An Introduction to Stochastic Pension Plan Modelling [1] [2]

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Abstract

In this paper we consider models for pension plans which contain a stochastic element. The emphasis will be on the use of stochastic interest models, although we will also consider stochastic salary growth and price inflation. The paper will concentrate primarily on defined benefit pension plans. In doing so we will look at how the size of the fund and the contribution rate vary through time and examine how these are influenced by factors which are within the control of a plan's managers and advisers. These factors include the term over which surplus is amortized; the period between valuations; the delay between the valuation date and the implementation of the new contribution rate; and the asset allocation strategy.

The paper will stress the importance of having a well defined objective for a pension plan: optimal decisions and strategies can only be made when a well defined objective is in place.

The paper will also consider, briefly, defined contribution pension plans. The primary decision here relates to the construction of suitable investment strategies for individual members. Again, a well defined objective must be formulated before a sensible strategy can be designed.

Finally, computer simulation methods will be discussed.

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Contents

1 Introduction

In this paper we will consider stochastic pension plans. Pension plans generally fall into one of two categories: defined benefit plans; and defined contribution plans. Both of these are common in countries such as Canada, the USA, the UK and Australia. In all of these countries defined contribution plans are growing significantly in number at the cost of defined benefit plans as employers shift the burden of investment risk over to employees.

In this work we consider how the effects of investment risk can be reduced by making effective use of factors which are within the control of the scheme. These are

- Defined benefit: the method and period of amortization; the intervaluation period; the delay in implementing a recommended contribution rate; the funding method; the valuation basis; the asset allocation strategy.
- Defined contribution: the asset allocation strategy (age dependent); the contribution rate.

In the following sections we will look at each of these factors and consider the effects which each has on levels of uncertainty. In attempting to analyse such problems, a stochastic framework is the only sensible one to use. Within a deterministic framework there is no concept of uncertainty: the very thing we are attempting to quantify and control. For some factors the effect is the intuitive one, while in others the effect may not be known until some sort of exact or numerical analysis can be carried out.

2 Defined Benefit Pension Plans

Defined benefit pension plans provide benefits to members which are defined in terms of a member's final salary (according to some definition), and the length of membership in the plan. For example,

$$\begin{aligned} \text{Annual pension} &= \frac{N}{60} \times FPS \\ \text{where } N &= \text{number of years of plan membership} \\ FPS &= \text{final pensionable salary} \end{aligned}$$

In defined benefit pension plans pension and other benefits do not depend on past investment performance. Instead the risk associated with future returns on the funds assets is borne by the employer. This manifests itself through the contribution rate which must vary through time as the level of the fund fluctuates above and below its target level. If these fluctuations are not dealt with (that is, if the contribution rate remains fixed) then the fund will ultimately either run out of assets from which to pay the benefits or grow exponentially out of control.

2.1 A simple model

A number of the factors which we will look at can be first investigated by looking at a very simple stochastic model. By doing so we are able to focus quite quickly on the problem and to give ourselves a good feel for what might happen when we look at more realistic and complex models. This approach follows that of Dufresne (1988, 1989 a,b, 1990), Haberman (1992, 1993 a,b, 1994), Zimbidis and Haberman (1993), Cairns (1995) and Cairns and Parker (1995).

Suppose, then, that we have a fund which has a stable membership and a stable level of benefit outgo. Assuming that all benefits and contributions are paid at the start of each year we have the following relationship:

$$AL(t+1) = (1+i'_v)\,(AL(t)+NC(t)-B(t))$$

where

$$\begin{aligned} AL(t) &= \text{actuarial liability at time } t \\ B(t) &= \text{benefit outgo at time } t \\ NC(t) &= \text{normal contribution rate at time } t \\ \text{and } \; i'_v &= \text{valuation rate of interest} \end{aligned}$$

Suppose that salary inflation is at the rate s per annum and that benefit outgo increases in line with salaries each year. Then

$$\begin{aligned} B(t) &= B.(1+s)^t \\ AL(t) &= AL.(1+s)^t \\ NC(t) &= NC.(1+s)^t \end{aligned}$$

giving

$$\begin{aligned} AL(1+s) &= (1+i'_v)(AL+NC-B) \\ \text{or } \; AL &= (1+i_v)(AL+NC-B) \end{aligned}$$

where $i_v = (1+i'_v)/(1+s)-1 = (i'_v - s)/(1+s)$ is the real valuation rate of interest. Hence

$$NC = B-(1-v_v)L$$

where $v_v = 1/(1+i_v)$.

For convenience we will work in real terms relative to salary growth. In effect this means that we may assume that $s=0$, without losing any level of generality.

Now let $F(t)$ be the actual size of the fund at time t. Then

$$F(t+1) = (1+i(t+1))\,(F(t)+C(t)-B)$$

where $i(t+1)$ is the effective rate of interest earned on the fund during the period t up to $t+1$, and $C(t)$ is the contribution rate at time t.

$C(t)$ can be split into two parts: the normal contribution rate, NC; and an adjustment $ADJ(t)$ to allow for surplus or deficit in the fund relative to the actuarial liability. Thus

$$C(t) = NC + ADJ(t)$$

We will deal with the calculation of this adjustment in the next two sections.

The deficit or unfunded liability at time t is defined as the excess of the actuarial libility over the fund size at time t. Hence we define

$$\begin{aligned} UL(t) &= \text{unfunded liability at time } t \\ &= AL - F(t) \end{aligned}$$

In North America it is common also to look at the loss which arises over each individual year. This is defined as the difference between the expected fund size (based on the valuation assumptions) and the actual fund size at the end of the year given the history of the fund up to the start of the year. This gives us

$$\begin{aligned} L(t) &= \text{loss in year } t \\ &= E[F(t)] - F(t) \;\text{ given the fund history up to time } t-1 \\ &= UL(t) - E[UL(t)] \;\text{ given the fund history up to time } t-1 \end{aligned}$$

We will make use of $UL(t)$ and $L(t)$ in the next section.

No mention has been made so far of the interest rate process $i(t)$. Initially we will assume that $i(1), i(2), \ldots$ form an independent and identically distributed sequence of random variables with

$$\begin{aligned} i(t) &> -1 \;\text{ with probability } 1 \\ E[i(t)] &= i \\ Var[i(t)] &= Var[1+i(t)] = \sigma^2 \\ \Rightarrow E[(1+i(t))^2] &= (1+i)^2 + \sigma^2 \end{aligned}$$

For notational convenience we will define

$$\begin{aligned} v_1 &= \frac{1}{E[1+i(t)]} = \frac{1}{1+i} \\ v_2 &= \frac{1}{E[(1+i(t))^2]} = \frac{1}{(1+i)^2+\sigma^2} \end{aligned}$$

These will be made use of in later sections.

2.2 Two methods of amortization

The Spread Method: This is in common use in the UK. The adjustment to the contribution rate is just a fixed proportion of the unfunded liability: that is,

$$\begin{aligned} ADJ(t) &= k.UL(t) \\ \text{where } k &= \frac{1}{\ddot{a}_{\overline{m|}}} \text{ at rate } i_v \\ \text{and } m &= \text{the period of amortization.} \end{aligned}$$

The period of amortization is chosen by the actuary, and commonly ranges from 5 years to over 20 years. For accounting purposes in the UK m must be set equal to the average future working lifetime of the membership.

The Amortization of Losses Method: This is in common use in the USA and Canada. The adjustment is calculated as the sum of the losses in the last m years divided by the present value of an annuity due with a term of m years calculated at the valuation rate of interest: that is,

$$ADJ(t) = \frac{1}{\ddot{a}_{\overline{m|}}} \sum_{j=0}^{m-1} L(t-j)$$

The interpretation of this is that the loss made in year s is recovered by paying m equal instalments of $L(s)/\ddot{a}_{\overline{m|}}$ over the next m years. These m instalments have the same present value as the loss made in year s.

Dufresne (1989b) showed that the unfunded liabilities and the losses are linked in the following way:

$$\begin{aligned} UL(t) &= \sum_{j=0}^{m-1} \lambda_j L(t-j) \\ \text{where } \lambda_j &= \frac{\ddot{a}_{\overline{m-j|}}}{\ddot{a}_{\overline{m|}}} \end{aligned}$$

Intuitively this makes sense, since $\lambda_j L(t-j)$ is just the present value of the future amortization instalments in respect of the loss made at time $t-j$. Hence $UL(t)$ is equal to the present value of the outstanding instalments in respect of all losses made up until time t.

The Spread Method can also be defined in terms of the loss function. Whereas the Amortization of Losses Method recovers the loss at time t by taking in m *equal* instalments of $L/\ddot{a}_{\overline{m|}}$, the Spread Method recovers this by making a geometrically decreasing, infinite sequence of instalments which starts at the same level.

We are now in a position to calculate the long term mean and variance of the fund size and of the contribution rate. Details of these are provided in Dufresne (1989) (in the case when the valuation and the true mean rate of interest are equal) and Cairns (1995) (covering the case when $i \neq i_v$). For the Spread method we find that

$$\begin{aligned}
E[F(t)] &= \frac{(1-k-v_v)AL}{(1-k-v_1)} \\
E[C(t)] &= B - \frac{(1-k-v_v)(1-v_1)AL}{(1-k-v_1)} \\
Var[F(t)] &= \frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2 \\
Var[C(t)] &= k^2\frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2
\end{aligned}$$

When $i = i_v$ these simplify to

$$\begin{aligned}
E[F(t)] &= AL \\
E[C(t)] &= B-(1-v_1)AL \\
Var[F(t)] &= \frac{(v_1^2-v_2)}{(v_2-(1-k)^2)}AL^2 \\
Var[C(t)] &= k^2\frac{(v_1^2-v_2)}{(v_2-(1-k)^2)}AL^2
\end{aligned}$$

Now $v_1 > v_2$ and we must have $Var[F(t)]$ and $Var[C(t)]$ greater than 0. Hence we must have $(1-k)^2 < v_2 \;\Rightarrow\; k > 1-\sqrt{v_2}$. This then automatically implies that $k > 1-v_1$ and if this is combined with $k > 1-v_v$ it ensures that the mean fund size is also positive.

Looking at the Amortization of Losses Method we have, when $i = i_v$,

$$\begin{aligned}
Var[L(t)] &= \frac{\sigma^2(1+i)^{-2}AL^2}{1-\sigma^2(1+i)^{-2}\sum_{j=1}^{m-1}\lambda_j^2} = V_\infty \text{ say} \\
Var[F(t)] &= V_\infty\sum_{j=0}^{m-1}\lambda_j^2 \\
Var[C(t)] &= \frac{m.V_\infty}{(\ddot{a}_{\overline{m|}})^2}
\end{aligned}$$

2.3 The period of amortization

We now consider the first factor which we have within our control: the period of amortization, m.

For the time being, assume that $i = i_v$: we will look at the more general case in a later section. The following results can be shown to hold for the Spread Method (for example, see Dufresne, 1989b)

- $Var[F(t)]$ increases as m increases.

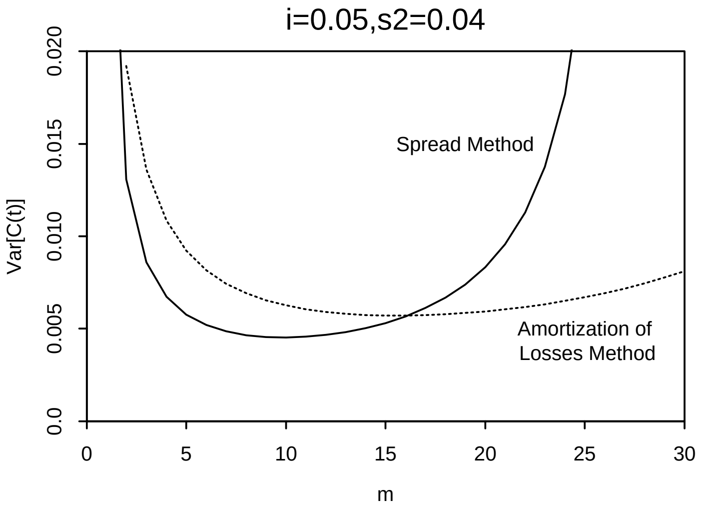

Figure 1: The effect of the period of amortization on the variance of the contribution rate with $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$.

- $Var[C(t)]$ decreases initially as m increases from 1 up to some value m^* and then increases as m increases beyond m^*. The optimal value, m^*, is such that
$k^* = 1/\ddot{a}_{\overline{m^*|}} = 1 - v_2$.

Looking at the Amortization of Losses Method no such analytical results have been proved but numerical examples show that the same qualitive behaviour holds, as illustrated in the following example.

Suppose $E[i(t)] = i = 0.05$ and $Var[i(t)] = \sigma^2 = 0.2^2$. Figure 1 illustrates how the variance of the contribution rate (with $AL = 1$) depends on m. The Spread Method has its minimum at about 10 while the Amortization of Losses Method has its minimum at about 16, and this minimum is higher.

In Figure 2 we compare the variance of the fund size against the variance of the contribution rate. We do this because we may be interested in controlling the variance of both the contribution rate *and* the fund size. As m increases each curve moves to the right, first decreasing and then increasing as m passes through m^*. Above m^* both the variance of the fund and the variance of the contribution rate are increasing. It is clear then that no value of m above m^* can be 'optimal' because the use of some lower value of m (say, m^*) can lower the variance of both the fund size and the contribution rate. The range $1 \leq m \leq m^*$ is the so-called *efficient* region: that is, given a value of m in this range there is no other value of m which can lower the variance of both the fund size and the contribution rate. There is therefore a trade-off between variability in the fund size and the contribution rate and settling on what we regard as an optimal spread period can only be done with reference to a more specific objective than 'minimize variance'.

It is significant that the Amortization of Losses Method curve always lies above the Spread Method curve. This means that the Spread Method is certainly more efficient than the Amortization of Losses Method: that is, for any value of m in combination with the Amortization of Losses Method there is a (different) value m' for which the variance of both the fund size and the contribution rate can be reduced by switching to the Spread Method.

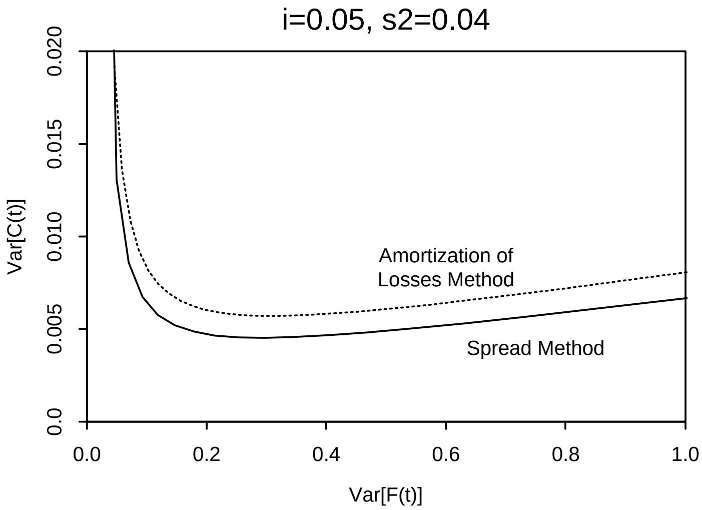

Figure 2: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. Comparison of $Var[F(t)]$ with $Var[C(t)]$. Notes: $Var[F(t)]$ increases as m increases; the efficient frontier for the Spread Method is always more efficient than that for the Amortization of Losses Method.

2.4 The intervaluation period

The time between valuations is nominally a factor which is within the control of the scheme. We have so far considered the case where valuations are carried out on an annual basis. Such an approach is common amongst larger funds but this is often felt to be uneconomic for smaller funds to carry out such frequent valuations. Instead smaller funds often opt for a three year period between valuations ($3\frac{1}{2}$ years being the statutory maximum in the UK).

The effects of changing from annual to triennial valuations have been considered by Haberman (1993b). He finds that under the Spread Method of amortization

- the optimal spread period for $Var[C(t)]$, m^*, increases by about 1 year;
- the variances of both $F(t)$ and $C(t)$ are increased for most values of m below about m^*.

Continuing the example of the previous section we looked at 1 and 3 year intervaluation periods. Figure 3 plots $Var[C(t)]$ against m. For low values of m lengthening the intervaluation period has the effecct of increasing the variance of $C(t)$: the intuitive effect. For higher values of m, however, the reverse is true. This perhaps reflects the fact that over each three year period $C(t)$ is being held fixed thereby reducing the overall variance.

Comparing the variances of $F(t)$ and $C(t)$ (Figure 4) we see that, in this example at least, the efficient range for annual valuations lies below that for triennial valuations. We conclude that annual valuations are preferrable, although for values of m close to m^* there is little difference in the variances, so the benefit of annual valuations is marginal.

2.5 The delay period

The original analysis asumes that the new contribution rate can be implemented at the valuation date. In reality the results of a valuation are often not known until 6 or even 12 months after the

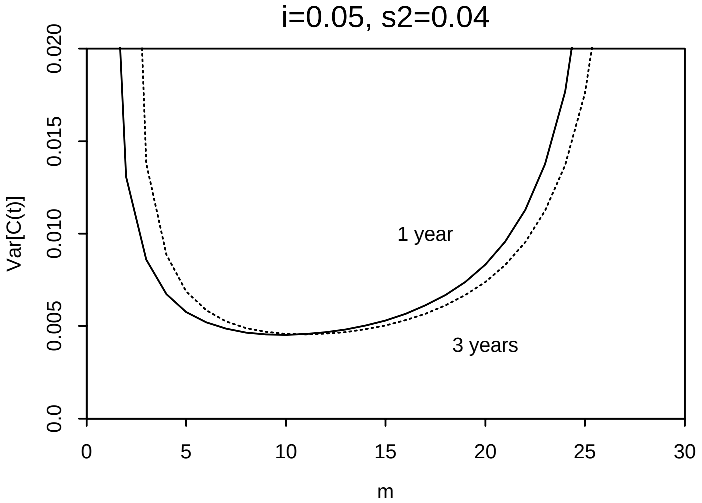

Figure 3: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]$ plotted against m for annual and triennial valuations.

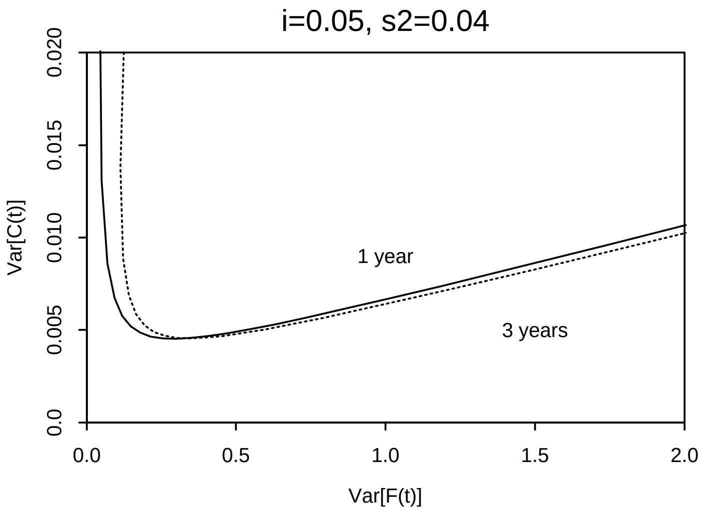

Figure 4: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. Comparison of $Var[F(t)]$ with $Var[C(t)]$. Note: the efficient frontier for the annual valuation case is, for most values of m less than m^*, below that for the triennial valuation case.

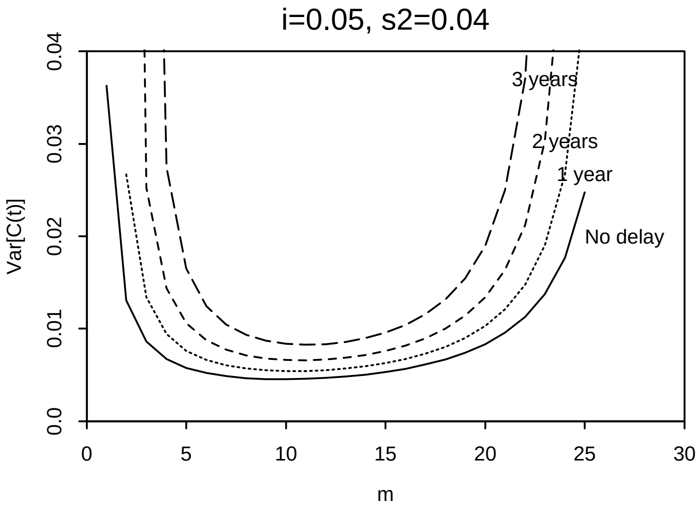

Figure 5: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]$ plotted against m for delay periods of 0, 1, 2 and 3 years.

valuation date. The new recommended contribution rate is therefore typically not implemented until one year later. There is a delay period of 1 year.

This problem has been investigated by Zimbidis and Haberman (1993). In the example under consideration each extra year's delay increases the variance of $F(t)$ and $C(t)$ by at least 20% and by much more substantial amounts for small values of m. Figures 5 and 6 illustrate the results for this example. One point to note is that where there is a delay period then $Var[F(t)]$ initially decreases with m before increasing as in the no-delay case. This has the effect of reducing the efficient range for m. For example, with a delay of 3 years the efficient range is $5 \leq m \leq 11$ as compared with $1 \leq m \leq 10$ when there is no delay.

In view of the substantial increases in variance caused by a delay it is felt that the delay should be kept as short as possible and perhaps that allowance should be made in the current rate even if the final results of a valuation are not known.

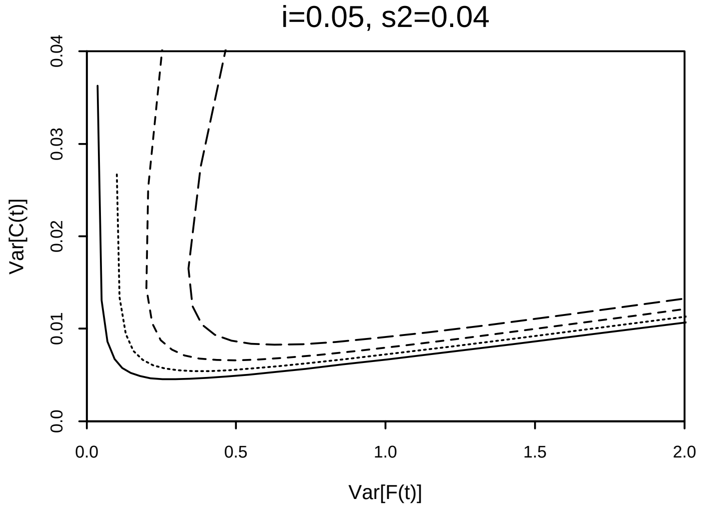

Figure 6: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. Comparison of $Var[F(t)]$ with $Var[C(t)]$. Increasing the delay period increases the variance of both $F(t)$ and $C(t)$.

2.6 The funding method

Recall the equilibrium equation relating AL to NC

$$AL = (1+i_v)(AL+NC-B)$$

If we increase AL then NC balances this by falling (this is because benefits are paid from contributions plus surplus interest on the fund, which has increased). Furthermore, AL is determined by the funding method. The normal ordering which we find is

$$AL_{CUC} < AL_{PUC} < AL_{EAN}$$

where the subscripts represent the Current Unit Credit (CUC), Projected Unit Credit (PUC) and Entry Age Normal (EAN) methods, these being the three main funding methods appropriate for a stable membership.

The Attained Age Method has the same actuarial liability as the Projected Unit Credit Method but normally has a higher normal contribution rate which is appropriate for a closed fund, but which will give systematic rise to surplus when the fund has a stable membership. In such a case the equilibrium equation is, therefore, not satisfied. Instead the system has a higher equilibrium fund size which depends on the method and period of amortization.

The variances of $F(t)$ and $C(t)$ are both proportional to AL^2. This means that a more secure funding method (higher AL) gives rise to greater variability, suggesting that a method with a low actuarial liability is to be preferred. Clearly this is not a prudent strategy. It jeopardizes member's security and is more likely to violate statutory solvency requirements.

This problem can be overcome by a number of methods, including:

- the use of the normalized variances $Var[F(t)]/E[F(t)]^2$ and $Var[C(t)]/E[F(t)]^2$;
- the use of further fund objectives (for example, by conditioning on the mean fund size being at a specified level).

2.7 The strength of the valuation basis

So far we have concentrated on the case where the valuation rate of interest, i_v, is equal to the mean long term rate of interest, i. It is common, however, for valuations to be carried out on a strong (occasionally weak) basis: that is, to set $i_v < i$ (or $i_v > i$). This gives rise to a wider variety of results.

Recall that

$$\begin{aligned}
E[F(t)] &= \frac{(1-k-v_v)AL}{(1-k-v_1)} \\
E[C(t)] &= B - \frac{(1-k-v_v)(1-v_1)AL}{(1-k-v_1)} \\
Var[F(t)] &= \frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2 \\
Var[C(t)] &= k^2\frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2
\end{aligned}$$

We concentrate on the variance of the contribution rate and look for the existence of a minimum with respect to the period of amortization, m. There are a number of cases to consider.

1. **Strong basis:** $i_v < i$ $(v_v > v_1)$

   (these are currently observations, and not proved)

   (a) $E(C_t)$ is an increasing function of k for $k > 1-\sqrt{v_2}$.

   (b) $Var(C_t)$ has a minimum for some $1-\sqrt{v_2} < k^* < 1$.

   (c) $Var(F_t)$ is a decreasing function of k.

   From this we can see that for $k > k^*$ both the expected value and the variance of the contribution rate are increasing so that increasing k above k^* is not worthwhile. If k is decreased then we trade off a lower contribution rate for a higher variance. The optimal value therefore depends on the pension fund's utility function or objectives. This goes slightly against the conclusions of Dufresne who indicates that k^* would be the *minimum* acceptable value of k.

   For some values of k the mean contribution rate will be negative, indicating that the fund is large enough to pay for itself and at times requiring refunds to the employer. Although this seems an ideal situation, the reality is that the company must first have built up the fund to this level. It would also be likely to violate statutory surplus regulations.

   It is possible to have smaller expected fund levels and higher contribution rates, but these do not arise if the projected unit method is used in the calculation of the funding rate and using a conservative valuation rate of interest.

2. **Best estimate:** $i_v = i$ $(v_v = v_1)$

   The results of Dufresne (1989) hold.

   (a) $E(C_t)$ is a constant function of k for $k > 1-\sqrt{v_2}$.

   (b) $Var(C_t)$ has a minimum for some $1-\sqrt{v_2} < k^* < 1$.

   (c) $Var(F_t)$ is a decreasing function of k.

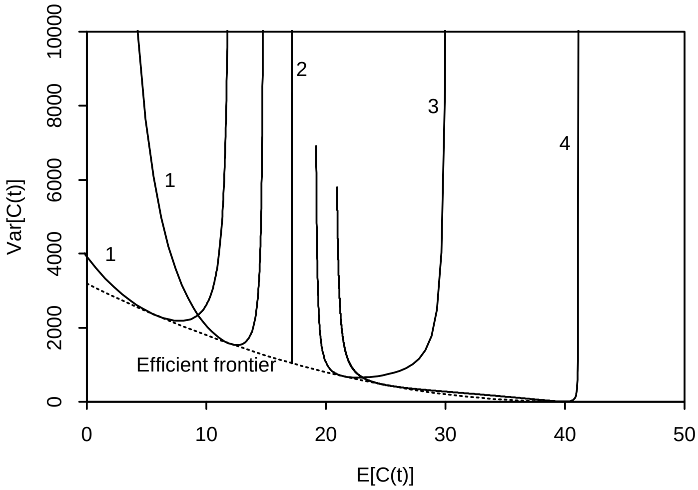

Figure 7: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]$ plotted against $E[C(t)]$ for different valuation rates of interest. Moving from left to right the curves represent: $i_v = 0.03, 0.04$ (type 1, strong basis); $i_v = 0.05$ (type 2, best estimate basis); $i_v = 0.06$ (type 3, weak basis); $i_v = 0.07$ (type 4, very weak basis). The dotted line is the efficient frontier.

3. **Weak basis:** $i < i_v < \sqrt{(1+i)^2+\sigma^2} - 1 \;\; (v_1 > v_v > \sqrt{v_2})$

   (a) $E(C_t)$ is a decreasing function of k for $k > 1 - \sqrt{v_2}$.

   (b) $Var(C_t)$ has a minimum for some $1 - \sqrt{v_2} < k^* < 1$.

   (c) $Var(F_t)$ is a decreasing function of k.

   This time we find that it may be acceptable to increase k above k^*, trading off lower contributions for higher variability.

4. **Very weak basis:** $\sqrt{(1+i)^2+\sigma^2} - 1 < i_v \;\; (\sqrt{v_2} > v_v)$

   (a) $E(C_t)$ is a decreasing function of k for $k > 1 - v_v$ at which point it equals B and the scheme is funded on a pay as you go basis. For $1 - v_v > k > 1 - \sqrt{v_2}$ $E(C_t)$ is still a decreasing function.

   (b) $Var(C_t)$ has a minimum equal to zero at $k = 1 - v_v$. This is because the scheme is now funded on a pay as you go basis and contributions equal the constant B.

   (c) $Var(F_t)$ has a local minimum at $k = 1$, a maximum at some $1 - v_v < k^* < 1$ and a global minimum equal to zero at $k = 1 - v_v$ when the fund stays constant at zero.

The efficient frontier

Pooling these results together we can determine a curve $m(\mu_c)$ where

$$m(\mu_C) = \min\{Var(C_t) : E(C_t) = \mu_C, 1 > k > \max(1 - v_v, 1 - \sqrt{v_2}), v_v < 1\}$$

That is, $m(\mu_C)$ gives us the minimum variance attainable for a given mean contribution rate. In fact, it can be shown that $m(\mu_C)$ is convex (quadratic).

These different types of outcome are illustrated in Figure 7, with $i = 0.05$ and $\sigma^2 = 0.2^2$.

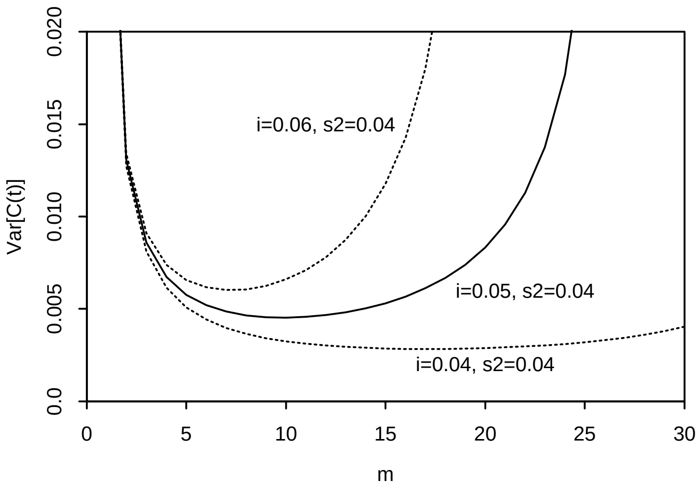

Figure 8: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]$ plotted against m for different long term rates of return. The valuation rate of interest is fixed.

2.8 Sensitivity testing

In carrying out such analyses it is important to realize that the model for the rate of return including its parameter values are uncertain. First, the model we use here is only one of a range of possible models of varying complexity which all fit past data reasonably well. All of these models are, however, only an approximation to a much more complex reality. Second, the parameter values which we have used (here $i = 0.05$ and $\sigma^2 = 0.04$) are not known with certainty: for example i could equally well be 0.04 or 0.06.

In fact this can have a very significant effect on level the variability. Figures 8 and 9 illustrate this point. i is allowed to take the values 0.04, 0.05 and 0.06. In Figure 8 the effect on $Var[C(t)]$ is very significant, particularly for larger values of m. However, these results are distorted by the fact that when $i \neq i_v$ the mean fund size ($E[F(t)]$) depends on m. The normalized variance of $C(t)$ is plotted in Figure 9 and the effect can be seen to be reduced but still significant.

A change in the value of i of 1% makes a difference in m^* of about 2 years (for example, moving from $i = 0.05$ to $i = 0.06$ changes m^* from 10 to 8).

The result of these changes is not as significant as might first appear. For example, suppose we settled upon $m^* = 10$ on the basis that $i = 0.05$. If in fact the long term mean turned out to be $i = 0.06$ then amortizing over 10 years would only turn out to have been only marginally worse than if the true optimum $m^* = 8$ had been used. The fact that the actual variance of the contribution rate was perhaps 20% higher than that expected is irrelevant since the lower value would never, in fact, have been attainable.

Figure 10 shows the effects of uncertainty in σ^2 (with σ^2 taking the values 0.03, 0.04 and 0.05). The effect is again substantial, but much more uniform over the whole range of values for m. This is because σ^2 has a much more direct effect on the variance of the fund size and the contribution rate. However, as with uncertainty in i, the normalized variance is relatively stable over a range of values about the minimum, so choosing the wrong value of m will only marginally increase the long term variance.

The point to take in from this section is that we need to take care in ensuring that we look at the right quantities. We therefore need to compare the *actual* outcome based on the decision which

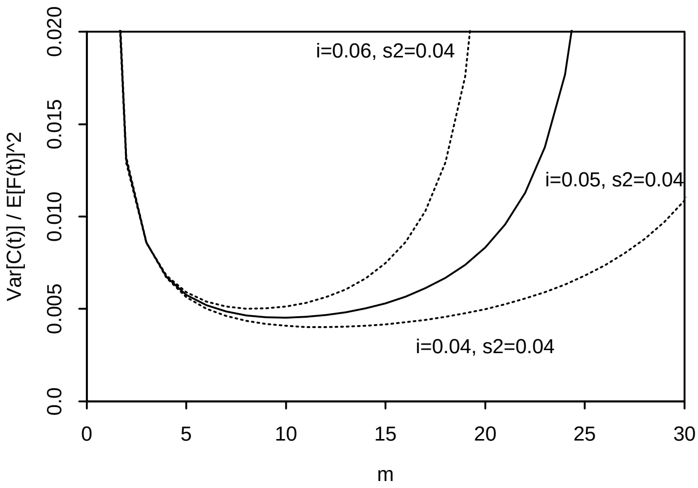

Figure 9: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]/E[F(t)]^2$ plotted against m for different long term rates of return. The valuation rate of interest is fixed.

was based on incorrect assumptions with the outcome which would have *actually* happened had the decision been based on the correct assumptions. Here the differences have been shown to be minimal but if we were to find that they were significant then we may need to look carefully at our estimates to see if they can be refined and improved upon.

2.9 Objectives

We have already discussed that within the efficient region for m ($1 \leq m \leq m^*$) there is a trade off between higher variance of $F(t)$ and higher variance of $C(t)$. To settle on an optimal spread period therefore requires a specific objective or utility function. For example, we may be concerned about containing the fund size within a specified band (bounded below, say, by the minimum solvency level and above by a statutory surplus limit). We could accomodate this by specifying that $E[F(t)]$ lie in the middle of this band and that the standard deviation of $F(t)$ be no more than 10% of this mean fund size. In this case the optimum would be m^{**} which pushes the variance of $F(t)$ up to the maximum level allowable or m^* if this is lower.

If a proper optimum is to be found then the fund must have a well defined objective which will allow optimization to take place. Examples of some objectives are:

- Minimize $Var[C(t)]$ subject to $Var[F(t)] \leq V_{max}$;
- Minimize $Var[C(t)]$ subject to $E[F(t)] = \mu_F$;
- Minimize the variance of the present value of all future contributions (that is, $\sum_{t=0}^{\infty} v^t C(t)$) subject to ......;
- Maximize $E[u(F(t))]$ where $u(f)$ is utility function which depends on the fund size. For example, if $u(f) = -(f - f_0)^2$ then $E[u(F(t))] = -Var[F(t)] - (E[F(t)] - f_0)^2$, the second term being a penalty for deviation of the mean from the target of f_0.

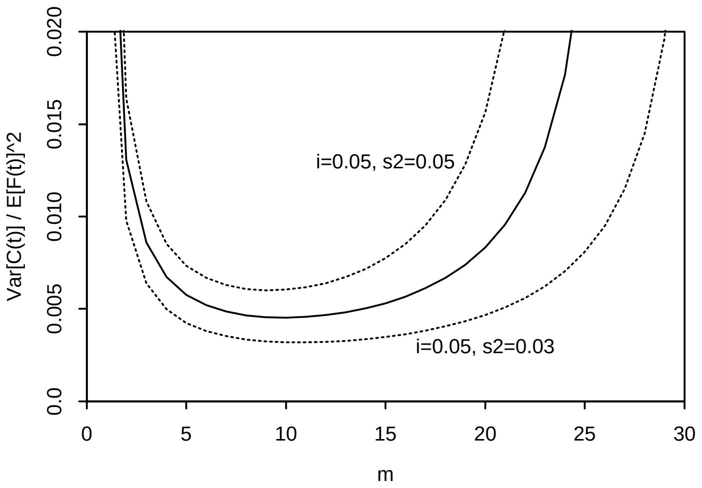

Figure 10: $E[i(t)] = 0.05$ and $Var[i(t)] = 0.04$. $Var[C(t)]/E[F(t)]^2$ plotted against m for varying levels of volatility in the rate of return. The valuation rate of interest is fixed.

Care should be taken when formulating an objective. For example, the last of these makes less sense if $E[F(t)]$ is constant for all values of m (that is if $i_v = i$); and constraints should have reasonable rather than extreme values.

2.10 Other stochastic investment models

We have used the simplest stochastic interest model here (independent and identically distributed returns) which allows us to obtain some intuitively appealing analytical results. A wide variety of more complex models are used in practice for which analytical results are not possible. However, it is expected that similar qualitative results should be available.

Autoregressive time series models: Haberman (1993a) has investigated the use of the AR(1) time series model:

$$\begin{aligned}
\delta(t) &= \delta + \alpha(\delta(t-1) - \delta) + \nu Z(t) \\
\text{where } \delta(t) &= \log(1 + i(t)) \\
Z(t) &\sim N(0,1) \\
|\alpha| &< 1 \text{ is the autoregressive parameter} \\
\delta &= \text{long term mean rate of return} \\
\nu^2 &= \text{variance parameter} \\
\text{Hence } E[\delta(t)] &= \delta \\
Var[\delta(t)] &= \sigma^2 = \frac{\nu^2}{1-\alpha^2} \\
E[1+i(t)] &= e^{\delta + \frac{1}{2}\sigma^2} \\
Var[1+i(t)] &= e^{2\delta+\sigma^2}\left(e^{\sigma^2} - 1\right)
\end{aligned}$$

It has been found that $\alpha > 0$ (positively correlated returns) decreases the value of m^* (for example, with $E[i(t)] = 0.05$ and $Var[i(t)] = 0.2^2$ m^* falls from 10 to 5 when α is changed from 0 (independent and identically distributed returns) to only 0.1). More likely is the case $\alpha < 0$ (a high return one year is followed by a low return the next year) which increases the value of m^*.

Note that such models seem more appropriate to fixed interest investments: past equity data do not show any significant signs of autocorrelation from one year to the next.

In summary the most widely used stochastic interest models are

- Independent and identically distributed returns: for example, Waters (1978), Dufresne (1990), Papachristou and Waters (1991), Parker (1993 a,b, 1994 a,b) and Aebi *et al.* (1994) give but a few examples.
- Simple autoregressive models, such as the $AR(1)$ time series model, and the Ornstein-Uhlenbeck process: for example, Dhaene (1989), Parker (1993 a,b, 1994 a,b) and Norberg and Møller (1994).
- Models for the term structure of interest rates: for example, Boyle (1978, 1980), Brennan and Schwarz (1979), Albrecht (1985), Cox, Ingersoll and Ross (1985), Beekman and Shiu (1988), Heath, Jarrow and Morton (1990, 1992), Reitano (1991), Sercu (1991) and Longstaff and Schwarz (1992).
- Models with several asset classes: for example, Wilkie (1987, 1992, 1994), and Chan (1994).

The last two of these classes are the most appropriate for the purposes of making an asset allocation decision. In an objective based setting, however, the asset allocation strategy must be considered simulataneously with other factors which are within our control (see the example in the next section).

Increasing complexity means that we need to resort to stochastic simulation in most of these cases.

2.11 Example: A two asset model

Suppose that the fund has two assets in which it can invest. The return in year t on asset j ($j = 1,2$) is $i_j(t)$ with

$$\begin{aligned} E[i_j(t)] &= i_j \text{ for } j = 1,2 \\ Cov[i_j(t), i_k(t)] &= c_{jk} = c_{kj} \quad j,k = 1,2 \end{aligned}$$

Suppose asset 1 carries a lower risk and a lower return: that is, $i_1 < i_2$ and $c_{11} < c_{22}$.

Let $i(t)$ be the overall return during year t, and suppose that a proportion p of the fund is invested in asset 1. Then

$$
\begin{aligned}
E[i(t)] &= pi_1 + (1-p)i_2 = \mu(p) \text{ say} \\
Var[i(t)] &= Var[pi_1(t) + (1-p)i_2(t)] \\
&= Var[pi_1(t)] + Var[(1-p)i_2(t)] + 2Cov[pi_1(t), (1-p)i_2(t)] \\
&= p^2 c_{11} + (1-p)^2 c_{22} + 2p(1-p)c_{12} \\
&= \sigma^2(p) \text{ say}
\end{aligned}
$$

(This is following the approach of Modern Portfolio Theory.)

We now put this new mean and variance into the original equations:

$$
\begin{aligned}
E[F(t)] &= \frac{(1-k-v_v)AL}{(1-k-v_1)} \\
E[C(t)] &= B - \frac{(1-k-v_v)(1-v_1)AL}{(1-k-v_1)} \\
Var[F(t)] &= \frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2 \\
Var[C(t)] &= k^2\frac{(1-k-v_v)^2(v_1^2-v_2)}{(1-k-v_1)^2(v_2-(1-k)^2)}AL^2 \\
\text{where } v_1 &= \frac{1}{E[1+i(t)]} = \frac{1}{1+\mu(p)} \\
v_2 &= \frac{1}{E[(1+i(t))^2]} = \frac{1}{(1+\mu(p))^2+\sigma^2(p)}
\end{aligned}
$$

We now have at our disposal:

- the period of amortization;
- valuation basis;
- asset mix.

We have seen from looking at the strength of the valuation basis that a wide range of fund sizes can be attained. Optimal choices must therefore be made with reference to some specific objectives. For example,

$$
\begin{aligned}
\text{minimize} \quad & Var[C(t)] \\
\text{subject to} \quad & E[F(t)] = AL' \\
& Var[F(t)] \leq (0.1AL')^2
\end{aligned}
$$

where AL' is, for example, a statutory minimum plus 20%.

To find an appropriate solution one must now use numerical methods to optimize over the factors within our control. The process of optimization may proceed as follows:

1. Fix the asset proportion and the valuation rate of interest (p and i_v). Then k (therefore m) is determined by the constraint on $E[F(t)]$:

$$E[F(t)] = \frac{(1-k-v_v)}{(1-k-v_1)} AL(i_v) = AL'$$

2. Find the range of values of i_v for which $Var[F(t)] \leq (0.1AL')^2$, and within that range which i_v minimizes $Var[C(t)]$. Let this minimum be $M(p)$.

3. Minimize $M(p)$ over $0 \leq p \leq 1$.

4. Check that the optimal values are reasonable: for exampe, is i_v reasonable when compared with $E[i(t)] = \mu(p^*)$; is m^* reasonable; is p^* acceptable? If the answer to any of these questions is no then we should ask ourselves why and reformulate the objectives accordingly.

2.12 Constraints on strategies

We have already mentioned in Sections 2.6 and 2.9 that our optimal strategy may be influenced by statutory funding levels. These may be

- a minimum solvency requirement;
- a maximum surplus regulation.

Different countries have different regulations for what happens when one of these limits is breached. Typically, however, there may be a requirement to amortize the difference between the limit and the current fund size over a shorter period than normal (in the UK and Canada this is 5 years).

Another constraint may be a limit on the ability of the employer to take a refund from the fund. If no refund at all is possible then ultimately the fund will reach a stage where the fund becomes large enough to be self funding (that is, interest exceeds benefit outgo) beyond which point the fund will grow exponentially out of control. This is a certain event in a stochastic environment. More common is a (statutory) constraint that contribution refunds may only be made while the asset/liability ration remains above a specified level.

When such constraints are in place exact analyses are no longer possible. Instead numerical investigations are necessary.

2.13 Salary growth and price inflation

We have already illustrated how salary growth can be incorporated into these models. This is done by indexing the actuarial libility, the normal contribution rate and the benefit outgo in line with the total salary roll $S(t)$, and treating $i(t)$ as a real rate of return.

Salary inflation can be adequately modelled by an autoregressive process of order 1 or alternatively it can be linked to price inflation (for example, see Section 3 and Wilkie, 1994).

Problems arise when benefit outgo is not proportional to the total salary roll. For example, if pensions are paid from the fund but linked to a prices index then benefit outgo is equal to a mixture of past salary rolls increased in line with the appropriate price index.

This can be approached in two ways: by carrying out a simulation study (described in the next section); or by assuming that pensions are matched at the date of retirement by index-linked securities. In the latter case

$$\begin{aligned} B(t) &= B \times S(t) \times A(t) \\ \text{where } B &= \text{base pension benefit} \\ S(t) &= \text{salary index} \\ A(t) &= \text{real annuity rate at time } t \end{aligned}$$

The annuity rate $A(t)$ is itself governed by a random process: for example, $A(1), A(2), \ldots$ may be independent and identically distributed positive random variables.

2.14 Simulation methods

Two simulation methods are available.

Method 1: (Ergodic method)

All of the interest rate processes described are examples of *ergodic* processes (for example, see Karlin and Taylor, 1975). A consequence of this (amongst other properties) is that the fund process will satisfy

$$\begin{aligned} \bar{f}_n &= \frac{1}{n}\sum_{t=1}^{n} F(t) \rightarrow E[F(t)] \text{ almost surely as } n \rightarrow \infty \\ s_n^2 &= \frac{1}{n}\sum_{t=1}^{n} \left(F(t) - \bar{f}_n\right)^2 \rightarrow Var[F(t)] \text{ almost surely as } n \rightarrow \infty \end{aligned}$$

(If salary growth is allowed for, then $F(t)$ above should be replaced by the asset/liability ratio $F(t)/AL(t)$.)

This means that a single, long simulation run of the pension plan will give us good estimates of the means and variances of the quantities of interest. Rough calculations suggest that this simulation should be of at least 2000 years.

The simulation should be repeated for each combination of decisions being examined. For consistency and efficiency the same realization of the interest rate process should be used for each combination of decisions.

Method 2: Repeated simulation

The objective of the fund may, amongst other things, aim to minimize variance over a short period, say 10 years, rather than over the longer term. Repeated simulation is more appropriate here: that is, simulate the fund for 10 years, given appropriate initial conditions; and then repeat this, say, 200 or more times. For consistency and efficiency the same 200 scenarios of the interest rate process should be used for each combination of decisions.

3 Defined Contribution Pension Plans

Defined contribution pension plans are becoming of ever increasing importance and as such they require some long overdue investigation in order that their reliability as a pensions vehicle can be improved upon. The principal distinctions with defined benefit pension plans are that benefits are no longer based upon final salary but depend on past contribution levels and past investment returns thereby passing investment risk from the employer to the individual members.

Whereas an employer as sponsor of a defined benefit plan is able to smooth out good and bad years' investment returns, defined contribution pension plan members are rather more at the mercy of variations in returns from one year to the next. For example, Knox (1993) carried out a simulation study using a simple model which illustrated the high degree of uncertainty in the final amount of a defined contribution pension relative to final salary. This risk is well known and is a major criticism of the defined contribution set-up. Further work is therefore required to see if this risk can be reduced.

Defined contribution pension plans can be divided into two categories:

- those sponsored by an employer;
- those taken out by individuals with an insurer and with no (or only indirect) involvement on the part of an employer (Retirement Savings Plan).

From a statistical standpoint, this is an artificial distinction. Any decision which can be applied to one type should be applicable to the other: for example, the use of investment strategies which depend on the age of the individual.

3.1 Objectives

Clearly defined objectives are perhaps even more important in the decision making process associated with a defined contribution pension plan than a defined benefit pension plan. Different, member oriented objectives are required and the situation may be complicated further by the possibility that different members may have different objectives and utility functions.

An objective is most likely to be defined in terms of the the amount of pension at retirement *as a proportion of final salary* rather than as an absolute amount. Thus we define

$$\begin{aligned}
P(t) &= \text{pension on retirement at time } t \\
S(t) &= \text{salary at time } t \\
\pi(t) &= P(t)/S(t) \\
&= \text{pension as a proportion of final salary}
\end{aligned}$$

Now $P(t)$ depends on past contributions, past investment returns and annuity rates at retirement. If contributions are paid at the start of each year then

$$
\begin{aligned}
P(t) &= \frac{1}{A(t)} \sum_{s=0}^{t} \rho(s) S(s) \frac{F(t)}{F(s)} \\
\text{where } \rho(s) &= \text{contribution rate at time } s \\
\frac{F(t)}{F(s)} &= \text{accumulation at time } t \text{ of an investment of 1 at time } s \\
A(t) &= \text{annuity factor applied on retirement at time } t
\end{aligned}
$$

Normally it will be assumed that the contribution rate $\rho(t)$ is constant through time, although this could be used as a method of reducing uncertainty.

Each of the processes $F(t)$, $S(t)$ and $A(t)$ are random. This will exaggerate the level of uncertainty at retirement unless a suitable strategy can be found which can use one process to offset the effects of another. For example, by investing in fixed interest bonds, a fall in bond prices close to retirement will be offset by a fall in the value of $A(t)$, the cost of purchasing an annuity.

Objectives may be divided into two categories

(A) ones in which the member is told of his or her pension only at the date of retirement;

(B) ones in which the member is given advance notice of the (likely) future amount of pension and then expects the final pension to be as close to this as possible (or not too much less than).

Possible objectives of type A are:

- maximize $E[\pi(t)]$;
- maximize $E[\pi(t)]$ subject to $Var[\pi(t)] = \sigma_\pi^2$;
- maximize $Var[\pi(t)]$;
- maximize $Var[\pi(t)]$ subject to $E[\pi(t)] = \mu_\pi$;
- minimize $Pr(\pi(t) < \pi_{\min})$;
- maximize $E[u(\pi(t))]$ where $u(\cdot)$ is some utility function.

Objectives of type B include

- minimize $E[(\pi(t) - \hat{\pi}(t))^2 \mid H_s]$ where H_s gives us the history of the fund up until time t and $\hat{\pi}(t)$ is the estimated future pension based on H_s;
- maximize $E[u(\pi(t)) \mid H_s, \hat{\pi}(t)]$.

It is questionable whether some such objectives may be reasonable. For example, suppose an objective results in a strategy which locks into a given level of pension some time in advance of retirement. The problem with this is that the level which we lock into may be just as variable as the pension which could be obtained had the fund been left alone until the date of retirement. So is it really in the member's best interests to lock into a pension at too early a stage?

3.2 Investment strategies

It may be difficult to examine all possible investment strategies. However, an appropriate starting point may be to examine a small number of possibilities. For example,

- strategies which are fixed through time:
  - equities only
  - equities and matching options
  - fixed interest bonds
  - equities, fixed interest bonds and cash
  - index linked bonds
  - equities, matching options, fixed interest bonds and cash
  - etc.
- strategies which vary through time:
  - equities switching into fixed interest bonds over the last 5 years
  - fixed interest bonds
  - equities and matching options
  - equities, matching options, fixed interest bonds and cash
  - etc.
- strategies which vary through time and depend on the past history of the fund.

3.3 A simple example

Here we look at a simple example which illustrates the fallacy of an early switch into fixed interest bonds.

We simplify the situation by considering a fund which is now of size $F(0)$ and which will receive no further contributions. We are interested in the lump sum which this fund will produce at retirement as a proportion of final salary.

Three options are available:

- a zero-coupon fixed interest investment which provides a guaranteed lump sum L at retirement;
- investment in long-term index linked bonds;
- investment in equities.

The model we will use is described in the Appendix. The model and its parameters were found to fit UK experience reasonably well.

The measure of risk for each option (the variance of the logarithm of the lump sum as a proportion of final salary) is plotted in Figure 11. We can see that although the fixed pension fares

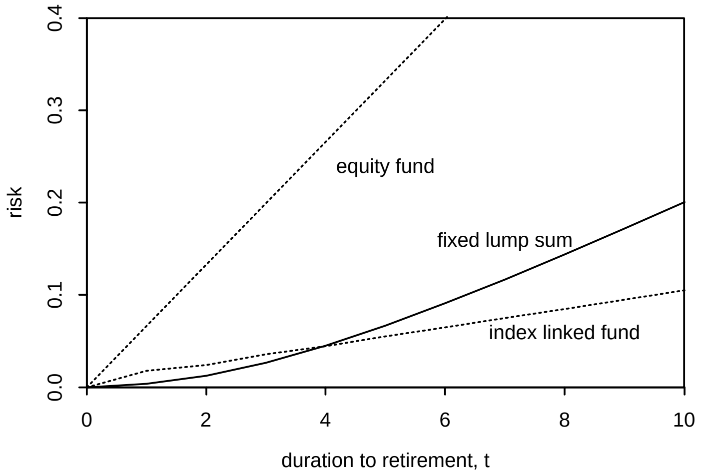

Figure 11: Risk relative to policyholder's salary for three different investment strategies. Risk is measured as $Var[L(t)/S(t)]$ where $L(t)$ is the lump sum at retirement and $S(t)$ is the final salary.

better early on the index linked option clearly becomes lower risk later on. (Note that this does not take account of uncertainty in the initial lump sum which would arise had we been considering the situation part of the way through a policy's lifetime.) The equity fund is, perhaps not surprisingly, well above the other two in terms of risk, but will also attract a reasonable risk premium. It is also likely that a fixed interest investment attracts a small risk premium over an index-linked investment so at later durations the ordering of the risks is in the order we might expect.

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5 Appendix

$$
\begin{aligned}
S(t) &= \text{salary at time } t \\
F_e(t) &= \text{equities fund at time } t \\
F_{il}(t) &= \text{index-linked fund at time } t \\
\delta_s(t) &= \log[S(t)/S(t-1)] \\
\delta_e(t) &= \log[F_e(t)/F_e(t-1)] \\
\delta_{il}(t) &= \log[F_{il}(t)/F_{il}(t-1)]
\end{aligned}
$$

$$
\begin{aligned}
\text{with } \delta_s(t) &= \delta_p(t) + \delta_{rs}(t) \\
\delta_e(t) &= \delta_p(t) + \delta_{rs}(t) + \delta_{re}(t) \\
\delta_{il}(t) &= \delta_p(t) + \delta_{ril}(t)
\end{aligned}
$$

$$
\begin{aligned}
\delta_p(t) &= \text{force of price inflation between } t-1 \text{ and } t \\
&= \delta_p + \alpha_p(\delta_p(t-1) - \delta_p) + \sigma_p Z_p(t) \\
\delta_{rs}(t) &= \text{real salary growth rate} \\
&= \delta_{rs} + \alpha_{rs}(\delta_{rs}(t-1) - \delta_{rs}) + \sigma_{rs} Z_{rs}(t) \\
\delta_{re}(t) &= \text{real equities rate of return over salaries} \\
&= \delta_{re} + \sigma_{re} Z_{re}(t) \\
\delta_{ril}(t) &= \text{real index linked return} \\
&= \delta_{ril} + \alpha_{ril}(\delta_{ril}(t-1) - \delta_{ril}) + \sigma_{ril} Z_{ril}(t)
\end{aligned}
$$

where $Z_p(t)$, $Z_{rs}(t)$, $Z_{re}(t)$ and $Z_{ril}(t)$ (for $t = 0, 1, 2, \ldots$) are independent and identically distributed sequences of standard Normal random variables.

Now let

$$
\begin{aligned}
y_p(t) &= \sum_{s=1}^{t} \delta_p(s) \\
y_{rs}(t) &= \sum_{s=1}^{t} \delta_{rs}(s) \\
y_{re}(t) &= \sum_{s=1}^{t} \delta_{re}(s) \\
y_{ril}(t) &= \sum_{s=1}^{t} \delta_{ril}(s) \\
\text{Then } E[y_p] &= \delta_p.t \\
Var[y_p(t)] &= \frac{\sigma_p^2}{(1-\alpha_p)^2}\left[t - \frac{2\alpha_p(1-\alpha_p^t)}{(1-\alpha_p)} + \frac{\alpha_p^2(1-\alpha_p^{2t})}{(1-\alpha_p^2)}\right] \\
E[y_{rs}] &= \delta_{rs}.t \\
Var[y_{rs}(t)] &= \frac{\sigma_{rs}^2}{(1-\alpha_{rs})^2}\left[t - \frac{2\alpha_{rs}(1-\alpha_{rs}^t)}{(1-\alpha_{rs})} + \frac{\alpha_{rs}^2(1-\alpha_{rs}^{2t})}{(1-\alpha_{rs}^2)}\right] \\
E[y_{re}(t)] &= \delta_{re}.t \\
Var[y_{re}(t)] &= \sigma_{re}^2.t \\
E[y_{ril}] &= \delta_{ril}.t \\
Var[y_{ril}(t)] &= \frac{\sigma_{ril}^2}{(1-\alpha_{ril})^2}\left[t - \frac{2\alpha_{ril}(1-\alpha_{ril}^t)}{(1-\alpha_{ril})} + \frac{\alpha_{ril}^2(1-\alpha_{ril}^{2t})}{(1-\alpha_{ril}^2)}\right]
\end{aligned}
$$

We also define

$$
\begin{aligned}
F_e(t) &= \exp[y_p(t) + y_{re}(t)] \\
F_{il}(t) &= \exp[y_p(t) + y_{ril}(t)] \\
S(t) &= \exp[y_p(t) + y_{rs}(t)]
\end{aligned}
$$

We are interested in the three quantities

$$
\begin{aligned}
L_1 &= L/S(t) \\
L_2 &= F_{il}(t)/S(t) \\
L_3 &= F_e(t)/S(t)
\end{aligned}
$$

Of particular interest is the level of risk associated with each option which we measure by taking the variance of the logarithm of each quantity.

$$
\begin{aligned}
Var[\log L_1] &= Var[y_p(t)] + Var[y_{rs}(t)] \\
Var[\log L_2] &= Var[y_{rs}(t)] + Var[y_{ril}(t)] \\
Var[\log L_3] &= Var[y_{re}(t)]
\end{aligned}
$$

These variances are described in the main text.

Parameter values

type, θ	δ_θ	α_θ	σ_θ^2
prices, *p*	0.05	0.7	0.05^2
real salary, *rs*	0.02	0.4	0.03^2
real index-linked, *ril*	0.036	-0.5	0.13^2
real equity, *re*	0.036		0.26^2