40

y	z	Maximum Gtee is Achieved	Option Exercised	UWP > UL	Ach Gtee > UL	Max Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	387	333	333	333	3845	10000
0.00	0.04	8200	2099	1041	1041	1255	4419	9965
0.00	0.08	5268	4421	1914	1914	3056	5138	8454
0.02	0.00	9997	982	720	720	720	4119	10000
0.02	0.04	6339	3880	1759	1759	2202	5166	9539

unaffordable when $y = 2\%$ on 46 occasions. This problem occurs in 22 out of 10,000 simulations, sometimes for only one cohort of business, but for one simulation it occurs for as many as 8 cohorts. Whenever the guaranteed growth rate of 2% is unaffordable, we set the guaranteed growth rate to 1%. In practice the insurer would have the right to change the guaranteed bonus rate on new business, and possibly even for additional premiums to existing policies, if interest rates fell.

The problem of unaffordable guaranteed growth rates is more common in this chapter because the simulations are run for longer and so are more likely to move substantially away from the neutral initial starting conditions. In this chapter we run the model for 50 years, whereas in Chapter 6 we projected for only 20 years.

We can compare Tables 8.44 to 8.51 where the risk-free rate is fixed with Tables 8.65 to 8.70 where the risk-free rate is stochastic and derived from the Wilkie Model.

As in Section 6.3 we find that the mean and standard deviation of payout are always higher under the stochastic risk-free rate. This is because the stochastic risk-free rate has both higher mean and variance than the deterministic rate.

Again as in Section 6.3, the mean achieved guarantee is little changed by the switch to a stochastic risk-free rate. The standard deviation of the achieved guarantee is also, as in Section 6.3, generally larger with a stochastic risk-free rate. However for policies starting at time 0 the standard deviation may actually be lower, perhaps because the maximum guarantee is achieved more often at time 0.

We see that in all cases the maximum guarantee is achieved more often when the stochastic interest rate is used. We commented in Section 6.3 that the options were in fact slightly cheaper on average using the stochastic risk-free rate because the average stochastic rate was more than the fixed deterministic rate of 6.77%. Hence the desired guarantee is more likely to be affordable. However in Section 6.3 we also commented that the added variability of the stochastic risk-free rate would cause the maximum guarantee to be affordable less often. In Section 6.3 we found that the effect of the greater variability dominated for lower bonuses, but perhaps because of the shorter term of policies considered in this chapter, the result is not repeated here.

We find the same result as in Section 6.3 that the option is generally exercised less often under the stochastic risk-free case. As the options are cheaper on average, we can retain a larger investment in shares, and so are more likely to have a share value in excess of the guarantee at maturity. In Section 6.3 we found that for low bonus rates it is possible that the option is exercised more often under stochastic risk-free rates, and this result is repeated here for start years 20 and 40, but not for the less variable start year zero.

Tables 8.65, 8.67 and 8.69 also show the figures for the unit-linked and risk-free accounts. As we noted in Section 6.3, the unit-linked account is unaffected by the stochastic risk-free rate.

The risk-free account invests the single premium at the then current risk-free rate *i.e.* the premium is invested in a zero coupon bond maturing in 10 years time.

Notice that unlike Section 6.3 (see Table 6.34) where the premiums were annual, we have no reinvestment risk with the single premium contract. Hence the return on the risk-free contract is known at outset. At time zero, all the simulations have the same zero coupon yield curve given by the neutral initial conditions, and so the standard deviation of the risk-free payout is zero. For contracts sold at later times, the yield curve at that time will be different for each simulation and so the standard deviation is non-zero. However, the payout on these later contracts is still known at the time they are actually sold.

The relative performance of the risk-free contract to the UWP contract is shown in Tables 8.66, 8.68 and 8.70 with a stochastic risk-free rate and Tables 8.45, 8.49 and 8.51 with a constant risk-free rate. We see in all cases that the number of simulations in which the risk-free contract outperforms the UWP payout is higher if we use the stochastic risk-free model because the average stochastic risk-free rate is higher than the constant risk-free rate.

The risk-free payout on a policy sold at time zero is the same in each simulation, and is higher than the maximum guarantee for the bonus rates we consider. Hence, the guarantees on the risk-free policy sold at time zero are always higher than the achieved guarantees on the UWP policies.

However, each simulation will use a different stochastic risk-free rate for policies sold after time zero. Therefore it is possible to have risk-free payouts in the stochastic case which are even lower than the guarantees achieved by UWP policies with low bonuses. For example, the risk-free payout from the constant risk-free model always exceeds the achieved guarantee when the guaranteed growth rate is zero and the desired bonus rate is 4%. While, under the stochastic risk-free model, we often find that the risk-free payout is lower than the achieved guarantee.

However, for policies with a high desired bonus rate of 8% and no guaranteed growth, the position is reversed. The probability that the risk-free payout is bigger than the achieved guarantee is higher under the stochastic risk-free model. Firstly, the average risk-free rate is higher under the stochastic model. Secondly, the variability of the risk-free rate leads to the possibility of some very high risk-free payouts.

8.5.2 Guarantee Account

We now look at the size of the guarantee account if the charges are based on the option pricing approach, but are invested in the risk-free asset.

In Section 8.3 we accumulated the guarantee account at the fixed risk-free interest rate of 6.77% (see equation 8.25). However, in this section we have a risk-free yield curve, giving us a choice of term for the risk-free asset. The guarantee account is meant to continue indefinitely making payouts at any time the guarantee bites. Therefore, there is no one term that is appropriate for the assets. (We could hedge the guarantees by buying zero coupon bonds of the appopriate term, but this would require a negative holding of equities. We are assuming in this chapter that the insurer either cannot, or does not want to, hedge the risk or buy options.) Hence, I will assume for simplicity that the mismatching accounts are invested in consols from now on as follows:

$$\begin{aligned} GA_t &= GA_{t-1} \cdot \frac{C[t-1]}{C[t]} \cdot (1 + C[t]) \\ &\quad + N_{t,t} \cdot E_{t,t} \\ &\quad + \sum_{d=1}^{n-1} N_{t,t-d} \cdot O^+_{t,t-d} - N_{t-1,t-d} \cdot O^-_{t,t-d} \\ &\quad - \max([N_{t-1,t-n} \cdot E_{t-1,t-n} - N_{t-1,t-n} \cdot S_t], 0). \end{aligned} \tag{8.27}$$

The quantiles of the guarantee account are given in Tables 8.71 to 8.73 below.

We can compare Tables 8.71, 8.72, and 8.73 with Tables 8.55, 8.56, and 8.58. The only difference is that Tables 8.55, 8.56, and 8.58 use a constant risk-free rate while Tables 8.71, 8.72, and 8.73 use a risk-free yield curve obtained from the Wilkie Model.

We do not show Tables 8.71, 8.72, and 8.73 graphically as the overall patterns are the same as Figures 8.17 to 8.21 for the constant risk-free case. However, the size of the guarantee account can be quite different under the stochastic risk-free model. Figures 8.28 to 8.30 show the 1st, 50th, and 99.01th quantiles under both the constant and stochastic risk-free rate models. Note that these figures use different scales to enable us to see the differences between the quantiles under each model clearly.

Table 8.71: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time	Quantiles of the Guarantee Account, QGA_t								
		t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.00	0	1	1	1	1	1	1	1	1	1
0.00	0.00	1	1	2	2	2	3	3	4	4	4
0.00	0.00	2	2	2	2	3	4	5	6	7	8
0.00	0.00	3	2	2	3	4	6	8	10	11	14
0.00	0.00	4	2	3	4	5	8	11	13	16	22
0.00	0.00	5	3	3	4	6	9	13	18	22	27
0.00	0.00	10	-7	1	9	14	21	32	45	59	74
0.00	0.00	15	-121	-67	-2	25	37	55	82	112	159
0.00	0.00	20	-204	-117	-21	42	61	87	127	174	264
0.00	0.00	25	-433	-179	-37	69	96	132	180	248	299
0.00	0.00	30	-622	-281	-49	107	149	197	264	350	559
0.00	0.00	35	-717	-370	-72	159	227	298	391	526	759
0.00	0.00	40	-1381	-590	-114	231	342	458	610	862	1082
0.00	0.00	45	-2886	-818	-143	329	507	715	1004	1460	1807
0.00	0.00	50	-6801	-1169	-205	465	748	1129	1687	2254	3160

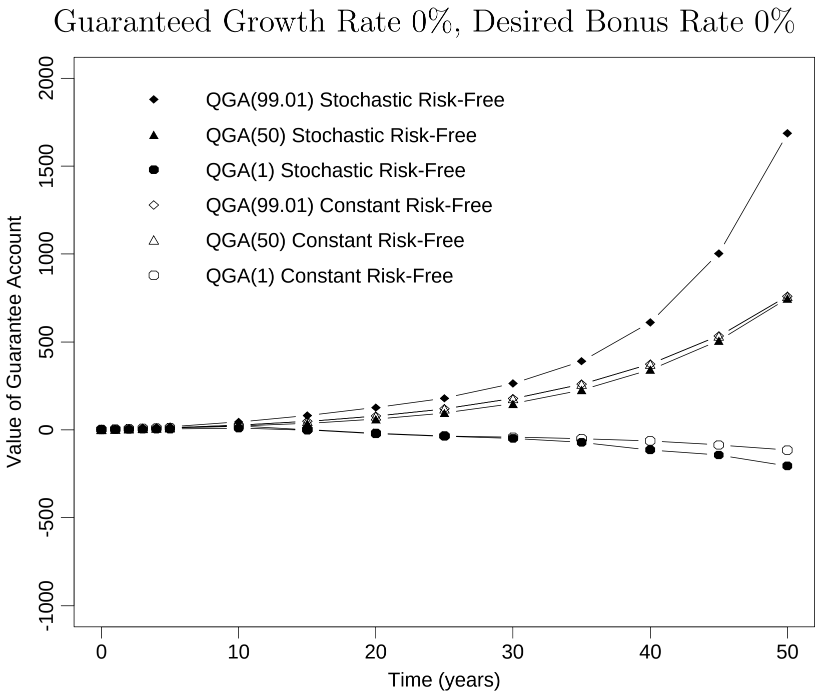

Figure 8.28: A Comparison of the Quantiles of the Guarantee Account using the Wilkie Model for Equities, the Fixed Bonus Strategy, and the Two Alternative Risk-Free Rates

Payments are made to the mismatching account GA_t whenever a bonus is declared. Hence cashflows occur every year including time zero. We can see at time zero that the quantiles are virtually the same in both sets of tables. The only cashflow is the charge for the initial guarantees; there is no payment out of the account at this time. The charge will be marginally less under the stochastic risk-free rate because the yield on a 10-year zero-coupon bond is greater than 6.77% at time zero.

At times 0 to 9 all the cashflows are positive. We can see that the greater variability of the stochastic risk-free rate causes the lower (higher) quantiles to be even lower (higher) than under the fixed risk-free rate, due to the greater variability in the cost of the options.

From time 10 onwards policies mature and in some simulations the guarantee bites. A number of factors affect the results as follows:

- The options are on average cheaper under a stochastic risk-free rate which will cause the guarantee account to be smaller on average.

- Lower charges mean that the policyholder retains more in equities and hence

Table 8.72: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.04	0	1	1	1	1	1	1	1	1	1
0.00	0.04	1	1	2	2	2	3	3	4	4	4
0.00	0.04	2	2	2	3	4	5	6	7	9	10
0.00	0.04	3	2	3	4	5	7	10	13	15	19
0.00	0.04	4	3	4	5	7	10	15	20	26	31
0.00	0.04	5	3	4	6	9	13	21	31	44	54
0.00	0.04	10	8	11	15	22	39	78	119	145	190
0.00	0.04	15	-126	-72	-6	42	74	126	187	255	381
0.00	0.04	20	-268	-208	-65	68	123	197	286	373	555
0.00	0.04	25	-521	-330	-108	104	194	300	418	567	811
0.00	0.04	30	-976	-504	-159	157	300	449	626	830	1155
0.00	0.04	35	-1298	-838	-228	223	454	679	945	1311	1770
0.00	0.04	40	-2433	-1136	-330	327	681	1023	1497	1958	2869
0.00	0.04	45	-5163	-1825	-463	470	1010	1586	2405	3411	4393
0.00	0.04	50	-12160	-2521	-679	689	1487	2492	3918	5674	8031

Guaranteed Growth Rate 0%, Desired Bonus Rate 4%

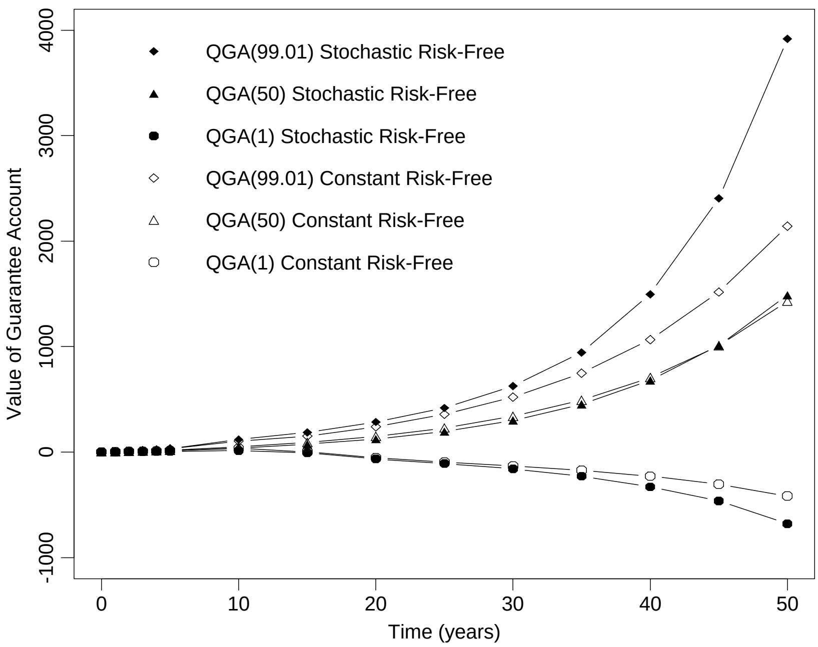

Figure 8.29: A Comparison of the Quantiles of the Guarantee Account using the Wilkie Model for Equities, the Fixed Bonus Strategy, and the Two Alternative Risk-Free Rates

Table 8.73: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time	Quantiles of the Guarantee Account, QGA_t								
		t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.02	0.04	0	3	3	3	3	3	3	3	3	3
0.02	0.04	1	3	4	4	5	6	7	8	9	9
0.02	0.04	2	4	5	6	8	11	13	16	18	21
0.02	0.04	3	5	6	8	11	16	21	27	33	42
0.02	0.04	4	6	8	11	15	22	32	44	55	74
0.02	0.04	5	8	10	14	20	30	46	67	82	106
0.02	0.04	10	18	24	34	52	91	152	211	266	320
0.02	0.04	15	-158	-106	-16	84	154	239	345	457	677
0.02	0.04	20	-409	-299	-103	108	246	375	522	680	1247
0.02	0.04	25	-771	-471	-197	142	377	559	762	952	1186
0.02	0.04	30	-1378	-803	-295	204	573	846	1141	1462	2230
0.02	0.04	35	-1776	-1242	-435	293	861	1288	1743	2346	3658
0.02	0.04	40	-3802	-1855	-612	419	1271	1997	2809	3553	6468
0.02	0.04	45	-8100	-3128	-956	603	1868	3050	4559	6331	9889
0.02	0.04	50	-19157	-4286	-1402	868	2719	4797	7513	10450	17102

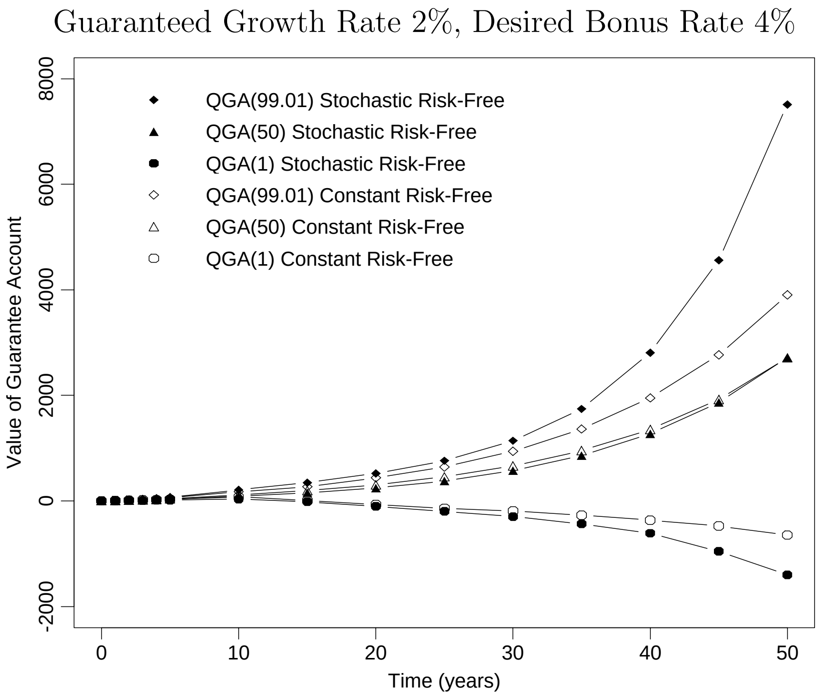

Figure 8.30: A Comparison of the Quantiles of the Guarantee Account using the Wilkie Model for Equities, the Fixed Bonus Strategy, and the Two Alternative Risk-Free Rates

the guarantee is exercised less often under the stochastic risk-free rate. This will cause the guarantee account to be larger on average.

- The guarantee account is rolled up at the yield on consols which is on average larger than the fixed risk-free rate used in Section 8.3.2. This causes the guarantee account to be larger on average under the stochastic risk-free rate model.

Hence we have two causes for the account to be larger on average under the stochastic risk-free rate model, and one cause for it to be smaller on average. Figures 8.28 to 8.30 show that in fact the 50th quantile of the guarantee account under the stochastic risk-free model is very similar to that under the constant risk-free model. However, the higher variability of the stochastic risk-free rate causes a greater spread between the 1st and 99.01th quantiles.

Figure 8.28 shows an unusual pattern. Table 8.55 shows that under the constant risk-free rate model the 50th and 99.01th quantiles are actually equal, because the charges are identical and the guarantee does not bite in these simulations.

8.5.3 Free Assets

We now look at the free asset ratio, as given by Equation 8.26, using the stochastic risk-free rate model.

The quantiles of the free asset ratios are given in Tables 8.74 to 8.76 below.

We can compare Tables 8.74, 8.75, and 8.76 with Tables 8.60, 8.61, and 8.63. The only difference is that Tables 8.60, 8.61, and 8.63 use a constant risk-free rate while Tables 8.74, 8.75, and 8.76 use a risk-free yield curve obtained from the Wilkie Model.

Again we do not show Tables 8.74, 8.75, and 8.76 graphically as the overall patterns are the same as Figures 8.23 to 8.27 for the constant risk-free case. However, the size of the free asset ratios can be quite different under the stochastic risk-free model. Figures 8.31 to 8.33 show the 1st, 50th, and 99.01th quantiles under both the constant and stochastic risk-free rate models. Note that these figures use different scales to enable us to see the differences between the models clearly.

Table 8.74: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	Quantiles of the Free Asset Ratio, FAR_t (%) 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.00	0	0	0	0	0	0	0	0	0	0
0.00	0.00	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.00	2	-9	-4	-3	-1	1	1	1	1	2
0.00	0.00	3	-12	-7	-4	-1	1	1	1	2	2
0.00	0.00	4	-15	-9	-5	-1	1	1	2	2	3
0.00	0.00	5	-15	-12	-6	-1	1	2	2	2	3
0.00	0.00	10	-30	-24	-9	0	2	3	3	5	6
0.00	0.00	15	-54	-26	-9	2	3	6	8	11	16
0.00	0.00	20	-45	-31	-8	3	6	10	14	19	32
0.00	0.00	25	-87	-30	-8	6	11	16	23	30	39
0.00	0.00	30	-136	-47	-8	10	17	26	36	48	83
0.00	0.00	35	-149	-65	-10	15	27	42	58	78	123
0.00	0.00	40	-260	-89	-13	22	41	66	93	134	193
0.00	0.00	45	-333	-114	-17	32	61	104	153	213	268
0.00	0.00	50	-1520	-161	-25	46	92	161	260	356	513

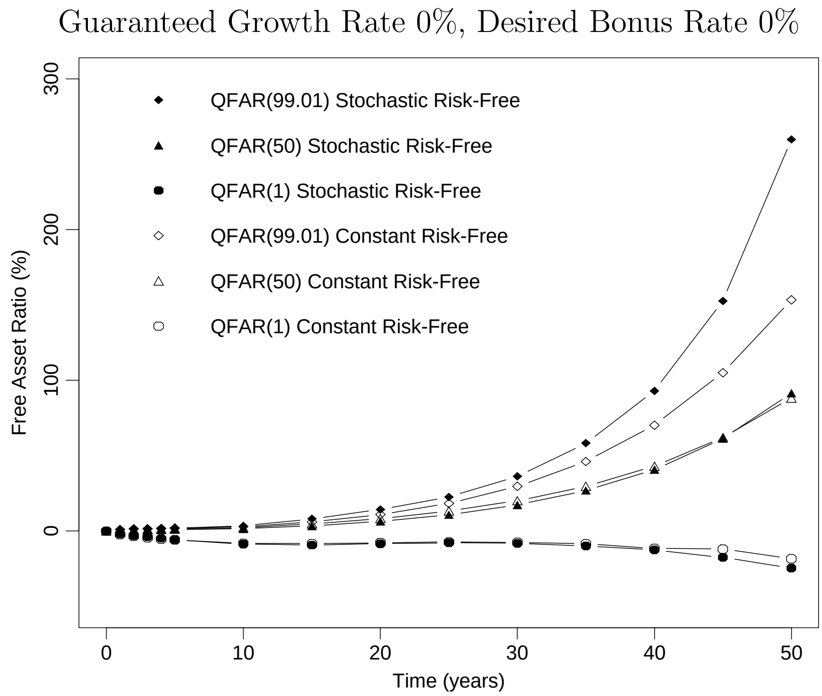

Figure 8.31: A Comparison of the Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, the Fixed Bonus Strategy, and the Two Alternative Risk-Free Rates

Table 8.75: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
			Quantiles of the Free Asset Ratio, FAR_t (%)								
0.00	0.04	0	0	0	0	0	0	0	0	0	0
0.00	0.04	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.04	2	-9	-5	-3	-1	1	1	1	2	2
0.00	0.04	3	-12	-8	-4	-1	1	1	2	2	2
0.00	0.04	4	-17	-10	-6	-1	1	2	2	3	3
0.00	0.04	5	-17	-13	-7	-1	1	2	3	3	4
0.00	0.04	10	-36	-29	-16	-0	3	5	8	11	15
0.00	0.04	15	-53	-38	-19	2	6	12	17	24	34
0.00	0.04	20	-68	-46	-20	4	12	22	33	42	66
0.00	0.04	25	-143	-56	-20	8	21	36	53	74	112
0.00	0.04	30	-169	-83	-24	14	34	57	85	120	168
0.00	0.04	35	-200	-118	-34	22	53	90	135	199	291
0.00	0.04	40	-450	-157	-49	33	81	143	212	296	358
0.00	0.04	45	-602	-228	-64	50	122	219	351	508	673
0.00	0.04	50	-2504	-332	-93	73	181	343	578	849	1304

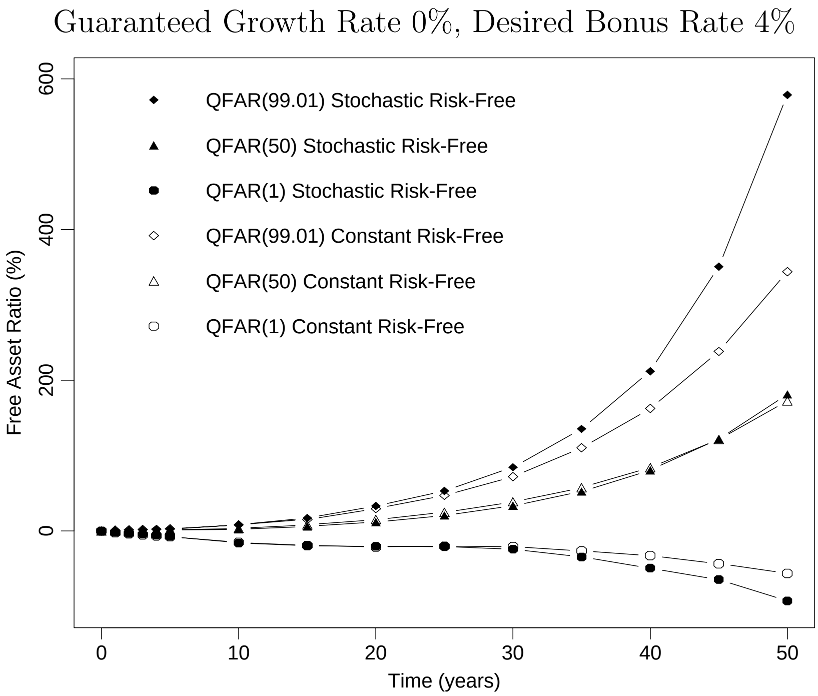

Figure 8.32: A Comparison of the Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, the Fixed Bonus Strategy, and the Two Alternative Risk-Free Rates

Table 8.76: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Fixed Bonus Strategy

y	z	Time t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
			Quantiles of the Free Asset Ratio, FAR_t (%)								
0.02	0.04	0	0	0	0	0	0	0	0	0	0
0.02	0.04	1	-6	-5	-3	-1	0	1	2	2	2
0.02	0.04	2	-13	-7	-5	-2	1	2	3	3	3
0.02	0.04	3	-16	-12	-7	-2	1	3	3	4	4
0.02	0.04	4	-21	-15	-9	-2	2	3	4	5	6
0.02	0.04	5	-22	-18	-11	-2	2	4	5	6	7
0.02	0.04	10	-40	-35	-21	-4	5	10	14	19	21
0.02	0.04	15	-66	-47	-28	-2	12	21	31	41	60
0.02	0.04	20	-83	-70	-34	3	23	39	56	72	156
0.02	0.04	25	-172	-86	-38	8	39	65	91	118	140
0.02	0.04	30	-230	-132	-51	16	63	106	150	204	306
0.02	0.04	35	-289	-187	-72	27	98	167	243	324	563
0.02	0.04	40	-670	-287	-97	43	149	264	386	493	692
0.02	0.04	45	-964	-426	-134	64	222	412	657	906	1308
0.02	0.04	50	-3449	-562	-192	98	329	649	1047	1557	2755

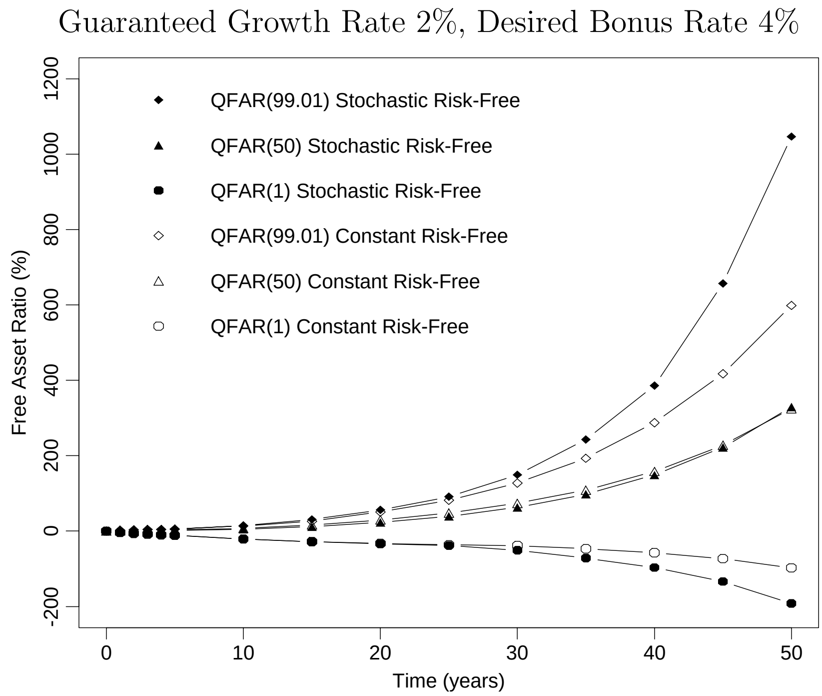

Figure 8.33: A Comparison of the Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, the Fixed Bonus Strategy, and the Two Alternative Risk-Free Rates

We see that the free asset ratios follow the same basic patterns under the stochastic risk-free rate model as under the constant risk-free rate in Section 8.4.1. The free asset ratio is zero at time zero. Then as time passes the higher (lower) quantiles become more positive (negative).

However, the results are different under the stochastic risk-free rate model due to the following factors:

- The options are on average cheaper under a stochastic risk-free rate which means lower charges and hence lower free assets on average.
- However, to calculate the free assets we must deduct the cost of options to match the outstanding guarantees. Hence cheaper options mean smaller deductions and hence higher free assets on average.
- Lower charges mean that the policyholder retains more in equities and hence the guarantee is exercised less often under the stochastic risk-free rate. This will cause the free assets to be larger on average.
- The guarantee account is rolled up at the yield on consols which is on average larger than the fixed risk-free rate used in Section 8.3.2. This causes the free assets to be larger on average under the stochastic risk-free rate model.

The first, third, and fourth factors described above also apply to the differences between the two risk-free models of the guarantee account quantiles. Hence the overall pattern of the free asset ratio is similar to that of the guarantee account. We see that the median of the free asset ratio under the stochastic risk-free model is very similar to that under the constant risk-free model. However, the higher variability of the stochastic risk-free rate causes a greater spread between the 1st and 99.01th quantiles.

Therefore, we can conclude that it is important for the insurer to use the more complex stochastic risk-free model. The constant risk-free model underestimates the risk of both excessively high free assets (*i.e.* the orphan estate problem), and of large negative free assets (which endangers solvency).

8.6 Dynamic Bonuses

We now extend the results of Section 8.5 by using the bonus mechanism considered in Section 7.3.2 such that bonuses are directly linked to the investment return subject to some smoothing.

8.6.1 Results

We obtain the following results using the Wilkie investment model with taxed dividends, and option pricing using a risk-free rate derived from a yield curve fitted to the Wilkie model and volatility of 20% in the Black-Scholes equation.

Summary statistics are shown in Tables 8.77 to 8.82.

Table 8.77: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

Start Year 0

y	bp	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		143.91	67.18	0.00	0.00
0.00	0.00	140.29	65.22	50.00	0.00
0.00	0.50	135.43	64.04	78.42	8.05
0.00	0.60	133.74	63.23	85.03	10.74
0.00	0.70	131.83	61.99	91.57	13.90
R-F		105.83	0.00	105.83	0.00

Table 8.78: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

Start Year 0

y	bp	Desired Bonuses Always Affordable	Option Exercised	UWP > UL	Ach Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	187	162	162	3420	10000
0.00	0.50	8687	1870	954	954	3998	9969
0.00	0.60	7877	2578	1251	1251	4280	9589
0.00	0.70	6857	3475	1529	1529	4534	8553

We can compare Tables 8.77 to 8.82 with Tables 8.65 to 8.70. The only difference between these two sets of tables is in the bonus mechanism used. Tables 8.65 to 8.70 use the simple bonus mechanism, whereas Tables 8.77 to 8.82 link the bonus directly to investment return with smoothing.

The results are not directly comparable due to the different bonus mechanisms. However it is interesting to compare smoothed investment linked bonuses with bp of 0.5 with fixed bonuses z of 4%, with the guaranteed bonus rate y equal to zero in

Table 8.79: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

Start Year 20

y	bp	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		148.78	83.07	0.00	0.00
0.00	0.00	145.00	81.16	50.00	0.00
0.00	0.50	140.22	80.51	78.25	13.63
0.00	0.60	138.81	79.76	84.44	17.61
0.00	0.70	137.43	78.71	90.37	21.84
R-F		110.33	21.77	110.33	21.77

Table 8.80: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

Start Year 20

y	bp	Desired Bonuses Always Affordable	Option Exercised	UWP > UL	Ach Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	364	308	308	3821	10000
0.00	0.50	7850	2553	1264	1264	4463	9607
0.00	0.60	7042	3190	1514	1514	4640	9018
0.00	0.70	6153	3865	1760	1760	4758	8276

Table 8.81: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

Start Year 40

y	bp	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		148.87	83.17	0.00	0.00
0.00	0.00	145.07	81.14	50.00	0.00
0.00	0.50	140.44	80.54	77.70	13.73
0.00	0.60	139.17	79.88	83.78	17.70
0.00	0.70	137.84	78.95	89.61	21.81
R-F		111.20	23.63	111.20	23.63

Table 8.82: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

Start Year 40

y	bp	Desired Bonuses Always Affordable	Option Exercised	UWP > UL	Ach Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	387	333	333	3845	10000
0.00	0.50	7875	2460	1235	1235	4493	9584
0.00	0.60	7138	3148	1494	1494	4675	9006
0.00	0.70	6319	3803	1736	1736	4757	8284

both cases, because they have very similar mean and standard deviation of payouts.

We saw in Section 7.3.2 that, for this set of parameters, the neutral initial bonus rate is given by $0.09979831 \cdot bp$ (see Equation 7.19). Hence in our example where bp is 0.5 the neutral initial bonus rate is approximately 5%. This means that the bonuses declared are likely to be larger than under the fixed bonuses of 4%. This explains why the mean achieved guarantee is larger under the investment-linked bonus mechanism.

Under the fixed bonus mechanism the bonus is always 4% except when it is unaffordable and hence reduced to zero. The investment-linked bonus is also set to zero when it is unaffordable. However there is another source of randomness for the investment-linked bonuses, namely they will change with investment returns, albeit subject to smoothing. This explains why the achieved guarantee is more variable under the investment-linked bonus mechanism.

The above point also explains why the mean payouts are similar despite the larger guarantees on average offerred by the investment-linked method. The fixed bonuses must be declared, if affordable, even if assets are performing badly, requiring expensive options to be bought, and hence leading to lower payouts. In contrast, the investment-linked bonuses are larger on average, which might be expected to cost more, but these bonuses will be reduced in years of poor investment returns, and so the options are more likely to be bought at modest costs.

Under the fixed bonus mechanism the desired bonuses are always affordable in more simulations than under the investment-linked bonus mechanism. For example, for policies sold at time zero the fixed bonus of 4% is affordable each year in 8917 simulations, whereas bonuses given by the investment-linked mechanism with bp of 0.5 are always affordable in only 8687 simulations. This might sound surprising given the investment-linked mechanism's ability to cut bonuses whenever investment returns are low in a particular year. However, the investment-linked bonuses are on average higher and smoothing limits any adjustment to no more than 20% year on year. So for example, several very good investment years leads to a very high bonus rate which may not be affordable if the market falls sharply. Also, even if the previous year's investment return is good, the desired bonus may be unaffordable if the earlier years suffered very poor returns, because the investment-linked mechanism considers only returns over one year rather than the policy term to date.

The higher guarantees on average under the investment-linked approach also lead to a greater number of simulations in which the option is exercised and the payout is greater than a unit-linked contract.

The above patterns apply for each start year. However, the later start years will have greater variability in investment conditions than start year zero. Under both bonus mechanisms this leads to greater variability of payouts and achieved guarantees, and in turn to more simulations where the guarantees are unaffordable and the options are exercised.

Increasing the bonus proportion bp under the investment-linked mechanism has the same effect as increasing the bonus rates y or z under the fixed bonus mechanism. The mean and standard deviation of the achieved guarantee are increased. Higher guarantees require a greater investment in options and hence a lower payout when the guarantee does not bite.

Which bonus mechanism is most appropriate depends on the needs of the individual investor. However for most people the more flexible investment-linked strategy would be better. It offers higher guarantees when investment returns allow these to be bought without reducing the expected payout. If investors want their guarantees to grow by a fixed 4% each year they should probably buy a combination of non-profit and unit-linked products.

The results for the unit-linked, and risk-free accounts are the same in both sets of tables as the only differrence is the with-profits bonus mechanism.

Again we can compare smoothed investment linked bonuses with bp of 0.5 with fixed bonuses z of 4%, with the guaranteed bonus rate y equal to zero in both cases. We see that the risk-free contract is less likely to outperform the achieved guarantee under the dynamic bonus strategy because of the potential for very high guarantees when investment returns are high. Conversely the risk-free contract is more likely to outperform the UWP payout under the dynamic bonus strategy.

8.6.2 Guarantee Account

We now look at the effect of the investment-linked bonus mechanism on the guarantee account GA_t. The guarantee account is calculated in the same way as in Section 8.5.2 as given by Equation 8.27.

The quantiles of the guarantee account are given in Tables 8.83 to 8.85 below.

Table 8.83: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

y	bp	Time	Quantiles of the Guarantee Account, QGA_t								
		t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	1	1	1	1	1	1	1	1	1
0.00	0.50	1	1	2	2	2	3	4	4	4	5
0.00	0.50	2	2	2	3	4	5	6	8	9	10
0.00	0.50	3	2	3	4	5	8	10	13	16	22
0.00	0.50	4	3	4	5	7	11	16	22	28	33
0.00	0.50	5	3	5	6	9	15	22	34	45	56
0.00	0.50	10	6	11	16	25	45	90	135	175	240
0.00	0.50	15	-174	-101	-24	44	84	146	218	285	466
0.00	0.50	20	-370	-243	-96	64	140	234	341	451	970
0.00	0.50	25	-707	-378	-157	89	220	354	518	676	805
0.00	0.50	30	-1293	-591	-226	128	340	537	792	1041	1245
0.00	0.50	35	-1696	-898	-327	194	515	816	1203	1599	2234
0.00	0.50	40	-2409	-1384	-483	284	770	1253	1880	2488	4721
0.00	0.50	45	-5102	-2001	-757	407	1137	1929	2966	4202	7241
0.00	0.50	50	-12030	-3109	-1043	595	1673	3020	4813	6715	11137

Table 8.84: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

y	bp	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.60	0	1	1	1	1	1	1	1	1	1
0.00	0.60	1	1	2	2	2	3	4	4	4	5
0.00	0.60	2	2	2	3	4	5	7	8	9	11
0.00	0.60	3	2	3	4	6	8	11	14	18	22
0.00	0.60	4	3	4	5	8	12	17	24	31	37
0.00	0.60	5	4	5	7	10	16	25	41	54	73
0.00	0.60	10	9	12	18	30	55	108	161	200	238
0.00	0.60	15	-179	-123	-34	50	101	177	266	349	541
0.00	0.60	20	-404	-278	-114	64	166	282	418	556	1003
0.00	0.60	25	-749	-421	-186	86	259	429	634	809	910
0.00	0.60	30	-1419	-655	-265	123	396	659	995	1234	1501
0.00	0.60	35	-1862	-1050	-385	186	596	998	1538	2019	2547
0.00	0.60	40	-2700	-1653	-561	276	892	1523	2384	3098	5252
0.00	0.60	45	-5744	-2660	-915	404	1307	2355	3710	5322	8063
0.00	0.60	50	-13528	-4099	-1288	595	1918	3697	6012	8637	12369

Table 8.85: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

y	bp	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.70	0	1	1	1	1	1	1	1	1	1
0.00	0.70	1	1	2	2	3	3	4	4	4	5
0.00	0.70	2	2	2	3	4	5	7	8	10	12
0.00	0.70	3	2	3	4	6	9	12	15	20	25
0.00	0.70	4	3	4	6	8	13	19	27	36	46
0.00	0.70	5	4	5	7	11	18	29	46	60	80
0.00	0.70	10	8	13	20	34	68	127	184	230	281
0.00	0.70	15	-209	-128	-42	53	119	210	317	408	569
0.00	0.70	20	-450	-317	-132	60	194	338	499	671	985
0.00	0.70	25	-842	-481	-223	81	298	515	772	999	1171
0.00	0.70	30	-1606	-846	-319	118	454	788	1211	1545	1951
0.00	0.70	35	-2089	-1257	-455	172	683	1202	1844	2489	3197
0.00	0.70	40	-4424	-2165	-666	263	1014	1842	2920	3947	5190
0.00	0.70	45	-8587	-3481	-1001	381	1491	2851	4643	6542	8100
0.00	0.70	50	-14676	-5046	-1448	579	2176	4418	7328	10263	12300

We can compare Tables 8.83 to 8.85 under the investment-linked bonus mechanism to Tables 8.71 to 8.73 under the fixed bonus mechanism.

We note that Table 8.71, which shows the results when the fixed bonus z equals zero, applies equally to the investment-linked bonus mechanism where bp equals zero. In both cases no bonuses are declared and the guarantee is simply a return of premiums.

We saw in Section 8.6.1 that the results were very similar for an investment-linked bonus with bp equal to 0.5 and a fixed bonus with z equal to 4%, and no guaranteed growth in both cases. We can see in Tables 8.83 and 8.72, and Figure 8.34, that the results are also very similar for the quantiles.

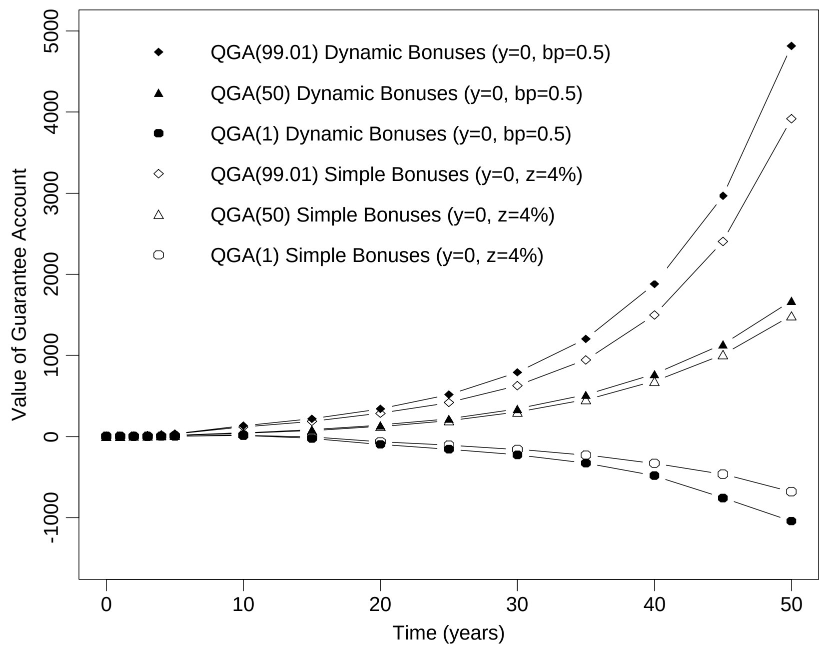

Figure 8.34: A Comparison of the Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Two Alternative Bonus Strategies

Before time 10 the guarantee account is only receiving charges (there have been no maturities) and the results are very close reflecting the very similar cost of options to match the bonuses and hence similar charges. The investment-linked bonuses are on average higher, but they will be reduced if investment returns are low which limits the cost of bonuses when options are expensive.

From time 10 onwards the guarantee account continues to receive charges but may also have to pay out if the guarantee bites. The option is exercised more often under the investment-linked bonuses which goes some way to explain why the lower quantiles $QGA(0.1)$, $QGA(1)$, and $QGA(10)$ are worse at later durations than under fixed bonuses. (Note that the lowest observed guarantee account actually occurs with fixed bonuses under this set of simulations. The stochastic nature of the results does mean that the overall pattern may not be repeated for the best or worst results which after all are given by a single simulation.) In the very best scenarios high investment returns mean the guarantee never bites and all the charges are retained in the guarantee account. The investment-linked bonuses will be higher than fixed bonuses in good investment years leading to higher charges and so a larger guarantee account.

We might have expected that by linking bonuses to investment returns that we could reduce the spread of results for the guarantee account, but in fact we see the opposite result. The policyholder gets similar mean and standard deviation of payouts under the two approaches. The policyholder benefits from higher mean guarantees under the investment-linked bonuses at the expense of greater variability of guarantees. Hence investment-linked bonuses bring no clear advantages for the policyholder, but have disadvantages for the insurer. The problem is caused by the smoothing mechanism which does not allow the bonuses to be cut rapidly enough when a period of poor investment returns follows a number of years of high returns.

Higher values of bp and z lead to a greater spread in the quantiles. The guarantees are larger and so potential losses are larger, but the charges are larger too.

8.6.3 Free Assets

We now look at the free asset ratio, using the dynamic bonus mechanism.

The quantiles of the free asset ratios are given in Tables 8.86 to 8.88 below.

We can compare Tables 8.86 to 8.88 under the investment-linked bonus mechanism to Tables 8.74 to 8.76 under the fixed bonus mechanism.

We note that Table 8.74, which shows the results when the fixed bonus z equals zero, applies equally to the investment-linked bonus mechanism where bp equals zero.

Table 8.86: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

y	bp	Time	Quantiles of the Free Asset Ratio, $QFAR_t$								
		t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	0	0	0	0	0	0	0	0	0
0.00	0.50	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.50	2	-9	-5	-3	-1	1	1	1	2	2
0.00	0.50	3	-12	-8	-4	-1	1	1	2	2	2
0.00	0.50	4	-17	-10	-6	-1	1	2	2	3	4
0.00	0.50	5	-18	-14	-7	-1	1	2	3	3	4
0.00	0.50	10	-37	-32	-17	-1	3	6	10	13	17
0.00	0.50	15	-56	-42	-22	0	7	13	20	27	42
0.00	0.50	20	-77	-51	-23	3	14	25	38	50	130
0.00	0.50	25	-182	-60	-25	7	23	42	64	82	115
0.00	0.50	30	-216	-106	-34	12	38	67	103	143	191
0.00	0.50	35	-253	-131	-47	19	59	106	166	252	336
0.00	0.50	40	-447	-206	-65	29	90	168	264	351	502
0.00	0.50	45	-592	-277	-96	43	136	259	426	644	1006
0.00	0.50	50	-2485	-455	-140	65	202	409	697	957	2078

Table 8.87: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

y	bp	Time	Quantiles of the Free Asset Ratio, $QFAR_t$								
		t	0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.60	0	0	0	0	0	0	0	0	0	0
0.00	0.60	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.60	2	-9	-5	-3	-1	1	1	1	2	2
0.00	0.60	3	-12	-8	-5	-1	1	1	2	2	2
0.00	0.60	4	-17	-11	-6	-1	1	2	2	3	4
0.00	0.60	5	-19	-14	-8	-1	1	2	3	4	5
0.00	0.60	10	-39	-33	-18	-2	3	7	11	15	18
0.00	0.60	15	-56	-47	-25	-1	8	16	24	32	47
0.00	0.60	20	-83	-57	-28	2	16	29	46	59	133
0.00	0.60	25	-195	-68	-31	5	27	50	75	103	118
0.00	0.60	30	-238	-118	-41	11	44	81	126	174	194
0.00	0.60	35	-280	-159	-60	18	68	127	206	298	385
0.00	0.60	40	-501	-248	-76	28	105	202	323	470	574
0.00	0.60	45	-669	-360	-118	43	157	313	531	764	1113
0.00	0.60	50	-2737	-558	-171	65	233	492	856	1237	2289

Table 8.88: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Smoothed Investment-linked Bonus Strategy

y	bp	Time t	Quantiles of the Free Asset Ratio, $QFAR_t$ 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.70	0	0	0	0	0	0	0	0	0	0
0.00	0.70	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.70	2	-10	-5	-3	-1	1	1	1	2	2
0.00	0.70	3	-12	-8	-5	-1	1	1	2	2	2
0.00	0.70	4	-18	-11	-6	-1	1	2	3	3	4
0.00	0.70	5	-19	-15	-8	-1	1	2	3	4	5
0.00	0.70	10	-40	-33	-20	-3	3	8	13	16	19
0.00	0.70	15	-59	-51	-27	-3	9	18	28	39	54
0.00	0.70	20	-91	-64	-30	0	18	35	53	71	137
0.00	0.70	25	-214	-73	-37	4	31	58	91	118	150
0.00	0.70	30	-267	-132	-48	9	50	96	152	198	246
0.00	0.70	35	-334	-191	-72	16	78	151	246	360	500
0.00	0.70	40	-619	-277	-95	26	120	242	399	577	863
0.00	0.70	45	-866	-460	-141	40	178	374	648	898	1434
0.00	0.70	50	-2901	-733	-201	62	265	583	1042	1579	2210

We saw in Section 8.6.1 that the results were very similar for an investment-linked bonus with bp equal to 0.5 and a fixed bonus with z equal to 4%, and no guaranteed growth in both cases. We can see in Tables 8.86 and 8.75 that the results are also very similar for the quantiles of the free asset ratio. Again, we see in Figure 8.35 a greater spread of results using the dynamic bonus mechanism.

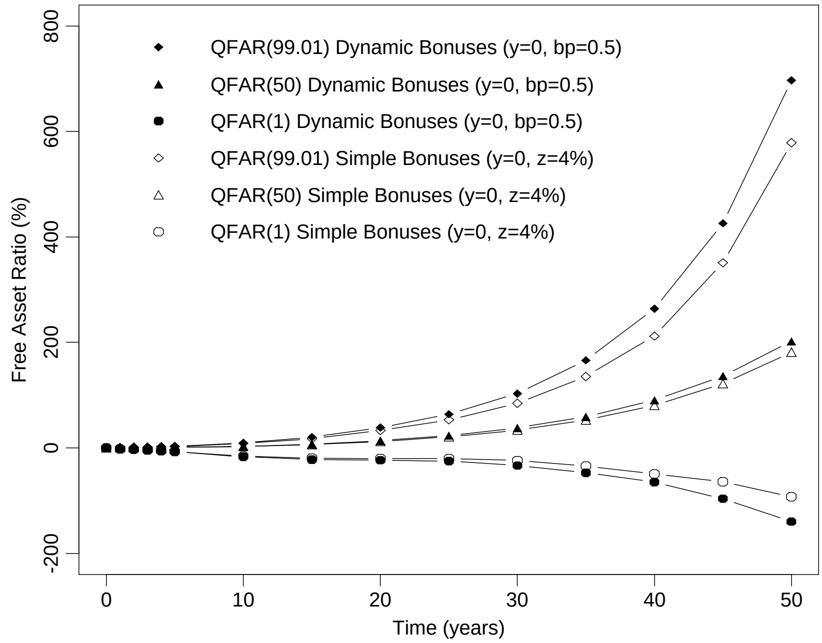

Figure 8.35: A Comparison of the Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, and the Two Alternative Bonus Strategies

8.7 Summary

In this chapter we have introduced a portfolio of UWP policies with different commencement dates. We then investigated the effects of mismatching the guarantees to see whether the insurer could diversify this risk through time.

The main conclusions from this chapter are:

- All the policies commencing at time zero have the same starting conditions, in this case the neutral initial conditions. However, the initial conditions for the policies starting at later times will be different because the simulations have already been running for some time. Hence, in Section 8.2.1 we saw that the standard deviation of the payouts and achieved guarantees was larger for policies issued after time zero.

- In Section 8.3.2 we considered mismatching the guarantees by investing the charges in the risk-free asset rather than matching options. We found that the guarantee account could grow very large, representing very high mismatching profits, but potentially leading to an orphan estate problem. However, more importantly, we saw that even after 50 years more than 1% of the simulations had negative guarantee accounts at that time. Higher bonuses resulted in more positive (negative) guarantee accounts in the best (worst) simulations.

- In Section 8.4.1 we calculated the free assets as the guarantee account less the cost of options to match the guarantees from in-force policies. We then compared the size of the free assets to the size of the liabilities via the free asset ratio. The median free asset ratio is positive and grows through time indicating that mismatching is expected to be profitable. However, in the cases with higher bonuses, we saw that by time 50 the accumulated losses exceeded the liabilities in around 1% of cases.

- In Section 8.5 we used the stochatic risk-free rate model. The stochastic risk-free rate has a higher mean than the constant risk-free rate we considered earlier, and of course has a higher variance. Hence, in Section 8.5.1 we found that the UWP payouts had both higher mean and standard deviation under the stochastic risk-free model than under the constant risk-free model.

- In Sections 8.5.2 and 8.5.3 we found that the stochastic risk-free rate resulted in more variability in both the guarantee account and the free asset ratio. This indicates that a simpler model using a constant risk-free rate may underestimate the size of potential problems. However, the median values were little changed.

- In Section 8.6 we used the dynamic bonus mechanism. Under the dynamic bonus mechanism, when investment returns are high (low), and hence the cost of matching options are low (high), we declare higher (lower) bonuses. We found in Section 8.6.1 that the dynamic bonus mechanism resulted in higher mean achieved guarantees compared to the fixed bonus mechanism with a similar mean payout.

- In Sections 8.6.2 and 8.6.3 we found that the dynamic bonus mechanism, compared to the fixed bonus mechanism with a similar mean payout, resulted in more variability in both the guarantee account and the free asset ratio. This demonstrates that smoothing bonuses under the dynamic bonus mechanism makes the problems of mismatching worse.

Chapter 9

Issues Affecting the With-Profits Industry

In Chapter 8 we outlined a model of an insurance company with a portfolio of UWP contracts. We introduced the idea of a guarantee account, which was built up from the charges levied on the UWP policies, and was used to make good any shortfall between the policyholders' equity assets and the maturity guarantee. We then investigated the effects on the guarantee account of investing in the risk-free asset rather than the matching options.

In Chapter 9 we will apply the model of Chapter 8 to a number of issues that have affected the with-profits industry in recent years. In Section 9.1 we will consider the effects of new business growth on the guarantee account. In Section 9.2 we will consider distributing excess free assets to the owners of the insurer. In Section 9.3 we will consider the effect of a move to a low inflation environment.

9.1 New Business Growth

We saw in Chapter 8 that the guarantee account could grow to very large values. In fact in each example we have considered, the 10th quantile has a positive free asset ratio by time 20 which continues to grow ever larger. Therefore at any given time a large proportion of the simulations look satisfactory with the trend being towards ever greater levels of solvency. We suggested in Section 8.4.1 that one way to use these free assets was to write more business.

However, so far we have assumed that the same level of business is sold each year. We have ignored expenses in our calculations assuming that these are matched by charges and that we consider only the investments after such charges. But in practice an insurer's new business must grow by at least the rate of inflation to maintain its expense ratio.

The insurer may well want to expand in real terms to obtain greater economies of scale and greater profits. In fact it may need to grow in real terms to remain competitive. However, we will only consider a real rate of new business growth of zero.

In this section we use the same methodology as in Section 8.6, but we now allow new business to increase at the rate of inflation derived from the Wilkie model. When inflation is negative the level of new business will actually decrease.

9.1.1 Results

We obtain the following results using the Wilkie investment model with taxed dividends. Option pricing is performed using a risk-free rate derived from a yield curve fitted to the Wilkie model and volatility of 20% in the Black-Scholes equation. Bonus rates are linked to the investment return with smoothing.

The mean and standard deviation of the payouts and achieved guarantees are shown in Table 9.89 for policies starting at time 20, and in Table 9.90 for policies starting at time 40.

We can compare the results with real new business growth in this section with the case of constant new business from Section 8.6.1.

The results for policies starting in year zero are the same as Tables 8.77 and 8.78 in Section 8.6.1 because the single premium is £50 at time zero in all simulations. Therefore we do not repeat the results for start year zero here.

The results in this section are also the same as Tables 8.80 and 8.82 in Section 8.6.1, and so again are not repeated here. In this section we have increased the premiums for all the contracts (unitised with-profits, unit-linked, and risk-free) by the same inflation rates. Hence guarantees and payouts have also been scaled up by the same proportions. Therefore, the number of simulations in which one amount is greater than another is unaffected by the rate of new business growth.

However, when we compare Tables 9.89 and 9.90 with Tables 8.79 and 8.81 we see some interesting differences.

Table 9.89: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, the Smoothed Investment-linked Bonus Strategy, and Real New Business Growth
Start Year 20

y	bp	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		436.65	362.96	0.00	0.00
0.00	0.00	427.51	359.02	140.62	62.76
0.00	0.50	414.58	354.78	227.05	127.64
0.00	0.60	410.36	351.42	246.45	145.66
0.00	0.70	406.46	347.67	265.14	164.18
R-F		326.38	198.22	326.38	198.22

Table 9.90: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, the Smoothed Investment-linked Bonus Strategy, and Real New Business Growth
Start Year 40

y	bp	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		1251.44	1263.10	0.00	0.00
0.00	0.00	1227.59	1250.46	400.96	267.45
0.00	0.50	1192.65	1232.54	648.24	505.12
0.00	0.60	1182.51	1223.14	704.03	568.22
0.00	0.70	1171.20	1210.17	757.68	629.28
R-F		958.75	807.46	958.75	807.46

Firstly we note that the achieved guarantee in the case of zero bonuses is equal to the premium, and so we can see that inflation increases the average premium to

£140.62 by time 20 and £400.96 by time 40. Compared with the fixed premium of £50 we see that inflation has increased the mean premium at time 20 by 181% and at time 40 by 702%.

In each simulation the payouts and achieved guarantees are increased by inflation in the same way as the premiums. However, the high inflation scenarios will also tend to be the scenarios with the highest guarantees and payouts. Hence by increasing the premiums with inflation we give greater weight to the high inflation scenarios when we calculate the mean over all the scenarios. For example, the mean unitised with-profit payout with a bonus proportion of 0.5 has grown by 206% by time 20 (*i.e.* from £135.43 to £414.58), whereas the mean premium has grown by only 181%.

Now we turn to the standard deviation of the payouts and achieved guarantees. The new business growth has considerably increased these standard deviations for two reasons. Firstly, even if the new business growth were deterministic, the premiums are larger so the standard deviation should go up in proportion. Secondly, the rate of new business growth is stochastic and so adds additional variability to the results. By considering the achieved guarantee of the unitised with-profits policy with no bonuses we see that the standard deviation of the premium is 62.76 by time 20 and 267.45 by time 40.

9.1.2 Guarantee Account

We now look at the effect of new business growth on the guarantee account GA_t. The guarantee account is calculated in the same way as in Section 8.5.2 as given by Equation 8.27. We will concentrate on the case where the bonus proportion is 0.5 throughout this section.

The quantiles of the guarantee account are given in Table 9.91 below.

We can compare Table 9.91 with new business growth to Table 8.83 with constant new business. The overall patterns are the same in both cases, increasing time leads to a greater spread of results. However, there are some interesting differences. Figure 9.37 compares the quantiles of the guarantee account under the two alternative rates of new business growth.

At time zero the results are of course the same whether we allow for new business

Table 9.91: Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, the Smoothed Investment-linked Bonus Strategy, and Real New Business Growth

y	bp	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	1	1	1	1	1	1	1	1	1
0.00	0.50	1	1	2	2	2	3	4	4	4	5
0.00	0.50	2	2	2	3	4	5	7	8	9	10
0.00	0.50	3	2	3	4	6	8	11	13	16	22
0.00	0.50	4	3	4	5	8	12	16	21	27	35
0.00	0.50	5	4	5	7	11	16	24	34	43	53
0.00	0.50	10	11	14	20	32	54	98	143	179	229
0.00	0.50	15	-210	-104	-22	61	112	184	267	355	448
0.00	0.50	20	-412	-274	-103	90	203	335	492	643	987
0.00	0.50	25	-814	-469	-173	131	344	573	863	1131	1430
0.00	0.50	30	-1857	-775	-250	197	564	961	1485	1949	2536
0.00	0.50	35	-2828	-1249	-383	304	896	1583	2503	3425	4433
0.00	0.50	40	-4457	-1893	-500	459	1400	2586	4088	5962	6975
0.00	0.50	45	-7968	-3178	-663	689	2155	4225	6878	10087	14592
0.00	0.50	50	-14857	-4311	-923	1046	3297	6833	11342	16653	23860

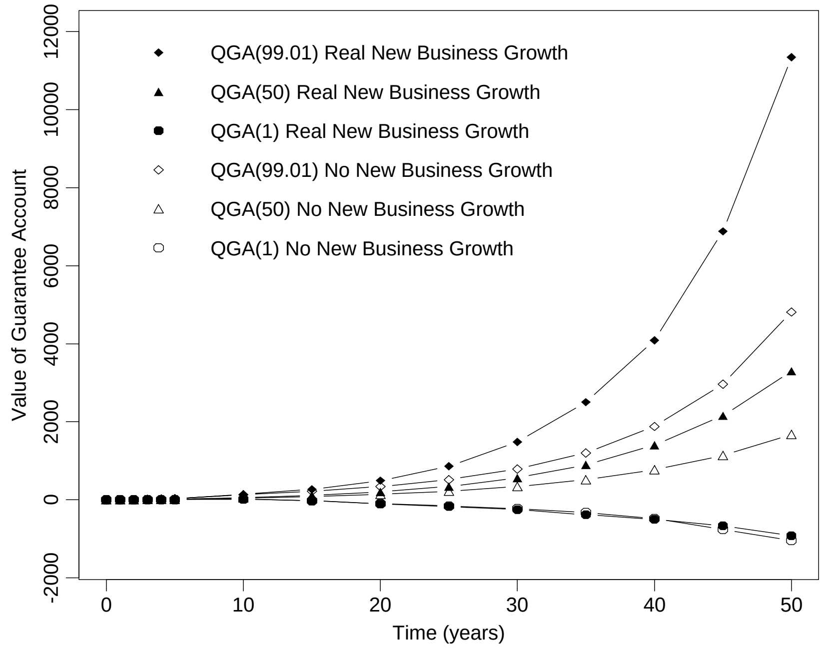

Figure 9.36: A Comparison of the Quantiles of the Guarantee Account using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, Smoothed Investment-linked Bonus Strategy, and the Two Alternative Rates of New Business Growth

growth or not. The rate of new business growth only affects the business sold from time 1 onwards.

Up to time 10 the guarantee account receives charges from the policies each year, but as yet no policy has matured so there is no possibility of a loss. The lower quantiles are a little better with new business growth because higher levels of business mean higher charges.

Surprisingly higher quantiles are sometimes lower with real new business growth. For example, in the case where the bonus proportion is 0.5, we see that real new business growth leads to a lower value for the 99.91th quantile at times 4 and 5. The highest charges will be received after equities have fallen in value and the risk-free rate is low, which often occurs alongside negative inflation and hence falling new business levels.

From time 10 onwards the guarantee account is exposed to possible losses from maturing policies. Negative inflation is unlikely to last more than a few years and so simulations will almost always show higher levels of new business in this section compared with the fixed business levels of Section 8.6.1. Hence charges will be greater, leading to higher values for the upper quantiles where the majority of policies are profitable.

The lower quantiles show a less clear pattern. The lower quantiles are sometimes more negative as when a loss occurs it will usually be on an increased amount of business. However the lower quantiles can also be higher because higher charges have been received from existing business.

9.1.3 Free Assets

We now look at the effect of new business growth on the free asset ratio. Again we concentrate on the case where the bonus proportion is 0.5 throughout this section.

Summary statistics of the free asset ratio with real new business growth are shown in Table 9.92.

Table 9.92: Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, the Smoothed Investment-linked Bonus Strategy, and Real New Business Growth

y	bp	Time t	Quantiles of the Free Asset Ratio, $QFAR_t$ 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	0	0	0	0	0	0	0	0	0
0.00	0.50	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.50	2	-10	-5	-3	-1	1	1	1	2	2
0.00	0.50	3	-13	-8	-4	-1	1	1	2	2	2
0.00	0.50	4	-17	-10	-6	-1	1	2	2	3	4
0.00	0.50	5	-18	-14	-7	-1	1	2	3	3	4
0.00	0.50	10	-38	-32	-16	-1	2	5	9	13	18
0.00	0.50	15	-65	-41	-21	0	5	11	19	27	49
0.00	0.50	20	-76	-50	-21	1	8	18	32	49	171
0.00	0.50	25	-154	-52	-21	3	12	27	47	71	90
0.00	0.50	30	-225	-84	-19	5	16	36	67	111	205
0.00	0.50	35	-236	-106	-22	6	21	47	92	169	236
0.00	0.50	40	-197	-100	-23	9	27	62	122	198	331
0.00	0.50	45	-199	-107	-25	12	34	77	149	237	414
0.00	0.50	50	-449	-135	-27	15	42	98	192	333	552

We can compare Table 9.92 with real new business growth to Table 8.86 with constant new business. Figure 9.37 compares the quantiles of the free asset ratio under the two alternative rates of new business growth.

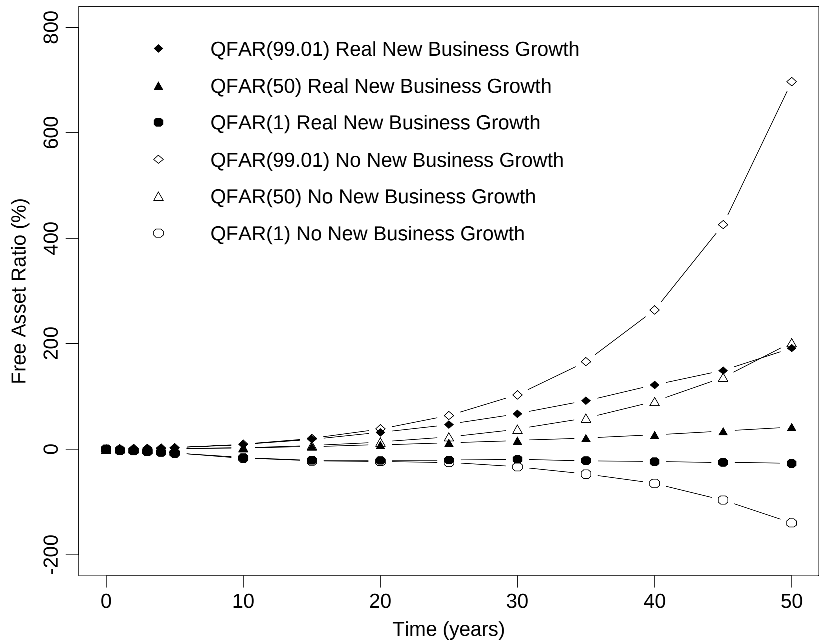

Figure 9.37: A Comparison of the Quantiles of the Free Asset Ratio using the Wilkie Model for Equities, a Stochastic Risk-Free Rate, Smoothed Investment-linked Bonus Strategy, and the Two Alternative Rates of New Business Growth

The free asset ratios take much less extreme values when we allow for real new business growth. For example, in the case where the bonus proportion is 0.5, we see that new business growth reduces the 99.91th quantile of the free asset ratio from 697% to 192%, and similarly improves the 1st quantile of the free asset ratio from −140% to −27%. New business growth is typically positive so that we are usually dividing the profits and losses made on past cohorts by the assets of the new larger cohorts. Hence new business growth tends to dilute past performance.

However for the extreme quantiles, either very high or very low, the results can be either better or worse. These quantiles depend very much on the timing of negative inflation rates which can either push up or down the free asset ratio.

In conclusion we saw in Section 8.6.3 that the free assets often grew to very high levels compared to the existing business. When we allow for new business growth we can still get very high free asset ratios, but in the majority of cases the free asset ratios are noticeably lower. If the office were to grow at a faster rate than inflation then the reduction in the free asset ratio would be even greater. Therefore an expanding guarantee account may in fact be a requirement for an office to continue selling business rather than a disadvantage. We will investigate in Section 9.2 at what point the guarantee account reaches a level where it can be returned to the owners of the insurer without adversely affecting new business growth.

Often new business growth has diluted losses on past business. This is clearly to the advantage of the insurer if they can spread past losses over a larger number of future policies. However, in practice new business levels are likely to decrease as financial advisers are unlikely to recommend insurers with low or negative free assets.

9.2 Distributing Excess Free Assets

In Section 9.1 we saw that despite new business growth the free asset ratio could still grow to very high levels. For example, by time 50 the 90th quantile of the free asset ratio, in the case with a bonus proportion of 0.5, was 98%. Clearly in many simulations the free assets become excessively large. This is the orphan estate problem, whereby the insurer has assets in excess of those required to continue in business. The build up of orphan estates in U.K. insurers is discussed in Smaller et al. (1996).

In this section we will investigate to what extent we can return excess free assets to the owners of the insurer without adversely affecting the insurer's ability to follow its chosen bonus strategy and to pay out the asset share (adjusted for cost of guarantees) at maturity.

We now choose a maximum free asset ratio. If the free asset ratio exceeds this level at the end of the year we remove the excess from the guarantee account and pay it to the owners of the insurer (who may be shareholders and/or policyholders depending on the type of company and the rules governing ownership of the guarantee account). We will consider maximum free asset ratios of 100%, 50%, 25% and 12.5%. Smaller et al. (1996) describe the legal process by which the orphan estate can be attributed to its owners.

The free asset ratio is given by the ratio of the guarantee account less the cost of options to match existing guarantees over the total asset shares (see equation 8.26). Therefore, if the free asset ratio is twice the maximum we need to remove less than half of the assets from the guarantee account. The larger the cost of the options relative to the guarantee account, the smaller the proportion of the guarantee account that will need to be removed.

We will assume that any excess free assets are distributed either to shareholders, or to policyholders as special one-off cash payments. Therefore we assume that a distribution of the free assets does not alter the UWP payout. When the guarantee account becomes negative, we assume that the insurer can call on other funds to maintain payouts. Hence the payout statistics are the same as in Section 9.1 and can be seen in Tables 9.89 and 9.90. The distribution of the free assets only affects the size of the guarantee account.

9.2.1 Guarantee Account and Free Assets

We use the same model as Section 9.1 *i.e.* the Wilkie model with new business growth linked to inflation. Option pricing is performed using a risk-free rate derived from a yield curve fitted to the Wilkie model and volatility of 20% in the Black-Scholes equation. Bonus rates are linked to the investment return with smoothing.

We first look at the effect of distributing excess free assets when the ratio exceeds 100%. The quantiles of the guarantee account and free asset ratio are given in Tables 9.93 and 9.94 respectively. We will only consider the case where the bonus proportion is 50%. Similar results would occur if we looked at different bonus proportions.

The first thing to note is that the free asset ratios are given before any distribution of the excess. For example, Table 9.94 shows us that the 99.01th quantile of the free asset ratio at time 45 was 110%. Assets would then be immediately removed from the guarantee account to bring the free asset ratio down to 100%. The guarantee accounts shown are the figures immediately after any distribution of the free assets.

We can compare Tables 9.93 and 9.94 where the free asset ratio is limited to 100% with Tables 9.91 and 9.92 where the ratio is unlimited. Figure 9.38 shows the sample path of the free asset ratio for a single simulation where the free asset ratio

Table 9.93: Quantiles of the Guarantee Account (Free Asset Ratio Limited to 100%)

y	bp	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	1	1	1	1	1	1	1	1	1
0.00	0.50	1	1	2	2	2	3	4	4	4	5
0.00	0.50	2	2	2	3	4	5	7	8	9	10
0.00	0.50	3	2	3	4	6	8	11	13	16	22
0.00	0.50	4	3	4	5	8	12	16	21	27	35
0.00	0.50	5	4	5	7	11	16	24	34	43	53
0.00	0.50	10	11	14	20	32	54	98	143	179	229
0.00	0.50	15	-210	-104	-22	61	112	184	267	355	448
0.00	0.50	20	-412	-274	-103	90	203	335	492	643	769
0.00	0.50	25	-814	-469	-173	131	344	573	863	1131	1430
0.00	0.50	30	-1857	-775	-250	197	564	960	1480	1925	2536
0.00	0.50	35	-2828	-1249	-383	304	894	1578	2475	3425	4433
0.00	0.50	40	-4457	-1893	-500	458	1391	2567	4037	5962	6975
0.00	0.50	45	-7968	-3178	-663	682	2114	4149	6659	10087	14592
0.00	0.50	50	-14857	-4311	-923	1011	3171	6610	10953	15916	23860

Table 9.94: Quantiles of the Free Asset Ratio (Free Asset Ratio Limited to 100%)

y	bp	Time t	Quantiles of the Free Asset Ratio, $QFAR_t$ 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	0	0	0	0	0	0	0	0	0
0.00	0.50	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.50	2	-10	-5	-3	-1	1	1	1	2	2
0.00	0.50	3	-13	-8	-4	-1	1	1	2	2	2
0.00	0.50	4	-17	-10	-6	-1	1	2	2	3	4
0.00	0.50	5	-18	-14	-7	-1	1	2	3	3	4
0.00	0.50	10	-38	-32	-16	-1	2	5	9	13	18
0.00	0.50	15	-65	-41	-21	0	5	11	19	27	49
0.00	0.50	20	-76	-50	-21	1	8	18	32	49	171
0.00	0.50	25	-154	-52	-21	3	12	26	47	71	90
0.00	0.50	30	-225	-84	-19	5	16	36	67	98	117
0.00	0.50	35	-236	-106	-22	6	21	47	88	120	140
0.00	0.50	40	-197	-100	-23	9	27	61	103	128	165
0.00	0.50	45	-199	-107	-25	11	34	75	110	132	169
0.00	0.50	50	-449	-135	-27	15	42	90	120	146	191

is not limited, and compares it to the sample paths for the same simulation when the free asset ratio is limited to various values.

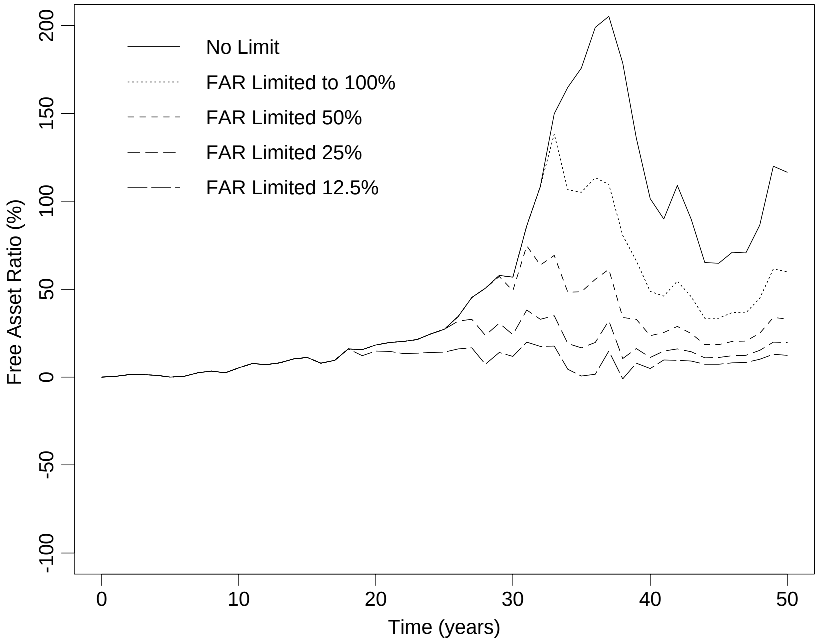

Figure 9.38: A Comparison of the Sample Paths of the Free Asset Ratios where the Free Asset Ratio has been Limited to Various Values

The quantiles are identical up to time 15 because no simulation has yet exceeded the maximum ratio.

From time 20 onwards a number of simulations have free asset ratios greater than 100%. For example, the 99.91th quantile is over 100% from time 30 onwards in the unlimited case in Table 9.92. Hence, from time 30 we see much lower values for the 99.91th quantile in the limited case in Table 9.94.

Notice that even if the free asset ratio quantile in the unlimited case is less than 100% the corresponding quantile in the limited case may still be reduced. Any simulation that has exceeded the maximum ratio in the past will have a smaller guarantee account from that date onwards, as can be seen in Figure 9.38. For example, at time 35 the 99.01th quantile of the free asset ratio is 92% in the unlimited case, but is only 88% in the limited case. This indicates that at time 35 over 90% of the simulations are below the maximum free asset ratio, but some of these simulations have been over 100% at some point in the past. Recall that the simulation with the qth largest free asset ratio at a given time need not be the simulation with the qth largest free asset ratio in the past.

We see that the 0.01th, 0.1th and 1st quantiles refer to simulations which have never distributed any free assets and so are unchanged. However it is still possible, although with low probability, for a simulation to have a free asset ratio in excess of 100% in the early years and negative in the later years — a position that will have been made worse by the earlier distribution of free assets.

Distributing the free estate when the free asset ratio exceeds 100% has been very effective at removing the very high free asset ratios we saw in Table 9.92. The upper quantiles of the guarantee account have been reduced to a lesser extent. This is because very high free asset ratios can be due to very low asset shares caused by negative inflation as well as high guarantee accounts. Some simulations show very high inflation and so a high guarantee account may correspond to only a moderate free asset ratio.

The lower quantiles are barely affected by the free asset distribution strategy. This is good news as it implies that we have been able to safely distribute assets to the owners without adversely affecting policyholders expectations regarding bonuses and payouts.

It can be argued in fact that we are being too cautious and more free assets could be distributed. A free asset ratio of 100% means that even if part of the guarantee account was used to buy matching options, the guarantee account still has risk-free investments left over which are equal in value to the policyholders asset share. Therefore, even if the share price were to collapse and the options writer were to default, the guarantee account would still be able to meet the guarantee. Hence a free asset ratio of 100% represents an incredibly high level of security for the current policyholders.

We now look at the effect of distributing excess free assets when the ratio exceeds 50%. The quantiles of the guarantee account and free asset ratio are given in Tables 9.95 and 9.96 respectively.

Comparing Table 9.96 with Table 9.94 we can see that the upper quantiles of the free asset ratios are of course smaller due to the upper limit of 50%. However, the median free asset ratio is little changed, falling from 42% at time 50 to 37% when

Table 9.95: Quantiles of the Guarantee Account (Free Asset Ratio Limited to 50%)

y	bp	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	1	1	1	1	1	1	1	1	1
0.00	0.50	1	1	2	2	2	3	4	4	4	5
0.00	0.50	2	2	2	3	4	5	7	8	9	10
0.00	0.50	3	2	3	4	6	8	11	13	16	22
0.00	0.50	4	3	4	5	8	12	16	21	27	35
0.00	0.50	5	4	5	7	11	16	24	34	43	53
0.00	0.50	10	11	14	20	32	54	98	143	179	229
0.00	0.50	15	-210	-104	-22	61	112	184	267	355	448
0.00	0.50	20	-412	-274	-103	90	203	335	492	640	769
0.00	0.50	25	-814	-469	-173	131	343	571	847	1131	1430
0.00	0.50	30	-1857	-775	-250	196	561	947	1436	1913	2536
0.00	0.50	35	-2828	-1249	-383	301	870	1530	2363	3350	4433
0.00	0.50	40	-4457	-1893	-500	437	1309	2439	3885	5890	6975
0.00	0.50	45	-7968	-3178	-663	606	1869	3811	6109	9505	14592
0.00	0.50	50	-14857	-4311	-923	832	2577	5799	9666	15406	19423

Table 9.96: Quantiles of the Free Asset Ratio (Free Asset Ratio Limited to 50%)

y	bp	Time t	Quantiles of the Free Asset Ratio, $QFAR_t$ 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	0	0	0	0	0	0	0	0	0
0.00	0.50	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.50	2	-10	-5	-3	-1	1	1	1	2	2
0.00	0.50	3	-13	-8	-4	-1	1	1	2	2	2
0.00	0.50	4	-17	-10	-6	-1	1	2	2	3	4
0.00	0.50	5	-18	-14	-7	-1	1	2	3	3	4
0.00	0.50	10	-38	-32	-16	-1	2	5	9	13	18
0.00	0.50	15	-65	-41	-21	0	5	11	19	27	49
0.00	0.50	20	-76	-50	-21	1	8	18	32	46	102
0.00	0.50	25	-154	-52	-21	3	12	26	46	60	78
0.00	0.50	30	-225	-84	-19	5	16	36	56	68	82
0.00	0.50	35	-236	-106	-22	6	21	45	60	69	78
0.00	0.50	40	-197	-100	-24	9	27	50	62	77	86
0.00	0.50	45	-199	-107	-25	11	33	53	65	75	85
0.00	0.50	50	-449	-135	-27	15	37	55	68	84	97

we apply the upper limit of 50%.

The lower quantiles of the free asset ratio have not been noticeably affected by the imposition of an upper limit of 50%. This indicates that the security of the policyholders at time 50 has not been adversely affected by any earlier distributions of the estate.

Comparing Table 9.95 with Table 9.93 we can see that the upper quantiles of the guarantee accounts are also smaller due to the upper limit of 50%, but the difference is less significant than with the free asset ratios.

The 0.01th, 0.1th and 1st quantiles of the guarantee account are not affected by the upper limit of 50% for the free asset ratio. However, the 10th quantile of the guarantee account falls noticeably from £1,011 at time 50 to £832 when we apply the upper limit of 50%. This is in contrast to the situation for the free asset ratios where the 10th quantile was unchanged. However, the simulation with the qth largest free asset ratio need not be the same as the simulation with the qth largest guarantee account.

We now look at the effect of distributing excess free assets with a much less cautious maximum ratio of 25%. The quantiles of the guarantee account and free asset ratio are given in Tables 9.97 and 9.98 respectively.

From time 30 we see that the upper quantiles of the free asset ratio have reached a steady state above the maximum ratio.

Comparing Table 9.98 with Table 9.96, we see that the 10th quantile of the free asset ratio has been noticeably reduced from 15% at time 50 to 11% under the maximum ratio of 25%. In addition we see a small drop in the 1st quantile from −27% to −29%.

Over 90% of the simulations at time 50 show positive free asset ratios, so the insurer may feel comfortable with distributing free assets when the ratio rises above 25%. However, we see that distributions of free assets early in the projection have adversely affected the lower quantiles by time 50. Therefore it may be dangerous to reduce the maximum free asset ratio much further.

Finally we look at the effect of a very low maximum free asset ratio of 12.5%. The quantiles of the guarantee account and free asset ratio are given in Tables 9.99

Table 9.97: Quantiles of the Guarantee Account (Free Asset Ratio Limited to 25%)

y	bp	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	1	1	1	1	1	1	1	1	1
0.00	0.50	1	1	2	2	2	3	4	4	4	5
0.00	0.50	2	2	2	3	4	5	7	8	9	10
0.00	0.50	3	2	3	4	6	8	11	13	16	22
0.00	0.50	4	3	4	5	8	12	16	21	27	35
0.00	0.50	5	4	5	7	11	16	24	34	43	53
0.00	0.50	10	11	14	20	32	54	98	143	179	229
0.00	0.50	15	-210	-104	-22	61	112	184	267	346	448
0.00	0.50	20	-412	-274	-103	90	202	329	479	622	710
0.00	0.50	25	-814	-469	-173	129	335	544	803	1078	1320
0.00	0.50	30	-1857	-775	-250	189	514	867	1333	1762	2285
0.00	0.50	35	-2828	-1249	-383	260	733	1330	2105	2933	4433
0.00	0.50	40	-4457	-1893	-500	334	981	2007	3205	4540	5986
0.00	0.50	45	-7968	-3178	-669	429	1280	2927	5060	6986	11608
0.00	0.50	50	-14857	-4311	-923	552	1626	4039	7625	11787	15798

Table 9.98: Quantiles of the Free Asset Ratio (Free Asset Ratio Limited to 25%)

y	bp	Time t	Quantiles of the Free Asset Ratio, $QFAR_t$ 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	0	0	0	0	0	0	0	0	0
0.00	0.50	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.50	2	-10	-5	-3	-1	1	1	1	2	2
0.00	0.50	3	-13	-8	-4	-1	1	1	2	2	2
0.00	0.50	4	-17	-10	-6	-1	1	2	2	3	4
0.00	0.50	5	-18	-14	-7	-1	1	2	3	3	4
0.00	0.50	10	-38	-32	-16	-1	2	5	9	13	18
0.00	0.50	15	-65	-41	-21	0	5	11	19	27	40
0.00	0.50	20	-76	-50	-21	1	8	18	28	35	66
0.00	0.50	25	-154	-52	-21	3	12	24	32	39	46
0.00	0.50	30	-225	-84	-20	4	16	27	33	40	47
0.00	0.50	35	-236	-106	-22	6	19	28	34	39	44
0.00	0.50	40	-197	-100	-24	8	21	29	35	41	52
0.00	0.50	45	-199	-107	-26	9	22	29	35	40	42
0.00	0.50	50	-449	-135	-29	11	23	29	35	42	52

and 9.100 respectively.

Table 9.99: Quantiles of the Guarantee Account (Free Asset Ratio Limited to 12.5%)

y	bp	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	1	1	1	1	1	1	1	1	1
0.00	0.50	1	1	2	2	2	3	4	4	4	5
0.00	0.50	2	2	2	3	4	5	7	8	9	10
0.00	0.50	3	2	3	4	6	8	11	13	16	22
0.00	0.50	4	3	4	5	8	12	16	21	27	35
0.00	0.50	5	4	5	7	11	16	24	34	43	53
0.00	0.50	10	11	14	20	32	54	98	142	177	229
0.00	0.50	15	-210	-104	-22	61	111	177	251	324	440
0.00	0.50	20	-412	-274	-103	88	189	290	413	536	643
0.00	0.50	25	-814	-469	-173	114	277	450	661	894	1112
0.00	0.50	30	-1857	-775	-250	144	371	669	1039	1409	1858
0.00	0.50	35	-2828	-1249	-389	172	475	960	1564	2234	2935
0.00	0.50	40	-4457	-1893	-516	204	602	1326	2344	3418	4270
0.00	0.50	45	-7968	-3178	-746	246	760	1776	3341	5049	8750
0.00	0.50	50	-14857	-4331	-1043	299	958	2356	4679	7649	10808

Comparing Table 9.100 with Table 9.98, we see that the 1st quantile of the free asset ratio has been noticeably reduced from -29% at time 50 to -38% under the maximum ratio of 12.5%. The 10th quantile is now dangerously close to zero.

By distributing the free assets when the ratio is as low as 12.5%, the insurer greatly increases the likelihood of a negative free asset ratio. Therefore there is a significant probability that the insurer will need additional capital to continue to trade. The size of the deficit in the free assets is also greatly increased. Therefore there is also an increased chance of the insurer's insolvency.

Given the significant risks to the insurer it is unlikely that they would distribute any free assets when the ratio was as low as 12.5%.

9.2.2 Probability of Negative Free Asset Ratios

So far we have looked at the quantiles of the free asset ratios to give us an idea of the distribution of possible values. However, we only have results for certain quantiles.

Table 9.100: Quantiles of the Free Asset Ratio (Free Asset Ratio Limited to 12.5%)

y	bp	Time t	Quantiles of the Free Asset Ratio, $QFAR_t$ 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	0	0	0	0	0	0	0	0	0
0.00	0.50	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.50	2	-10	-5	-3	-1	1	1	1	2	2
0.00	0.50	3	-13	-8	-4	-1	1	1	2	2	2
0.00	0.50	4	-17	-10	-6	-1	1	2	2	3	4
0.00	0.50	5	-18	-14	-7	-1	1	2	3	3	4
0.00	0.50	10	-38	-32	-16	-1	2	5	9	13	18
0.00	0.50	15	-65	-41	-21	0	5	11	16	20	28
0.00	0.50	20	-76	-50	-22	1	8	14	18	23	49
0.00	0.50	25	-154	-52	-21	1	10	15	19	23	31
0.00	0.50	30	-225	-84	-23	2	11	15	19	24	33
0.00	0.50	35	-236	-106	-26	2	12	15	19	23	29
0.00	0.50	40	-197	-100	-28	3	12	15	19	24	32
0.00	0.50	45	-199	-107	-31	3	12	15	19	23	25
0.00	0.50	50	-449	-135	-38	3	12	15	19	24	28

An event we are particularly interested in is the probability of the insurer having negative free assets. In this section we will investigate the probability that the insurer has negative free assets both at a given point in time and at any point up to and including a given time.

Using the definition in Section 8.4, negative free assets represent an insurer who has received less in charges than the cost of the guarantees. The guarantee account is simply an accounting device to keep track of profits and losses from providing maturity guarantees. It was considered in Clay et al. (2001) that this would make the workings of the with-profits fund more transparent.

Negative free assets do not necessarily imply an insolvent insurer. The insurer may be able to continue to pay out the full value at maturity by drawing on other funds. The insurer may be quite comfortable running a small deficit in the guarantee account for a few years.

However, a negative guarantee account does represent a genuine loss to the owners. The insurer will want to minimise the size of this loss and the probability of its occurrence.

A negative guarantee account will also be seen by financial advisers as a negative feature. The advisers may expect such insurers to increase their charges at some time in the future.

We consider first the number of simulations which show negative free assets at a given point in time (starting at year zero with 5 yearly intervals). Insurers with negative free assets are allowed to continue trading as normal. No funds are injected into the guarantee account to bring it back to zero. We limit the maximum size of the account in the same way as Section 9.2.1.

The results can be seen in Table 9.101.

Table 9.101: The Number of the 10,000 Simulations which have Negative Free Assets at the Given Time

Max FAR_t (%)	Time 0	5	10	15	20	25	30	35	40	45	50
12.5	0	1909	1318	1024	900	877	788	795	760	778	761
25	0	1909	1318	1019	836	712	582	499	444	412	363
50	0	1909	1318	1019	832	701	553	465	399	364	305
100	0	1909	1318	1019	832	700	553	465	399	359	301
No Limit	0	1909	1318	1019	832	700	553	465	398	359	301

First we consider the case where there is no upper limit on the free asset ratio and hence no distributions from the free assets. The probability of negative free assets in the early years is a high 19%. The guarantee account starts at zero so any adverse event is likely to create a deficit. In practice the insurer may wish to inject some capital into the guarantee account when the first policies are sold. However, as time goes by the guarantee account tends to rise and the probability of negative free assets falls to 3%.

Now we consider distributing the free assets to the owners whenever a maximum free asset ratio has been reached. Initially the figures will be the same as the case with no upper limit. Differences occur when simulations have negative free assets which have occurred due to distributions of free assets at some point in their past.

The insurer will want to make distributions only when it is reasonably confident that the funds will not be required at a later date.

The probability of negative free assets is little affected by distributing free assets when they exceed 50% of asset share *e.g.* at time 50 the probability has only increased from 3.01% to 3.05%.

Lower maximum ratios have a more noticeable effect on the probabilities. A maximum free asset ratio of 12.5% would probably be considered too low as it leads to considerably higher probabilities of negative free assets, in this case 7.61% at time 50.

So far we have considered only the probability of negative free assets at a given point in time. This is important to an insurer who can call on additional funds to allow it to continue trading through short periods of negative free assets, but is still worried by the negative publicity it will bring.

However, if the insurer is unable to call on further assets then it may have to stop trading as soon as its free assets become negative. For this insurer it is the probability that the free assets are negative at any point up to the given time which is important.

The number of simulations that have ever shown negative free assets up to a given time can be seen in Tables 9.102.

Table 9.102: The Number of the 10,000 Simulations which have had Negative Free Assets at Least Once by the Given Time

Max FAR_t (%)	Time 0	5	10	15	20	25	30	35	40	45	50
12.5	0	6782	7855	8203	8395	8492	8580	8638	8719	8786	8848
25	0	6782	7855	8203	8379	8446	8486	8507	8521	8531	8535
50	0	6782	7855	8203	8379	8441	8473	8485	8491	8492	8492
100	0	6782	7855	8203	8379	8441	8473	8485	8491	8492	8492
No Limit	0	6782	7855	8203	8379	8441	8473	8485	8491	8492	8492

We can see that with annual charging 68% of the simulations have had negative free asset ratios by time 5. Even without distributions of the free estate this figure rises to 85% by time 50. Most of the affected simulations will have been slightly negative in the early years because share prices increased too slowly at some point leading to the cost of the options being higher than the accumulated charges.

These probabilities could be substantially reduced if the insurer injected even a modest amount of capital into the guarantee account at outset. For example, even if the insurer paid only £1 of capital into the guarantee account at time zero, the probability of negative free assets by time 5 drops to 38%.

Therefore we see that the probability of negative free assets at a given time is much lower than the probability of negative free assets at least once before that time. This indicates that many simulations have negative free assets for a short period and then recover. The calculation of the additional capital required to reduce the probability of negative free assets is left to future research.

In conclusion, we have seen in Section 9.2 that distributing free assets when the ratio exceeds 50% or more has little effect on the number of simulations in which negative free assets occur, and little effect on the size of the losses. However, if the maximum free asset ratio is reduced to 25%, we begin to see an increased number of losses and some losses increase in size. Therefore, the results suggest that an insurer is unlikely to feel comfortable with distributing free assets until the free asset ratio is somewhere in the range of 25% to 50%. These results though are likely to be very sensitive to the choice of the investment model, its parameters, and the bonus policy.

9.3 Transition to a Lower Inflation Environment

So far we have been modelling the investment returns using the Wilkie model with the taxed dividend parameterisation shown in Table 5.24. These parameters were obtained in Wilkie (1995) by fitting the model to U.K. data from the period 1923 to 1994. Therefore, the results we have obtained so far are appropriate for an insurer modelling their business in 1994.

However, we see from the data provided by David Wilkie shown in Figure 9.39 that the U.K. has experienced many years of relatively low and stable inflation since

1994. This has lead to lower investment returns, and hence a greater likelihood that the guarantees will bite.

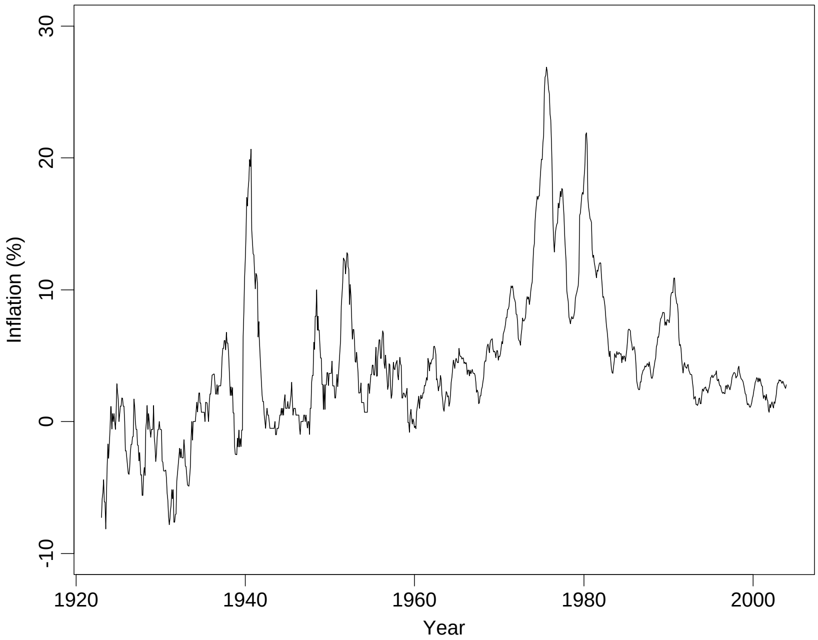

Figure 9.39: U.K. Inflation

Nowell et al. (1999) consider the reasons behind the recent trends in U.K. inflation and discuss the effects on U.K. life insurers. In particular, they use stochastic modelling to calculate the probability of insolvency for an insurer which has experienced high inflation in the past but expects lower inflation in the future.

In this section we will consider the effect on the projected policy payout and guarantee account which would be obtained if the real world model now allows for a greater probability of low inflation in the future.

For the first 10 years of each simulation we will continue to use the high inflation parameters with taxed dividends from Table 5.24. We will use the corresponding neutral initial starting conditions from Table 5.25. Hence, by time 10 we will have 10,000 simulated portfolios of policies built up under the high inflation parameters.

From time 10 we want to switch to a lower inflation parameterisation. The parameter QMU governs the mean force of inflation. Hence at time 10 we reduce the parameter QMU from 0.047 to 0.024. We leave all the other parameters unchanged in order to focus on the effects of a change in the mean force of inflation only.

The 10,000 simulations are then continued with the lower value of QMU for the remainder of each simulation.

Note that under the lower inflation parameterisation it is still possible for high inflation scenarios to occur. However, high inflation will now occur with lower probability. Our model reflects the belief at time zero that low inflation will become more likely from time 10 onwards.

We can see the mean, standard deviation and correlation of the annualised returns produced using the low inflation parameters in Table 9.103. The annualised returns, $GX(t)$, of the total return indices (with income reinvested), $X(t)$, are given as in Equation 5.14, for prices Q, equities PR, consols CR, and cash BR as follows:

$$GX(t) = 100\left[\left(\frac{X(t)}{X(0)}\right)^{1/t} - 1\right]$$

Table 9.103: Summary Statistics from the Wilkie Investment Model with Low Inflation Parameters

	Term					
	1	2	5	10	20	50
$M(GQ)$	4.91	4.93	4.90	4.74	3.69	2.94
$SD(GQ)$	4.45	4.14	3.60	2.95	2.23	1.45
$M(GPR)$	11.55	10.65	10.26	10.30	9.42	8.26
$SD(GPR)$	19.60	12.71	7.03	4.91	3.54	2.31
$M(GCR)$	7.63	7.61	7.73	7.91	8.33	7.66
$SD(GCR)$	7.96	5.34	2.94	1.64	1.13	1.08
$M(GBR)$	6.16	6.23	6.36	6.47	6.37	5.80
$SD(GBR)$	0.00	0.63	1.11	1.30	1.30	1.14
$C(GPR, GQ)$	-0.31	-0.14	0.16	0.38	0.53	0.64
$C(GCR, GQ)$	-0.32	-0.41	-0.54	-0.55	-0.26	0.34
$C(GCR, GPR)$	0.33	0.25	0.06	-0.06	0.00	0.28
$C(GBR, GQ)$	0.00	0.11	0.20	0.31	0.43	0.62
$C(GBR, GPR)$	0.00	-0.05	0.01	0.15	0.26	0.40
$C(GBR, GCR)$	0.00	-0.27	-0.33	-0.25	0.16	0.67

We can now compare the results from the low inflation parametrisation in Table 9.103 with the higher inflation parameterisation in Tables 5.26 and 5.27.

First of all we notice that the results are identical prior to time 10. The low inflation model has been run with the high inflation parameters until time 10.

The mean inflation rate over a given term, $M(GQ)$, has indeed reduced from around 5% p.a. over the first year, to 3% p.a. over the 50 year term. Note that $M(GQ(50))$ represents the mean rate of inflation over the 50 year period, and so averages over the initial high inflation and the later low inflation. The annual rate of inflation at time 50 has actually fallen to 2.5% p.a.. Negative inflation rates are permitted and occur much more frequently with the low inflation parameters.

The mean return on equities, $M(GPR)$, over 10 years is a slightly higher 10.30% under the low inflation model, than the 10.17% under the high inflation model. Lower inflation causes the dividend yield to fall at time 10, and hence the price rises. However, from time 20 the mean return is lower under the low inflation model because of the lower dividend growth.

The mean return on consols, $M(GCR)$, over 10 and 20 years is higher under the low inflation model. Lower inflation leads to lower yields and hence to higher prices. However, over longer time periods the lower reinvestment rate becomes more important, so that the mean return on consols over 50 years is lower under the low inflation model. Note that the consols yield continues to be set to a minimum of 0.5%. This minimum value is used on many more occasions for the low inflation parameters.

The mean base rate, $M(GBR)$, decreases in line with the lower consols yield under the low inflation model. Hence the return on cash over 20 or more years is lower under the low inflation model.

The parameters affecting the standard deviation and correlation structure remain unchanged and indeed we see only small differences between Table 9.103 and Tables 5.26 and 5.27 in respect of second order moments.

9.3.1 Results

In this section we consider the resulting payouts and guarantees for the UWP, unit-linked, and risk-free policies using the Wilkie model with low inflation parameters discussed above as our model of the real world. We use the bonus mechanism considered in Section 8.6 such that bonuses are directly linked to the investment return subject to some smoothing.

The option pricers continue to use a different model than the real world model, but set their parameters to be consistent with their observations of the real world.

The standard deviation of the equity returns has not substantially changed, so option pricers continue to use a volatility of 20% in the Black-Scholes equation.

The risk-free rate is still derived from a yield curve fitted to the Wilkie model, but the lower consol and base rates will lead to correspondingly lower risk-free rates. Hence options will become more expensive under the low inflation parameters. This is an advantage of the option pricing approach to setting charges in that charges will rise to reflect the increased cost of the guarantees.

The summary statistics are shown in Tables 9.104 to 9.109.

Table 9.104: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model with Low Inflation Parameters
Start Year 0

y	bp	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		145.65	68.01	0.00	0.00
0.00	0.00	141.98	66.04	50.00	0.00
0.00	0.50	136.91	64.98	78.42	8.05
0.00	0.60	135.12	64.22	85.03	10.74
0.00	0.70	133.08	63.01	91.57	13.90
R-F		105.83	0.00	105.83	0.00

Table 9.105: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model with Low Inflation Parameters
Start Year 0

y	bp	Desired Bonuses Always Affordable	Option Exercised	UWP > UL	Ach Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	10000	177	146	146	3307	10000
0.00	0.50	8687	1798	893	893	3913	9969
0.00	0.60	7877	2515	1185	1185	4202	9589
0.00	0.70	6857	3374	1462	1462	4464	8553

We can compare Tables 9.104 to 9.109 which use the low inflation parameters with Tables 8.77 to 8.82 which use the higher inflation parameters.

First of all we look at policies sold at time zero, and so compare Tables 9.104 and 9.105 with Tables 8.77 and 8.78.

In both cases we have used high inflation parameters up to and including time 9. The risk-free payout is determined by the yield on a 10-year zero coupon bond at time zero, and so is the same in both cases. The UWP bonuses are declared up to the ninth policy year, and so the achieved guarantee is also the same in both cases.

However, the low inflation parameters are used at time 10. We saw in Table 9.103 that this leads to a slightly higher return on equities that year. Hence, under the low inflation parameters we see a slightly higher mean unit-linked and UWP payout. Correspondingly, under the low inflation parameters, the option is exercised a little less often, the unit-linked payout is more likely to exceed the UWP achieved guarantee, and the UWP payout is more likely to exceed the risk-free payout, than under the high inflation parameters.

Table 9.106: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model with Low Inflation Parameters
Start Year 20

y	bp	Maturity Payout Mean	Maturity Payout SD	Achieved Gtee Mean	Achieved Gtee SD
UL		118.83	66.34	0.00	0.00
0.00	0.00	115.34	63.66	50.00	0.00
0.00	0.50	111.69	61.29	73.45	11.98
0.00	0.60	110.90	60.18	78.07	15.19
0.00	0.70	110.20	58.79	82.31	18.35
R-F		102.20	20.29	102.20	20.29

Table 9.107: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model with Low Inflation Parameters
Start Year 20

y	bp	Desired Bonuses Always Affordable	Option Exercised	UWP > UL	Ach Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	9998	963	789	789	5032	10000
0.00	0.50	6572	3927	2130	2130	5794	9606
0.00	0.60	5702	4628	2431	2431	5943	9145
0.00	0.70	4845	5245	2715	2715	6004	8586

We now look at policies sold at time 20, and so compare Tables 9.106 and 9.107

with Tables 8.79 and 8.80.

Bonuses are declared as a proportion of the investment return, so the bonuses are lower under the low inflation parameters. Hence the mean and standard deviation of the achieved guarantees are lower. The achieved guarantee is higher for higher values of bp, but the differences are smaller than we saw under the higher inflation parameters.

The lower investment returns and lower guarantees lead to a lower mean and standard deviation of the unitised with-profit payouts under the low inflation parameters. Lower inflation also leads to a lower risk-free rate and hence to an increase in the cost of options. Therefore, higher charges also reduce the UWP payouts.

There are more occasions when the bonuses are unaffordable under the low inflation parameters than under the higher inflation parameters. The expected investment return is lower, but the variance is unchanged. Hence we will have more occasions when investment returns are negative over a period of years, making even a small positive bonus impossible. This fact is made worse by the smoothing mechanism which limits the speed at which the bonuses can be cut as the investment returns fall.

The greater likelihood of poor investment returns under the low inflation parameters means that the option is exercised more often. Whenever the option is exercised the policyholder receives just the guarantee with no additional terminal bonus. In fact for a bonus proportion of 70% we see that the option is now exercised in more than half of the simulations. Policyholders expect a terminal bonus to be paid, and so the insurer should be using a strategy that leads to lower guarantees within a low inflation environment.

The unit-linked and risk-free policies are also affected by the parameter changes.

Lower equity returns lead to much lower unit-linked payouts. The relatively high guarantees mean that the UWP policy has a higher probability of outperforming the unit-linked policy under the low inflation parameters.

Similarly, lower risk-free returns lead to lower risk-free payouts. The risk-free payout is determined at time 20 by the yields on zero coupon bonds at that time, which are lower due to the lower inflation. The average inflation rate continues to fall between times 20 and 30. This leads to a lower mean return on equities, but has no effect on the risk-free payout. Hence, the risk-free payout has fallen much less than the unit-linked payout. For example, the move from high inflation to low inflation parameters reduces the mean unit-linked payout from £148.78 to £118.83, while the corresponding reduction for the risk-free policy is from £110.33 to £102.20.

Note that the standard deviation of the risk-free payout is positive only because the state of the model at time 20 will have moved away from the initial conditions to different extents in each simulation. However, at the time the policy is sold the risk-free rate is known. So for the policyholder buying a policy at any given time the risk-free account always has zero risk.

We see that the risk-free policy has a higher probability of outperforming the UWP policy under the low inflation parameters. In fact, for a bonus proportion of 70%, the risk-free policy outperforms the UWP policy in 60% of simulations. This reflects the fact that the risk-free policy has been able to lock into a high rate of return before inflation has fallen further.

Table 9.108: Mean and Standard Deviation of the Payout and Achieved Guarantee using the Wilkie Model with Low Inflation Parameters
Start Year 40

y	bp	Maturity Payout Mean	SD	Achieved Gtee Mean	SD
UL		116.87	65.28	0.00	0.00
0.00	0.00	111.89	61.76	50.00	0.00
0.00	0.50	108.02	59.68	70.14	11.05
0.00	0.60	107.26	58.75	74.06	13.91
0.00	0.70	106.45	57.66	77.72	16.70
R-F		94.72	20.40	94.72	20.40

Table 9.109: The Number of the 10,000 Simulations where the Payouts and Guarantees of a 10-year Single Premium Policy Display Certain Features using the Wilkie Model with Low Inflation Parameters
Start Year 40

y	bp	Desired Bonuses Always Affordable	Option Exercised	UWP > UL	Ach Gtee > UL	RF > UWP	RF > Ach Gtee
0.00	0.00	9997	1093	848	848	4702	10000
0.00	0.50	6658	3898	2003	2003	5473	9414
0.00	0.60	5854	4483	2250	2250	5591	8894
0.00	0.70	5112	5121	2464	2464	5683	8317

Finally we look at policies sold at time 40. The overall patterns are the same as for policies sold at time 20. However, there are some interesting differences due to the continuing fall in the average rate of inflation.

The lower rate of inflation between times 40 and 50 compared to that between times 20 and 30 means that the payout on the unit-linked and UWP policies has continued to fall. However, the risk-free payout has fallen much more quickly. The inflation rate is not falling so sharply at time 40 compared to time 20, so the risk-free policy no longer benefits from being able to lock into higher interest rates.

There is a higher probability that the desired bonuses will be affordable for a policy sold at time 40 than at time 20. At time 20 the smoothing mechanism meant that the bonuses were often cut too slowly to be affordable. However, by time 40 there has been adequate time for the bonuses to fall to a more appropriate level given the lower expected investment returns.

The risk-free policy sold at time 40 has a lower probability of outperforming the UWP policy than for a policy sold at time 20. As we commented above, the risk-free policy no longer has the benefit of locking into high yields before inflation falls.

9.3.2 Guarantee Account

We now look at the effect of the low inflation parameters on the guarantee account GA_t. The guarantee account is calculated in the same way as in Section 8.5.2 as given by Equation 8.27. We will concentrate on the case where the bonus proportion is 0.5 throughout this section.

The quantiles of the guarantee account are given in Table 9.110 below.

We can compare Table 9.110 under the low inflation parameters to Table 8.83 under the higher inflation parameters. Certain quantiles are compared in Figure 9.40.

Up to and including time 9, both models use the high inflation parameters and so the results are identical.

From time 10 onwards the low inflation parameters have the following effects on the guarantee account:

- The average yield on zero coupon bonds is lower. Hence, options are more expensive leading to higher charges being passed into the guarantee account for a given level of guarantees.
- The lower expected return on equities means that the guarantees bite more often, and so the guarantee account must pay out more often.
- However, the above effect is offset by the lower guarantees that build up due to the lower investment returns.

Table 9.110: Quantiles of the Guarantee Account using the Wilkie Model with Low Inflation Parameters

y	bp	Time t	Quantiles of the Guarantee Account, QGA_t 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	1	1	1	1	1	1	1	1	1
0.00	0.50	1	1	2	2	2	3	4	4	4	5
0.00	0.50	2	2	2	3	4	5	6	8	9	10
0.00	0.50	3	2	3	4	5	8	10	13	16	22
0.00	0.50	4	3	4	5	7	11	16	22	28	33
0.00	0.50	5	3	5	6	9	15	22	34	45	56
0.00	0.50	10	6	11	16	26	45	90	136	184	243
0.00	0.50	15	-168	-104	-22	48	90	157	241	326	541
0.00	0.50	20	-380	-260	-98	69	154	263	399	549	1552
0.00	0.50	25	-798	-439	-172	84	239	397	624	900	1325
0.00	0.50	30	-1494	-653	-250	103	357	595	946	1431	2932
0.00	0.50	35	-1839	-902	-357	147	514	877	1397	2333	4886
0.00	0.50	40	-2400	-1488	-507	208	727	1274	2034	3533	5692
0.00	0.50	45	-4762	-1986	-748	281	1011	1831	2967	4569	7718
0.00	0.50	50	-12276	-2857	-1021	387	1383	2688	4549	7436	12356

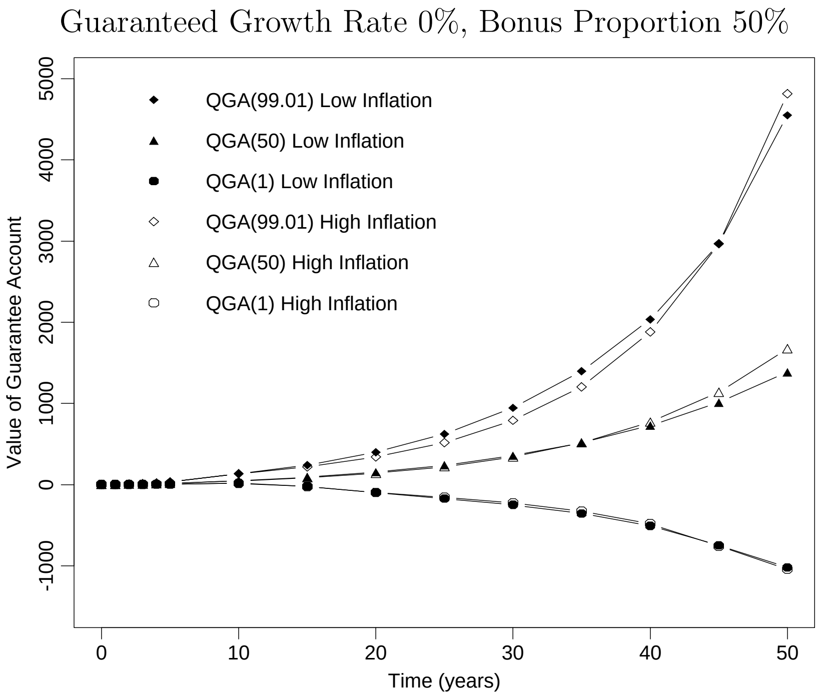

Figure 9.40: A Comparison of the Quantiles of the Guarantee Account using the Two Alternative Wilkie Model Parameterisations

- The average rate of return on consols is lower, and so the guarantee account rolls up at a lower rate of interest. This reduces the accumulated profits made on the upper quantiles, but also reduces the accumulated losses on the lower quantiles.

Some of the above effects cause the guarantee account to be higher under the low inflation model, while other effects cause the guarantee account to be lower. As a result, the quantiles are very similar under the two parameterisations. However, there are a few differences.

The 99.91th and 100th quantiles represent simulations where the guarantees have never bitten and so benefit from the higher charges.

Similarly, the 10th through to the 90.01th quantiles initially show higher values, but at later dates are lower than under the high inflation model.

We might expect the lowest quantiles to show larger losses under the low inflation model due to the greater likelihood of the guarantees biting. However, the results are in fact very similar under both parameterisations, due to the smaller guarantees and the lower rate at which the guarantee account is rolled up.

Hence, the insurer is certainly no worse off under a lower inflation environment as long as lower bonuses are declared and the charges are adjusted for a lower risk-free rate in the Black-Scholes equation. A low inflation environment does not increase the probability of large mismatching losses.

9.3.3 Free Assets

We now consider the effect of the low inflation parameters on the free asset ratios. The results are shown in Table 9.111.

We can compare Table 9.111 under the low inflation parameters to Table 8.86 under the higher inflation parameters. Certain quantiles are compared in Figure 9.41.

The high inflation parameters are used in both models up to and including time 9, so that we again see identical results up to this time.

The free asset ratio is given by the guarantee account less the cost of matching options, all divided by the asset share. Hence the free asset ratio is effected by the

Table 9.111: Quantiles of the Free Asset Ratio using the Wilkie Model with Low Inflation Parameters

y	bp	Time t	Quantiles of the Free Asset Ratio, $QFAR_t$ 0.01	0.1	1	10	50	90.01	99.01	99.91	100
0.00	0.50	0	0	0	0	0	0	0	0	0	0
0.00	0.50	1	-4	-3	-2	-1	0	1	1	1	1
0.00	0.50	2	-9	-5	-3	-1	1	1	1	2	2
0.00	0.50	3	-12	-8	-4	-1	1	1	2	2	2
0.00	0.50	4	-17	-10	-6	-1	1	2	2	3	4
0.00	0.50	5	-18	-14	-7	-1	1	2	3	3	4
0.00	0.50	10	-37	-32	-16	-1	3	6	10	13	18
0.00	0.50	15	-56	-44	-23	0	7	14	22	31	53
0.00	0.50	20	-85	-58	-28	1	15	28	44	63	232
0.00	0.50	25	-202	-75	-34	4	26	48	78	123	203
0.00	0.50	30	-272	-133	-48	8	42	78	129	205	481
0.00	0.50	35	-314	-163	-65	14	63	120	206	396	853
0.00	0.50	40	-459	-254	-85	21	94	183	314	503	1046
0.00	0.50	45	-645	-314	-118	32	133	268	460	768	1361
0.00	0.50	50	-2576	-469	-160	46	185	395	717	1268	2130

same factors as the guarantee account which we described above. In addition, the lower rate of inflation means a higher cost of matching options which will reduce the free asset ratio. However, the lower return on equities will reduce the asset share and hence increase the free asset ratio.

The result is that from time 10 onwards the quantiles of the free asset ratios are also very similar under the two parameterisations. However, we see that for the 10th quantile and below, low inflation has resulted in lower free asset ratios. While for the 99.01th quantile and above, low inflation has resulted in higher free asset ratios.

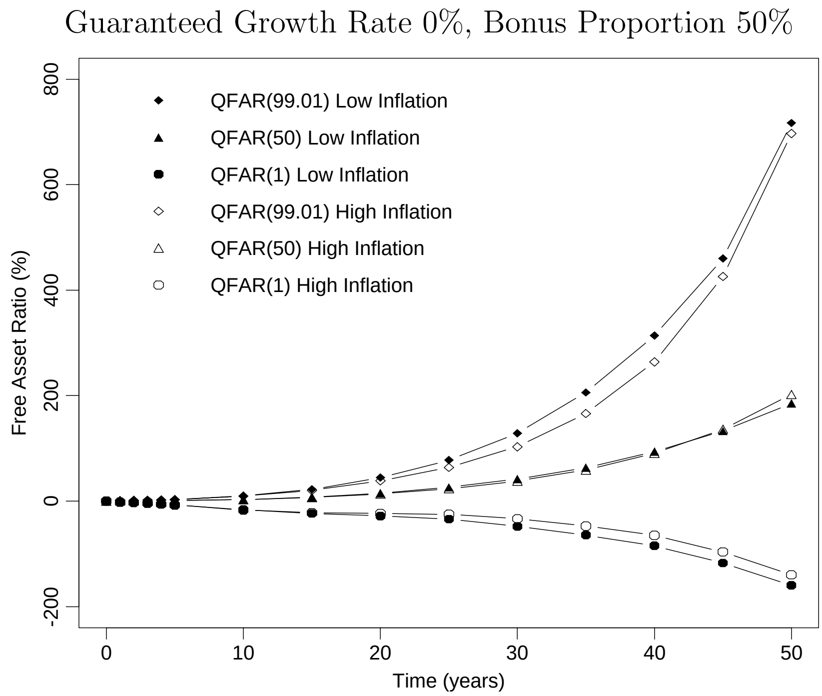

Figure 9.41: A Comparison of the Quantiles of the Free Asset Ratio using the Two Alternative Wilkie Model Parameterisations

Chapter 10

Conclusions and Further Research

In this thesis we have investigated how to charge a with-profits policyholder for investment guarantees in a way that is consistent with the cost of matching assets. Wilkie (1987) had shown how to do this for conventional with-profits policies. In this thesis we have extended this approach to unitised with-profits policies.

In a MSc project under my supervision, Yap (1999) was the first to apply this approach to UWP policies, but did not perform stochastic simulations.

Hare et al. (2000) also considered setting UWP charges with reference to the cost of options. However, they considered a put spread strategy which only ensured that the guarantee could be met in 99% of cases. They stochastically modelled the payouts to the policyholder, but assumed that no further bonuses would be declared.

We can summarise the main areas of new research performed in this thesis as follows:

- Survey of the literature covering the modelling of policies with investment guarantees.
- Simulations of UWP policies following the matched portfolio of shares and put options with annual bonus declarations.
- Investigation of possible bonus strategies based on this matched portfolio.
- Comparison of maturity guarantees and payouts for UWP policies under a variety of different bonus rates, with a unit-linked policy and a risk-free investment.

- Consideration of a portfolio of UWP policies with charges passed to a guarantee account.

- The effects of mismatching within the guarantee account.

- The effects on the guarantee account of new business growth, distribution of free assets, and transition to a low inflation environment.

We summarise the main conclusions of this thesis in Section 10.1. Then in Section 10.2 we discuss possible extensions to this research.

10.1 Conclusions

Chapter 1 described the operation of with-profits policies and financial derivatives.

Chapter 2 reviewed the literature on pricing and reserving for investment guarantees of both unit-linked and with-profits policies.

Chapter 3 began with a description of the option pricing mechanism suggested by Wilkie (1987) for conventional with-profits policies. We then described how Wilkie's methodology can be applied to unitised with-profits policies. We concluded that the option pricing approach to charging for the investment guarantees of UWP policies had the following advantages:

- A portfolio of shares and put options can be constructed that replicates the payoff from a UWP policy. The policholder's asset share is the value of this portfolio.

- The cost of purchasing the options is an implicit charge for the investment guarantees given. Hence the asset share is a fair value to pay the policyholder at maturity as it will already include an appropriate deduction for the cost of guarantees.

- The asset share also represents a fair surrender value in that it represents the value of the guarantee at the time of surrender.

- The option pricing approach maintains equity between policies with different guarantees. The charge paid by the policyholder represents the cost of their bonuses at that time. Hence, the policyholder could even be permitted to choose their own bonus rates up to the maximum affordable.

- The option pricing approach can also be modified to calculate the equitable charge from the policyholder whenever the insurer changes the investment portfolio.
- The conventional with-profits option pricing mechanism suggested by Wilkie (1987) suffers from the possibility that the amount of the guarantee actually matched by options can fall in some years. The UWP option pricing mechanism ensures that whatever guarantees have been declared can always be matched by appropriate options.

Chapter 4 projected the performance of several unitised with-profits policies with different bonus rates and compared them with a unit-linked policy and a risk-free policy. The projections were performed using geometric Brownian motion as the real world investment model, which was consistent with the assumptions underlying the Black-Scholes formula being used by the option pricers. Our aim was to see how changes in the guaranteed growth rate, and desired bonus rate, affected the payouts and guarantees of the UWP policies. The main conclusions were:

- Increasing the guaranteed growth rate or the desired bonus rate generally increases the achieved guarantee. However, in some simulations the final achieved guarantee is lower, because the guarantee is increased too quickly in the early years, hence too many units of shares must be sold to buy matching options, and later guarantees become unaffordable.
- Increasing the guaranteed growth rate or the desired bonus rate decreases the mean payout because a larger number of shares must be sold to buy options to match the higher guarantees.
- Increasing the guaranteed growth rate or the desired bonus rate decreases the standard deviation of the payout. However, to obtain a substantial reduction in the standard deviation requires a high guaranteed growth rate which also

substantially reduces the mean payout. Policyholders may not be prepared to pay such a high price for investment guarantees.

Chapter 4 used Geometric Brownian Motion as the real world model. Then in Chapter 5 we compared these results with those obtained using the Wilkie Model to simulate the real world. We continued to assume that options were priced according to the Black-Scholes formula, so that we could observe the effects of model error. Even though the two alternative real world models had been parameterised to give the same mean and standard deviation of log equity returns over a single year, we still observed significant differences in the results as follows:

- The annualised return on the equity index over a 20 year term has higher standard deviation under the GBM model than the Wilkie model, but similar mean. The effects of compounding these annualised returns over the 20 years will result in both a higher mean and standard deviation of payout from a unit-linked policy invested entirely in equities under the GBM model.

- The higher mean and standard deviation of unit-linked payouts under the GBM model is repeated for UWP policies. Typically the mean UWP payout is 10% higher, and the standard deviation is 50% higher, under the GBM model compared to the Wilkie model.

- The higher variability of the equity returns under the GBM model leads to a higher probability of very low returns. Hence, under the GBM model the desired bonuses are unaffordable more often, leading to a lower mean achieved guarantee, and the guarantees bite more often at maturity.

The differences in the results obtained under the two different real world models demonstrates the problems of model error. In particular, the lower probability of the guarantee biting under the Wilkie model than under the GBM model indicates that the option prices (and hence charges) set by the Black-Scholes formula (which is also based on the GBM model) may be too high.

Chapter 6 extended the work in Chapter 5 to use a stochastic risk-free rate of return derived from the Wilkie model. We assumed that the market continued to price options using the Black-Scholes formula, but that the risk-free rate used in the formula was the rate derived from the Wilkie model at that time. Hence the variability in the risk-free rate of interest causes greater variability in the charges deducted from the policyholder, and hence a higher standard deviation of the UWP payout.

In Chapter 6 we also calculated the payout on a policy where all premiums were invested at the risk-free rate. Both the mean and standard deviation of the payout were considerably lower than for the UWP policies.

In Chapter 7 we considered the effectiveness of a number of different bonus strategies for use with the option pricing technique. We found that flexibility in the bonus strategy was of benefit to the policyholder if they were correctly charged for their guarantees.

Firstly in Section 7.3.1, we found that the investment-linked bonus strategy resulted in both a higher mean payout and lower variability of payout than the fixed bonus strategy. The flexibility to change bonuses with investment returns is of benefit to the policyholder because larger guarantees are declared when the cost of the matching options is relatively cheap.

Secondly in Section 7.3.2, we found that the unsmoothed strategy resulted in both a higher mean payout and lower variability of payout than the smoothed strategy. This implies that retaining flexibility to change bonuses quickly without smoothing is of benefit to the policyholder.

Chapter 8 began by introducing a portfolio of policies with different start dates. The insurer could match the liabilities with a portfolio of equities and put options in the same way as for a single policy. Hence, the policyholder's payout would be unaffected by the other policies and the insurer would make neither a profit nor a loss.

However, insurers in the U.K. have not traditionally invested in options. Therefore in Chapter 8 we considered the case where the insurer did not match assets to liabilities. Whenever guarantees were increased, each policyholder was charged an amount equal to the cost of buying options to match the increase in their guarantees. Hence, the policyholder was unaffected by the mismatching strategy.

However, the charges were passed to a guarantee account owned by the insurer. Rather than invest in matching options we assumed that the insurer invested the guarantee account in the risk-free asset. We investigated the distribution of the mismatching profits and losses of this approach. We also calculated the free assets as being the value of the guarantee account less the cost of buying matching options. The free asset ratio was then calculated as the free assets as a proportion of the total asset shares of the portfolio. The main results from Chapter 8 of mismatching assets and liabilities were:

- The median free asset ratio is positive and grows through time. This indicates that mismatching is expected to be profitable.
- The free asset ratio can become negative. This requires the insurer to have access to additional capital to make good the loss in order to avoid insolvency.
- The free asset ratio can also become very large and positive. This is the orphan estate problem whereby the insurer has assets in excess of those required to continue in business.
- Declaring higher bonuses leads to higher median profits, but also leads to more extreme high and low values for the free asset ratio.
- Modelling using a stochastic risk-free rate leads to more extreme high and low values for the free asset ratio. This indicates that a simpler model using a constant risk-free rate may underestimate the size of potential problems.
- Similarly, using a dynamic bonus rate leads to more extreme high and low values for the free asset ratio. This demonstrates that smoothing bonuses under the dynamic bonus mechanism makes the problems of mismatching worse whenever these problems arise.

In Chapter 9 we applied the multiple generation model of Chapter 8 to a number of issues that have affected the with-profits industry in the U.K. in recent years. The main conclusions were as follows:

- Allowing for new business growth leads to less extreme high and low values for the free asset ratio. Even if new business grows only at the rate of inflation, the problems of the orphan estate and insolvency are both greatly reduced. A positive real rate of new business growth would reduce these problems further. Therefore, in theory, the insurer can use the rate of new business growth to control the free asset ratio. This approach is likely to be highly successful to control the orphan estate problem as a strong insurer should have no difficulty finding new customers. However, an insurer with negative free assets in the guarantee account may have difficulty attracting new customers, and may even contract making the free asset ratio even worse.

- Even for an insurer with new business growth at the rate of inflation it is still possible for very large free asset ratios to arise. We found that returning excess free assets to the owners of the insurer whenever the free asset ratio exceeded 50% had a negligible effect on both the number of simulations showing a negative free asset ratio in the future and the size of the negative free assets. Hence, the insurer could safely distribute all free assets in excess of 50% without adversely affecting the guarantee account. However, if assets are distributed when the ratio exceeds 25% then there is a noticeable increase in both the number and size of future negative free asset ratios. This suggests that policyholders should oppose any distribution of assets to the insurer's owners unless the ratio is above 25%.

- The transition to lower inflation causes investment returns to drop. It also causes the risk-free rate to fall, and hence the cost of options to rise. Both these factors cause UWP payouts to fall. However, falling inflation has very little effect on the guarantee account as long as bonuses are cut to reflect the lower investment returns, and charges rise to reflect the increased cost of options.

10.2 Suggestions for Further Research

The modelling of the insurance business and its investments in this thesis has necessarily required a number of simplifying assumptions. Suggestions for improvements are made for the modelling of the insurance business in Section 10.2.1, and for the modelling of the assets in Section 10.2.2.

10.2.1 More Realistic Modelling of the Insurance Business

The option pricing approach to charging for investment guarantees outlined in this thesis makes several simplifications in the way that the insurer's business is assumed to operate. We have ignored mortality, surrenders, and expenses. We have assumed that the with-profits fund invests only in units of an equity index and put options written on this index. We have also assumed that the insurer pays out the unsmoothed asset share at maturity. We discuss below how we could model the insurance business in a more realistic manner to allow for each of these factors.

An allowance for mortality could be included in a similar way to that described by Brennan and Schwartz (1976) for unit-linked policies as follows. We assume that mortality is independent of the investment risk and that we can diversify away the mortality risk by selling a large number of policies. Further, mortality is not the main risk if we are considering endowment style policies. Hence, it will be reasonable to assume that a deterministic number of lives die each year. Notice that the maturity guarantee G_t at time t for a UWP policy includes the compulsory bonuses rolled up to maturity, but death claims would only receive these bonuses rolled up to the time of death. Therefore, we buy options to match a fixed proportion of policyholders receiving guaranteed death benefits each year and the remainder receiving guaranteed maturity benefits.

In this thesis we have not explicitly allowed for surrenders. If we pay out asset share on surrender, as suggested in Section 3.4.4, then these policies will not be a source of profit or loss for the insurer. (Recall that we define the asset share as the value of the policyholder's assets if they had been invested in equities and put options. Hence the surrender value would be the number of units multiplied by the value of both the shares and current price of the options.) However, further work is required if the insurer offers guaranteed surrender values. Care is required in modelling surrender guarantees as the proportion of policyholders who surrender should be linked to the performance of the assets such that more surrenders occur when the value of the assets is low.

Charges for expenses could be included. Care is needed in the pricing of the options — we would need to adjust the assets in an appropriate way *e.g.* a 1% p.a. fund management charge would mean that we have only 0.99^{T-t} of our current units available to meet the guarantee.

We have assumed that the with-profits fund invests only in units of an equity index. Whereas, in practice, the fund may invest in a variety of assets including property and fixed interest securities as well as equities. We could consider the effects of holding a proportion of the fund in differrent asset categories, with the proportions varied through time according to some dynamic asset allocation rule. Section 3.4.6 discussed some of the issues that need to be considered when the option pricing technique to charging for guarantees is applied to alternative asset allocations.

We have seen that the guarantee account can become very large and positive and have investigated the effects on the guarantee account of removing some of the excess assets. Further, we could investigate the effects on the policyholders' payout if excess assets in the guarantee account were used to enhance UWP bonuses.

Finally, the guarantee account may become negative. Further research is required to investigate the implications for the reserves required to avoid insolvency.

10.2.2 More Realistic Modelling of the Assets

In this thesis we have required two different asset models. The real world model simulates the actual investment returns. The option pricing model represents the way the writers of the options expect the world to behave in order to set option prices. We will consider possible modifications to these models in turn below.

We initially modelled the real world using geometric Brownian motion for share prices with a constant risk-free rate of return.

We then used the more sophisticated Wilkie model which allowed us to stochatically model inflation, consols, and base rates, as well as shares. We used a simple equation to derive a par bond yield curve from the base rates and consols yield. This approach gives us a stochastic risk-free rate. However, further research is required to investigate how a more realistic yield curve which allows for changes in shape through time would effect our results.

It would also be interesting to consider other models of the real world. For example, Hardy (1999), Hardy (2001), and Hardy (2002) consider the regime switching lognormal model.

We have priced the options with the most simple form of the Black-Scholes equation where shares follow geometric Brownian motion with constant volatility and the return on the risk-free asset is constant.

In Chapter 6 we showed how to derive the risk-free rate of return from the real world model. The real world model produces a stochastic risk-free return, but the Black-Scholes formula assumes that it will be constant in the future. It is reasonable to assume that the risk-free rate is constant over a short time horizon. However, given the long term of the options we use we could consider deriving the price of the options using a model which assumes that the risk-free rate is stochastic.

Similarly, shares go through periods of high volatility followed by periods of low volatility. We could consider deriving the price of options using a model where the volatility of shares is expected to vary through time. The estimate of the volatility parameters in the option pricing model could be derived from the past history of the real world model in a similar way to the risk-free asset.

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