

Pensionmetrics 2: Stochastic pension plan design during the distribution phase

By

David Blake, Andrew J. G. Cairns, and Kevin Dowd\*

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\* Pensions Institute, Birkbeck College, 7-15 Gresse Street, London W1T, 1LL, UK; Department of Actuarial Mathematics and Statistics, Heriot-Watt University, Edinburgh EH14 4AS, UK; and Nottingham University Business School, Jubilee Campus, Nottingham NG8 1BB, UK, respectively. The authors thank an anonymous referee for helpful comments, and the BSI Gamma Foundation and UBS Global Asset Management for financial support for this project.

## **Abstract**

We consider the choices available to a defined contribution (DC) pension plan member at the time of retirement for conversion of his pension fund into a stream of retirement income. In particular, we compare the purchase at retirement age of a conventional life annuity (i.e., a bond-based investment) with distribution programmes involving differing exposures to equities during retirement. The residual fund at the time of the plan member's death can either be bequeathed to his estate or revert to the life office in exchange for the payment of survival credits while alive. The most important decision, in terms of cost to the plan member, is the level of equity investment. We also find that the optimal age to annuitise depends on the bequest utility and the investment performance of the fund during retirement.

**Keywords:** Stochastic pension plan design; defined contribution; discounted utility; life annuity; income drawdown; asset allocation; optimal annuitisation age.

## 1. Introduction

In many countries, the principal retirement income vehicle in defined contribution (DC) pension plans is the life annuity. This is a bond-based investment with longevity insurance, and is the only financial instrument in existence that protects the retiree from outliving his resources: no other distribution programme will guarantee fixed retirement payments for however long an individual lives. Consequently, it is optimal – given a single safe asset and no bequest motive – for an individual to use all his wealth to purchase an annuity as soon as he retires (Yaari (1965)).

In the UK, the accumulated pension fund must be used to buy a life annuity from a life office by the time the plan member reaches the age of 75. The amount of the annuity will depend on the size of the fund, the long-term bond yield on the purchase date, the type of annuity (i.e., whether the payments are fixed or variable<sup>1</sup>), the age, sex and (occasionally) state of health of the annuitant, and a margin to cover the life office's profit and costs of marketing, administration, and investment management.<sup>2</sup>

However, relatively few individuals voluntarily annuitise their DC fund at the time of retirement, particularly in the US.<sup>3</sup> One reason might be the high loading factor in quoted annuity rates (Friedman & Warshawsky (1990)), although Mitchell *et al.* (1999), using US data, and Finkelstein & Poterba (2002), using UK data, show that annuities are better value for money than is commonly supposed. Another possible reason is that many people have a strong bequest motive that reduces their desire to annuitise their wealth (Bernheim (1991)), although Brown (2001) finds that the

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<sup>1</sup> It has become possible for life offices to sell index-linked annuities (which link payments to the variability in the retail price index) as a result of the introduction of long-dated index-linked government bonds that provide the essential matching assets. For more on index-linked bonds, see, for example, Anderson *et al.* (1996) and Cairns (1997).

<sup>2</sup> Annuities involve risks for both the buyer and the seller. The plan member bears the risk of retiring when interest rates are low, so that the retirement annuity is permanently low. After he retires, he can also bear inflation risk: the risk of losses in the real value of his pension due to unanticipated later inflation. For their part, life offices selling annuities face reinvestment risk (the risk of failing to match asset cash flows with liability outgoings) and mortality risk (the risk that annuitants might live longer than expected). For a more detailed analysis of the problems facing annuity markets and some potential solutions to these problems, see, e.g., Blake (1999).

bequest motive is not a significant factor in the annuitisation decision. A third possibility is that people in poor health try to avoid buying annuities, because they do not expect to live very long (Brown (2001), Finkelstein & Poterba (2002)). Their findings confirm the predictions from Brugiavini's (1993) theoretical model in which the health status of the individual follows a stochastic model, the parameters of which are known only to the individual.

DC plans typically involve a sudden switch from an investment strategy based primarily on equities to one based entirely on bonds (see, for example, Blake, Cairns & Dowd (2001) and Cairns, Blake & Dowd (2000)). However, commentators have begun to question whether it is sensible to have a substantial bond-based investment over a long retirement period: after all, substantial improvements in longevity over the last century mean that retirees can typically expect to live for 15 years or more, and there are likely to be further improvements in the future. Furthermore, substantial falls in bond yields over the last decade have not only made bonds less attractive investment vehicles, but have also made annuities much more expensive to pension-plan members.<sup>4</sup>

The perceived poor value of traditional annuities has motivated a search for new investment-linked retirement-income programmes that involve the provision of retirement income from a fund with a substantial equity component. The attraction of such vehicles is obvious: there are very few historical periods where equities do not outperform bonds over long horizons.<sup>5</sup> Nevertheless, equity prices tend to be much more volatile than bond prices, so the higher expected returns from equities involves greater risk. Further, some of these alternatives do not hedge mortality risk and so do not satisfy the basic requirement of a pension plan to provide a retirement income for however long the plan member lives. For example, any income drawdown programme that draws a fixed income from a fund heavily invested in equities has a strictly positive probability of running down to zero before the plan member dies (Milevsky

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<sup>3</sup> In the US there is no mandatory requirement to purchase an annuity by any age. Brown & Warshawsky (2001) predict that the switch in employer sponsorship in the US from defined benefit (DB) plans to DC plans will lead to even lower annuity purchases in the future.

<sup>4</sup> For example, in the UK long-bond yields reached a forty-year low in 1999 pushing up annuity prices to corresponding highs.

<sup>5</sup> Siegel (1997) shows that US equities generated higher average returns than US Treasury bonds and bills in 97% of all 30-year investment horizons since 1802. CSFB (2000) shows that similar results hold for the UK.

& Robinson (2000); Albrecht & Maurer (2001)). However, this danger of running out of assets before death can be ameliorated by requiring that the amount drawn from the fund be linked to the fund size at each point in time, or by imposing a requirement that the plan member annuitise by a certain age, as in the UK.

In this paper we compare three different distribution programmes<sup>6</sup> for a male DC plan member retiring at age 65:<sup>7</sup>

- **Purchased life annuity (PLA):** On retirement at age 65 the plan member transfers his retirement fund to a life office in return for a level pension, and no bequest is payable at the plan member's time of death. This programme is the benchmark against which the other programmes below are compared.
- **Equity-linked annuity (ELA) with a level, life annuity purchased at age 75:** The assets are held in a managed fund containing both equities and bonds, and the plan member is protected from running out of money before age 75 by the requirement that annuity income fall in line with any fall in the fund value. We consider five different levels of equity exposure in the managed fund: 0%, 25%, 50%, 75% and 100%. At the start of each year, the life office pays an actuarially fair survival credit to the plan member if he is still alive. The survival credit accounts for anticipated mortality over the coming year, and involves an extra return arising from the mortality risk-sharing implicit in an annuity: those who die early on create a profit that is shared amongst those annuitants who die later. This extra return is equal to the expected proportion of surviving

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<sup>6</sup> In an earlier version of this paper (Blake, Cairns and Dowd, 2000), we analysed a larger range of distribution programmes. These included: (a) A programme in which income is fixed. The result is stability of income coupled with the risk of ruin before death. With most forms of plan member utility function analysed, the possibility of ruin results in very low discounted expected utility (in some cases minus infinity) making such programmes extremely unattractive. (b) Variants on the equity-linked annuity and income-drawdown programmes involving the use of derivatives to limit downside risk in the fund. Such programmes were found to give results similar to, but slightly worse than, funds excluding derivative investments with a similar annual standard deviation in returns. (c) A programme which purchased at 65 a deferred annuity from age 75 and consumed the remaining fund entirely between ages 65 and 75. None of these alternatives proved to be as effective as those discussed in detail in this paper.

<sup>7</sup> We treat any non-pensions-related personal savings by adding them to the pension fund at retirement. We also treat the residential home as a fixed asset that becomes a bequest on death. This bequest is the same under all of the programmes described below and so has no differential effect. We also assume for simplicity that the pensioner consumes all his pension income each year, and (generally) ignore any other sources of income – although at the end of Section 4 we also look at the impact on our results of a fixed (e.g., State) annuity that cannot be commuted for cash.

annuitants who die in that year, and is therefore increasing in age.<sup>8</sup> In return for these survival credits the residual fund reverts to the life office when the plan member dies, so he leaves no bequest.

- **Equity-linked income-drawdown (ELID) with a level, life annuity purchased at age 75:** This programme is otherwise similar to the ELA programme, except for the fact that the plan member does not receive any survival credit or surrender his bequest to the life office if he dies before age 75. Should he die before that age, his residual fund is paid as a bequest to his estate.

Our analysis leads to a number of significant conclusions and – to anticipate the later discussion – the most important ones are:

- The optimal programme depends on the plan member's attitude to risk: the greater his risk appetite, the greater his preferred exposure to equities.
- The cost of adopting a suboptimal programme is generally much less significant than the cost associated with an inappropriate equities exposure: it is therefore very important for the plan member to get the equities exposure right.
- The optimal choice of distribution programme is fairly insensitive to the plan member's bequest motive.
- The plan member's optimal choices are relatively insensitive to differences between his own and the life office's assessment of his mortality prospects.
- Compulsory annuitisation by any particular age can be costly for plan members with a relatively high appetite for risk, but impose no costs on members who are more risk averse.
- The optimal annuitisation age is very sensitive to the plan member's degree of risk aversion, moderately sensitive to the bequest motive, and dependent on the size of the retirement fund accumulated by the time the annuitisation decision is made.

The layout of this paper is as follows. Section 2 explains the stochastic framework underlying our analysis, Section 3 analyses the three key pension distribution programmes available to a DC plan member, and Section 4 presents and discusses numerical results. Section 5 then presents our conclusions.

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<sup>8</sup> For more details, see Blake (1999).

## 2. Stochastic Assumptions

### 2.1 Asset returns

We assume that there are two assets available for investment: risk-free bonds and equity. The bond fund,  $M(t)$ , grows at the continuously compounded constant risk-free rate of  $r$  per annum, so that at time  $t$ ,  $M(t) = M(0)\exp(rt)$ , where  $M(0)$  is the initial price. Equity prices,  $S(t)$ , satisfy geometric Brownian motion, so that  $S(t) = S(0)\exp(\mu t + \sigma Z(t))$ , where  $S(0)$  is the initial price and  $Z(t)$  is standard Brownian motion. It follows that the gross annual returns on equities are independent and identically distributed log-normal random variables with mean  $\exp(\mu + \sigma^2/2)$  and variance  $\exp(2\mu + \sigma^2)[\exp(\sigma^2) - 1]$ .

For simplicity, we assume that the pension is drawn at the start of each year and that pension plan assets are rebalanced annually to maintain predetermined proportions in each asset class. We assume that the annual life office charge on equity investment for all distribution programmes is constant at 1% of fund value, implying a reduction in yield of 1%.<sup>9</sup>

In our simulations, we used the following parameters (net of expenses):

### Table 1 hereabouts

These figures imply that the expected gross return on equities is 1.11 and the standard deviation of the total return is 0.275. These parameter values are consistent with historical returns on UK Treasury bills and equities over the last half century and include the 1% reduction in equity returns.

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<sup>9</sup> Some of the distribution programmes that we consider, such as income drawdown, can be very expensive with charges considerably in excess of 1% p.a. Nevertheless, to preserve comparability, we assume a 1% charge for income drawdown as well. See Blake (1999) or Appendix A of Blake, Cairns & Dowd (2000).

## 2.2 Mortality and financial functions

We also make use of the following mortality and financial functions:

- $q_x$  and  $p_x$  are the year-on-year mortality and survival probabilities respectively.<sup>10</sup> The values of these probabilities are the same as those used in the most appropriate UK mortality table for compulsory-purchase, male annuitants (i.e., PMA92Base: see McCutcheon *et al.* (1998, 1999)).
- ${}_t p_x = p_x \times \dots \times p_{x+t-1}$  is the probability that the pensioner survives to age  $x+t$  given that he is alive at age  $x$ .
- ${}_t | q_x = {}_t p_x - {}_{t+1} p_x$  is the probability that the pensioner will die between ages  $x+t$  and  $x+t+1$  given that he is alive at age  $x$ .
- $\ddot{a}_y = \sum_{t=0}^{\infty} {}_t p_y e^{-rt}$  is the fair price at age  $y$  of a level single-life life annuity of £1 per annum, payable annually in advance.

We assume in this study that mortality rates will not improve over time.<sup>11</sup>

## 3. Distribution Programmes

Our analysis is based on a typical 65-year old male who is assumed to have accumulated a personal pension fund on his retirement date (denoted  $t = 0$  below) of  $F(0) = £100,000$  and is about to retire. Our plan member has to choose between the following three distribution programmes.

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<sup>10</sup> Thus  $p_x$  is the probability that an individual aged  $x$  survives for one year, and  $q_x = 1 - p_x$  is the corresponding probability of death.

<sup>11</sup> For recent work on stochastic mortality improvements, see Milevsky & Promislow (2001), Yang (2001) and Wilkie *et al.* (2003).



### 3.1 Programme 1: Purchased life annuity (PLA)

In Programme 1,  $F(0)$  is used immediately to purchase a level life annuity at a price of  $\ddot{a}_x$  per £1 of pension. The pension is therefore  $P(t) = P_B = F(0)/\ddot{a}_x$  for  $t=0,1,2, \dots$  and is payable until death. No bequest is payable, but the surviving plan member receives implicit survival credits instead.

### 3.2 Programme 2: Equity-linked annuity (ELA)

Programme 2 is designed to benefit from equity investment but also adjusts the pension paid to remove the possibility of running out of funds before age 75. Under this programme, the pension,  $P(t)$ , is adjusted each year to reflect both the fund size available at the beginning of the year as the plan member ages. The procedure for calculating each year's pension payment ensures that both  $P(t)$  and  $F(t)$  are always positive. The programme also allows for different degrees of equity weighting ( $\omega$ ). Hence:

$$P(t) = \frac{F(t)}{\ddot{a}_{x+t}} \quad (1)$$

$$B(t) = (1 - \delta) \frac{q_{x+t}}{P_{x+t}} (F(t) - P(t)) \quad (2)$$

$$F(t+1) = \left( \omega \frac{S(t+1)}{S(t)} + (1 - \omega)e^r \right) (F(t) - P(t) + B(t)) \quad (3)$$

$$D(t+1) = \delta F(t+1) \quad (4)$$

for  $t=0,1,\dots,9$ , where  $\delta = 0$  if survival credits are payable (ELA) and  $\delta = 1$  if bequests are payable (ELID; see section 3.3 below).  $B(t)$  is the actuarially fair survival credit paid into the plan member's fund at the start of every year if the plan member is still alive (i.e., the fund is increased from  $F(t)$  to  $F(t) + B(t)$  at time  $t$  if the plan member is still alive at that time).  $F(t+1)$  is the residual fund which reverts

to the life office if the plan member dies during the year.

At time  $t = 10$  (age 75), if the plan member is still alive, the residual fund,  $F(10)$ , is used to purchase a level, single-life annuity at prevailing terms. No bequests are payable after time 10, but the plan member continues to receive the survival credits implicit in the annuity.

Note that for the case when  $\omega = 0$  we get  $P(t) = P_b$  for all  $t$  (i.e., the PLA and ELA programmes are identical).

### 3.3. Programme 3: Equity-lined income-drawdown (ELID)

The above set of equations can also be used to describe Programme 3 if we set  $\delta = 1$ .  $D(t+1)$  represents the bequest payable to his estate at  $t+1$  if the plan member dies between times  $t$  and  $t+1$ .

This programme will provide a level pension if the return on the assets is equal to the risk-free rate adjusted for the mortality drag<sup>12</sup>: i.e., if:

$$\omega \frac{S(t+1)}{S(t)} + (1-\omega)e^r = \frac{e^r}{P_{x+t}} \quad (5)$$

This condition for returns in equation (5) to achieve a level pension lies in contrast with Programme 2. In Programme 2 risky assets need only generate a return equal to the risk-free rate (i.e.,  $S(t+1)/S(t) = e^r$ ) to provide a level pension, because of the survival credits embodied in the ELA.

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<sup>12</sup> This is the additional return required on an investment to compensate for giving up the survival credit implicit in a life annuity. In a given year, it equals the percentage of the annuitant group alive at the beginning of the year who die during the year and hence increases steeply with age.

## 4 Numerical Results

### 4.1 The value function

To compare these programmes, we use the log-normal distribution of  $S(t+1)/S(t)$  and the translated log-normal distribution of  $(1-\omega)e^r + \omega S(t+1)/S(t)$  to calculate the plan member's discounted lifetime utility. This notion of utility captures the plan member's welfare throughout retirement and is similar to that employed by Merton (1990) and others.

We measure value relative to the standard life annuity that pays a fixed amount  $P_B = F(0)/\ddot{a}_{65}$  per annum for life with no bequest (i.e., Programme 1: PLA). We refer to this annuity as the *benchmark* pension. For the given parameters and mortality rates and an initial fund of £100,000,  $P_B = £7,551.53$  p.a. For Programmes 2 and 3 we consider fixed 0%, 25%, 50%, 75% and 100% proportions of equity investment. Naturally, we recognise that strategies based on dynamic optimisation might give superior results, but these are also difficult to implement in practice. Instead we choose to restrict ourselves to a set of programmes that are straightforward to implement and easily understood by plan members.

Now let  $K$  be the curtate future lifetime<sup>13</sup> of the plan member from age 65. The plan member's value function (or expected discounted utility) is assumed to take the form:

$$V(s, f) = E \left[ \sum_{t=s}^K e^{-\beta t} J_1(P(t)) + k_2 e^{-\beta(K+1)} J_2(D(K+1)) \mid F(s) = f, \text{ alive at } s \right] \quad (6)$$

where:

$$J_1(P(t)) = h_1(\gamma_1) \left( \frac{P(t)}{P_B} \right)^{\gamma_1} \quad (7)$$

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<sup>13</sup> That is, the random future lifetime of the plan member rounded down to the previous integer.

$$J_2(D(t)) = h_2(\gamma_2) \left[ \left( \frac{D(t) + d_2}{d_2} \right)^{\gamma_2} - 1 \right], \text{ with } d_2 > 0. \quad (8)$$

The value function used in equation (6) is typical of those used in optimal stochastic control theory (e.g., Merton, 1971) for general increasing and concave utility functions  $J_1(\cdot)$  and  $J_2(\cdot)$ , and is also consistent with Merton (1983), Kapur & Orszag (1999) and Milevsky & Young (2002) in the pensions context.<sup>14</sup>  $J_1(P(t))$  comes from the constant relative risk aversion (CRRA) class of utility functions (i.e., power and log utility functions). The plan member's relative risk aversion (RRA) parameter is  $1 - \gamma_1$ .  $J_2(D(t))$  comes from the hyperbolic absolute risk aversion (HARA) class (which includes the CRRA class as a special case). The parameter  $\beta$  measures the plan member's subjective rate of time preference and  $k_2$  is used to specify the appropriate balance between the desire for income and the desire to make a bequest.

The functions  $h_1(\gamma_1)$  and  $h_2(\gamma_2)$  have a considerable impact on the analysis, unless  $k_2 = 0$ , in which case the specifications of  $h_1(\gamma_1)$  and  $h_2(\gamma_2)$  irrelevant.<sup>15</sup> We assume:

$$h_1(\gamma_1) = \frac{1}{1 - d_1^{\gamma_1}} \quad (9)$$

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<sup>14</sup> A different approach to assessing the value to an individual of a series of cashflows has been proposed by Epstein & Zin (1989) and which also allows a clear separation of risk attitudes from intertemporal substitution effects. However, investigation of the present pensions problem using the Epstein-Zin framework is beyond the scope of this paper.

<sup>15</sup> An essential requirement for each function is that it takes the same sign as its argument, and a conventional parameterisation of  $J_1$  and  $J_2$  would be  $h_1(\gamma_1) = 1/\gamma_1$  and  $h_2(\gamma_2) = 1/\gamma_2$ . We experimented with these forms in preliminary work, but found that they resulted in strongly dichotomous preferences. Plan members with a low RRA had a very strong preference for the ELA programme (offering no bequests), while plan members with a high RRA had a very strong preference for the ELID programme (offering bequests). However, we do not believe that real world behaviour is so extreme: we would expect to see some individuals with low RRA still wishing to make a bequest and vice versa. We believe it is important to choose forms for  $h_1(\gamma_1)$  and  $h_2(\gamma_2)$  that avoid such extreme swings in preference over the range of RRA parameter values, leaving  $k_2$  as the main parameter determining the choice between the different programmes.

$$h_2(\gamma_2) = \frac{1}{\left(\frac{F(0) + d_2}{d_2}\right)^{\gamma_2} - 1}. \quad (10)$$

The parameter,  $d_1$ , can be freely determined within the range  $0 < d_1 < 1$  for  $h_1(\gamma_1)$  to take the correct sign, and the parameter,  $d_2$ , can be interpreted as the value of the plan member's non-pension assets such as his house.

We can now make the following remarks about the properties of  $J_1(\cdot)$  and  $J_2(\cdot)$ :

- $J_1(d_1 P_B) = d_1^{\gamma_1} / (1 - d_1^{\gamma_1}) = J_1(P_B) - 1$ : Thus,  $J_1(P)$  increases by 1 in absolute terms when  $P$  increases from  $d_1 P_B$  to  $P_B$ .
- $J_2(0) = 0$ : This is a consequence of our requirement that  $d_2 > 0$ .<sup>16</sup> Imposing  $J_2(0) = 0$  means that when we value the benchmark PLA programme, the  $J_2(\cdot)$  component of the discounted utility is unaffected by the timing of death. Additionally, any strictly positive bequest has a strictly positive impact on the discounted utility function relative to the benchmark.
- $J_2(F(0)) = 1$ :  $J_2(\cdot)$  increases by 1 as the size of the bequest changes from 0 to  $F(0)$  (i.e., if the member died on the same day that the programme started).

The value of  $k_2$  (and, to a lesser extent,  $\gamma_2$ ) will reflect the family characteristics of the plan member: for example, a married man with young children is likely to have a greater bequest motive and hence a higher value of  $k_2$  than a single man with no children. The choice for  $k_2$  will be discussed further in Section 4.3.

So far as the authors are aware, no standard definition exists for the relative risk aversion parameter attached to the value of a series of cashflows rather than to a single cashflow using a value function of the type in equation (6). We therefore propose the following definition. Consider the PLA programme where  $P(t) = P_B$  is

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<sup>16</sup> Suppose, in contrast, we used  $d_2 = 0$  in combination with  $\gamma_2 < 0$ . This implies that  $J_2(0) = -\infty$ . However, we know that many plan members do choose to annuitise their entire liquid assets thereby leaving a bequest of zero. Making such a choice is inconsistent with having  $J_2(0) = -\infty$ .

constant and no bequest is payable. Then the value function is a function of  $P_B$  only:<sup>17</sup>

$$\tilde{V}(P_B) \equiv V(0, F(0)) = E \left[ \sum_{t=0}^K e^{-\beta t} J_1(P_B) \mid F(0), \text{ alive at } 0 \right] \quad (11)$$

We now define the relative risk aversion parameter to be  $-P_B \cdot \tilde{V}''(P_B) / \tilde{V}'(P_B)$ , where primes indicate derivatives. For the value function in Equation (6) the relative risk aversion parameter is therefore  $1 - \gamma_1$  for all  $P_B$ . Where a bequest is payable we still take  $1 - \gamma_1$  as the RRA parameter while acknowledging that this reflects only the pre-death risks.

The optimal programme maximises the value function  $V(0, F(0))$ .

#### 4.2 General comments on the results

In the following experiments we have assumed  $\gamma_1 = \gamma_2 = \gamma$ , and, as plausible illustrative values,  $k_2 = 5$ ,  $d_1 = 0.75$  and  $d_2 = 10000$ .<sup>18</sup> Mortality is assumed to be independent of the investment scenario and we also assume for the moment that the true mortality model for (and known to) the plan member is the same as that used by the life office in calculating annuity rates. We refer to this as standard mortality. We then evaluated the value function  $V(0, F(0))$  across a range of values for the RRA parameter,  $1 - \gamma$ , varying from 0.25 to 25. This range embraces both very risk-averse and very risk-tolerant preferences and is consistent with values found in studies by Blake (1996) for the UK and (among others) Brown (2001) for the US.<sup>19</sup> Finally, we also fixed the value of  $\beta$  at  $\log 1.05$ .<sup>20</sup>

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<sup>17</sup> The bequest utility function is not included here since the PLA programme means that  $D(k+1) = 0$  for all  $k_2$ , and  $J_2(0) = 0$ .

<sup>18</sup> The sensitivity of our results to these particular parameter values is assessed below.

<sup>19</sup> Some of the published values are controversial, and Feldstein and Rangelova (2001) have recently suggested that the literature tends to over-estimate RRAs. However, even their 'plausible' estimates (that is,  $\text{RRA} < 3$ ) are still within our assumed range.

<sup>20</sup> Blake (forthcoming) estimates the marginal rate of time preference of a typical UK household to be

Numerical results are presented in Table 2 for the discounted utilities of the different programmes for an illustrative RRA coefficient of 3.96. These values were calculated using a backwards recursion:

$$V(s, F(s)) = e^{-\beta s} J_1(P(s)) + p_{65+s} E[V(s+1, F(s+1)) | F(s), \text{alive at } x+s+1] + q_{65+s} e^{-\beta(s+1)} E[\delta F(s+1) | F(s), \text{alive at } x+s] \quad (12)$$

The optimal strategy for the plan member in this case is to select the ELA programme with 25% in equities.

### Table 2 hereabouts

Figure 1 shows the plot of the differences between the value functions for each equity-linked programme and the benchmark PLA programme (i.e.,  $V_p(0, F(0)) - V_B(0, F(0))$ ) over our assumed range of values for the RRA parameter  $1 - \gamma$ . Panel (a) shows the plot for the ELA programme, panel (b) the plot for the ELID programme, and the top bold line indicates the best programme for a given degree of relative risk aversion. The Figure reveals the following:

- For our illustrative value of  $k_2 = 5$ , all the optimal policies use survival credits rather than bequests.
- For an RRA less than about 1.25, the best programme is an equity-linked annuity with 100% equities; for an RRA greater than 1.25 but less than about 10, the ELA programme remains optimal provided the equity weighting is gradually lowered as RRA increases; and for an RRA greater than about 10 (i.e., for more risk averse plan members), the best programme is the PLA.<sup>21</sup>

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about 3%. This is consistent with the use here of  $\beta = \log 1.05$  in combination with an assumed rate of inflation of 2% (although an inflation assumption is not required for the present analysis).

<sup>21</sup> However, Blake's (1996) estimates of RRA values against wealth suggest that only about 5% of individuals will want any equities exposure, and an even smaller proportion will want 100% equities exposure.

### **Figure 1 hereabouts**

The optimal equity proportions for different RRA's are shown in Figure 2. The solid line indicates the best portfolio mix in the unrestricted range 0% to 100% as a function of the RRA. With  $k_2 = 5$ , the optimal distribution programme for all levels of RRA is the ELA rather than the ELID programme. The dots indicate the best portfolio mix when the plan member is restricted to choosing one of the five ELA programmes with 0%, 25%, 50%, 75% or 100% equities. For example, the best available programme for a plan member with an RRA of 10, who would otherwise choose an equity proportion of 15% if free to do so, is the ELA with an equity proportion of 25%. The best available programme for a plan member with a slightly higher RRA is the one with a 0% equity proportion.

### **Figure 2 about here**

For any specific RRA, Figure 1 can be used to rank the programmes in order of preference. However, the differences in expected discounted utilities give us little feel for how much worse, say, the 75%-equities ELID programme is relative to the optimal (i.e., 25%-equities ELA) programme. A more intuitive comparison is given by the cash compensation criterion: how much extra cash (measured proportional to the initial fund value) would a plan member need at time 0 in order for the 75%-equities ELID to have the same expected discounted utility as the optimal ELA programme? This question is answered in Figure 3: a plan member with an RRA of 3.96 would require an extra 25% in his retirement fund for a 75%-equities ELID programme to match the optimal ELA outcome attainable with the retirement fund he actually has.

### **Figure 3 hereabouts**



Figure 3 is more informative than Figure 1 because it gives a cash-equivalent comparison of the relative quality of each programme. Thus, we can see that for plan members with a relatively strong appetite for risk (i.e., a low RRA of around 0.25), programmes such as the PLA would require as much as 50% to 70% extra cash to match their preferred Programme 2 (ELA with 100% equities). This finding should not be too surprising: (relatively) risk-loving plan members have a strong preference for equity exposure, and therefore need considerable compensation if they are to adopt a bond-based investment strategy.

At the other end of the spectrum, Figure 3 shows that a very risk-averse plan member would prefer Programme 1 (PLA) (or equivalently the ELA programme with 0% equities), and therefore need considerable compensation to accept any equity exposure. The compensation needed also rises with the degree of risk aversion and the specified equity exposure, and can be very large indeed.

To summarise: the plan member must decide on the programme type (PLA, ELA or ELID) and (for the latter two programmes) the equity proportion, and our findings suggest that the choice of equity proportion is far more important than the choice of programme type.<sup>22</sup>

#### *4.3. Importance of bequests*

Empirical studies (see Brown (2001) and the references cited therein) are inconclusive on the importance attached by retirees to bequests, so it is difficult to judge with the current state of knowledge what typical values for  $k_2$  should be. We examined the impact of changing  $k_2$  to 20 and 100 (see Figure 4, and Figure 3(b) for  $k_2 = 5$ ) and found that the effect of  $k_2$  on costs is quite gradual. We can therefore infer that our results and observations are not over-sensitive to changes in  $k_2$  and, hence, that mis-

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<sup>22</sup> These conclusions are not particularly sensitive to the size of the equity risk premium: the only notable difference associated with a decreased equity risk premium (we considered  $\mu = 0.0285$  instead of the standard value of  $\mu = 0.0746$  used elsewhere) is to decrease (respectively increase) the cost of suboptimality for plan members with low (respectively high) degrees of risk aversion by a relatively small amount. However, this difference aside, our broad conclusions remain much the same.

specification of  $k_2$  is unlikely to be as costly for the plan member as an inappropriate choice of equity mix.

**Figure 4 about here**

#### *4.4 Adverse mortality selection: impaired lives*

The calculations above assumed that the plan member's mortality probabilities equal those used by the life office to calculate annuity prices. However, a typical group of plan members will include some in good health and others in poor health. For the latter group, the purchase of a life annuity at retirement at standard rates represents poor value relative to other plan members in better health. It is often suggested, therefore, that those in poor health should defer annuitisation for as long as possible.

Consider an individual for whom mortality rates are approximately four times those assumed by the life office.<sup>23</sup> This degree of impairment is consistent with, for example, an individual who has just been diagnosed as suffering from Alzheimer's Disease (Macdonald & Pritchard (2000)). Results for such an individual are presented in Figure 5, which was constructed using the same set of parameters ( $k_2 = 5$ ,  $d_2 = 10000$  and  $\beta = 0.0488$ ) as Figure 3. We make the following observations.

First, if the choice of programme (ELA or ELID) is given, the optimal equity proportion (0%, 25%, 50%, 75% or 100%) for plan members with different levels of risk aversion is unaffected by differences between impaired and standard mortality. This is a direct consequence of power utility, which generates constant portfolio proportions.

Second, comparing Figures 5(a) and 3(a), we see that, if we are restricted to the use of an ELA programme, then the optimal equity mix is largely unaltered. However, the cost of choosing a sub-optimal ELA programme with impaired mortality is about one-third lower than with standard mortality.

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<sup>23</sup> Strictly we assume that,  $p_x^{impaired} = (p_x^{PLA})^4$ , that is, the force of mortality is 4 times the standard force used by the life office in its annuity pricing. For small  $q_x^{PLA}$ , this implies that  $q_x^{impaired} \approx 4(q_x^{PLA})$ .

Third, Figure 5(b) compares each of the ELID programmes with the best ELA programme for each level of risk aversion. The cost of choosing a sub-optimal ELID programme with impaired mortality is about 50%+ lower than with standard mortality (cf., Figure 3(b)). As the degree of impairment increases, plan members are more likely to prefer the ELID programme. However, except for those with very strong bequest motives (i.e., high  $k_2$ ), the degree of impairment needs to be quite strong before the plan member switches from the ELA to the ELID programme. It is also possible that an impaired life would benefit more from programmes that allow for the accelerated payment of pension. The most beneficial improvement from the point of view of the plan member would be the payment of higher (i.e., fairer) survival credits from the life office to reflect the higher mortality rates (as is the case with impaired life annuities) rather than from a programme with bequests. At the same time, individuals with a lower degree of impairment are still likely to prefer the ELA programme to the ELID one: the survival credits can still be worth having even though they are actuarially unfair.

**Figure 5 here**

#### *4.5 Optimal annuitisation and the cost of regulation*

A number of authors have tackled the problem of when a plan member would choose to annuitise, assuming he were free to annuitise or not as he wished. Different authors have tackled this issue in different ways, and Table 3 lists six key features of the various models to address this issue. Table 4 indicates how these features are incorporated into each study (and also into this one), and gives the main conclusions reached regarding the optimal annuitisation age.

From these tables we can see that the optimal annuitisation age depends on (a) the annuity options available to the plan member, (b) his risk aversion and, as we will see in Section 4.6, (c) the existence and form of the bequest utility.

**Table 3 here**

**Table 4 here**

To investigate this issue further, we compared the following three choices:

- Annuitise immediately (i.e., Programme 1).
- Employ the ELA or ELID programme with the optimal equity mix up to age 75 and then annuitise.
- Employ the ELA or ELID programme with the optimal equity mix and annuitise at the optimal age between age 65 and 85, with compulsory annuitisation at age 85 if voluntary annuitisation has not occurred beforehand. The annuitisation age is decided at age 65 and this decision is, for the moment, assumed to be irreversible.

We begin by comparing the ELA and PLA programmes. We find that it is optimal either to annuitise immediately or to wait until age 85, but never to annuitise at some intermediate age. This is consistent with Merton's (1983) approach. The cost of compulsory annuitisation at age 75 then turns out to lie between 0% and 15% of the initial fund value depending on the level of risk aversion.

Next we compare the ELID and PLA programmes. Table 5 shows results for the case where the plan member has the right to invest freely up to age 85 and to annuitise at any age up to 85, but attaches no value to bequests (i.e.,  $k_2 = 0$ ). This means that the decision to defer annuitisation is driven purely by a comparison between the loss of future expected excess equity returns and the gain from future survival credits under the PLA. The final column of the Table reports the cost of a regulation compelling plan members to annuitise at age 75. For example, a plan member with a very low RRA of 0.25 would require an extra 1.6% of his retirement fund to compensate for annuitising at 75 rather than 85. The Table indicates that the optimal annuitisation age is very sensitive to the level of risk aversion, but that the overall cost of forced annuitisation at 75 (when there is no bequest motive) is relatively small and declines to zero for RRAs above unity.

## Table 5 here

It is also interesting to note that at very low levels of RRA, the optimal annuitisation age of 79 is close to the age we would get (namely, 81) by applying Milevsky's (1998) rule, which specifies that we switch at the point where the mortality drag matches the expected excess return on equities over bonds. However, our analysis shows that this decision rule matches the one presented here only for a plan member who is risk neutral (i.e.,  $RRA = 0$ ). Our more general analysis demonstrates that decision making is much more complex than the Milevsky rule suggests, with the equity mix and the optimal annuitisation age critically dependent on both the level of risk aversion and the bequest motive.

Table 6 shows that, when the bequest motive is positive, the optimal age to switch from an ELID programme to the PLA generally increases, since the option to delay annuitisation becomes more valuable. The switching age is greater for very low and high RRAs (i.e., below 4.99 and above 12.53, respectively) than is the case without a bequest motive, and is not much affected for intermediate RRAs.<sup>24</sup> This increase in the optimal annuitisation age is not surprising. Under the PLA that the plan member switches into, the expected discounted utility is not affected by the inclusion of a bequest utility since the utility attached to a zero bequest is zero. In contrast the expected discounted utility under the ELID programme immediately increases as a result of the inclusion of a bequest utility. This makes continuation with the ELID programme for at least another year relatively more attractive at all ages. However, the results in Table 6 also indicate that even where it is positive, the cost of compulsory annuitisation by age 75 is also fairly low: for example, even in the 'worst case' where  $RRA=0.25$  and the optimal annuitisation age is 80, enforced annuitisation by age 75 costs only 2.4% of initial fund value.

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<sup>24</sup> The U-shaped pattern of optimal annuitisation ages in Table 6 is an artefact of the way in which the functions  $h_1(\gamma)$  and  $h_2(\gamma)$  have been parameterised. As the RRA increases ( $\gamma$  decreases), plan members, besides investing more conservatively, place greater emphasis on the bequest, since  $h_2(\gamma)$  is increasing in RRA.

## Table 6 here

### 4.6 Annuitisation under dynamic stochastic optimisation

The preceding discussion assumed for simplicity that plan members decide at 65 the future age at which they will annuitise, and then adhere to this decision regardless of future circumstances. However, it is more plausible to suppose that when they are 65 they merely anticipate the age at which they will annuitise – unless of course they annuitise immediately – and then make a firm annuitisation decision later. Suppose therefore that at the end of each year any pre-annuitised plan member reconsiders annuitisation taking into account the information available at that time. Within the present modelling framework, this means that the decision at time  $t$  will depend on the current fund size,  $F(t)$ , and the current age of the plan member,  $65 + t$ . Since  $F(t)$  is random, the plan member does not necessarily know in advance the optimal age at which to annuitise.

The optimisation process then proceeds as follows:

- Let the optimal value function at time  $t$  be denoted  $\hat{V}(t, F(t))$ .
- Start at the age,  $x = 65 + T$ , by which annuitisation is compulsory. For each possible fund size at that time calculate the value function  $V(T, F(T)) = \hat{V}(T, F(T))$ .
- Next work backwards recursively:
  - Assume that the optimal value function  $\hat{V}(t+1, F(t+1))$  is known for all  $F(t+1)$ . Now consider the decision at time  $t$  when the fund size is  $F(t)$ . We need to compare the value function (a) assuming that the plan member annuitises immediately with the value function (b) assuming that the plan member defers annuitisation until at least time  $t+1$  and then acts optimally thereafter. Under (b) we have several factors to take into account: the probability of survival, the pension payment at time  $t$ , and the possible bequest if the plan member dies before time  $t+1$ . The plan member chooses the option (a) or (b) that maximises the value function,

thus producing the optimal value for  $\hat{V}(t, F(t))$ .

- This procedure is repeated over the full range of possible values for  $F(t)$ .
- Once this has been done, we can step backwards by one year, repeat the previous step and continue in this way until we reach the age of 65 at which point we stop.<sup>25</sup>

One reason why the annuity decision is likely to depend upon the fund size  $F(t)$  is that the bequest utility does not exhibit constant RRA. Our results indicate that a plan member is more likely to prefer to delay (bring forward) annuitisation if his investments have been performing well (badly). To illustrate this, suppose the fund size is almost zero and a plan member is considering a switch from the ELID programme to the PLA programme. On the one hand, the negative impact on the bequest utility will be negligible because the fund size is very small. On the other hand, the payment of survival credits through the PLA will have a strong beneficial impact on the utility of consumption, because the marginal utility of consumption gets large as the fund size gets small. So both bequest and marginal utility of consumption considerations make the plan member keener to annuitise, relative to a plan member with more wealth.

The dependence of the annuitisation decision on fund size is illustrated in Figure 6, which shows the outcome of the above optimisation process at selected RRAs. Consider a plan member with an RRA of 3.15 who is now aged 75 and who has not previously annuitised. If his current fund size is below about £90,000, he should annuitise immediately. But if his current fund is above this level, then it is optimal for him to defer annuitisation. We can also see that the annuitisation region varies considerably with the RRA. We also observe from these graphs that for any given age and RRA, annuitisation will either:

- not be optimal for any fund size;
- be optimal for all fund sizes; or
- be optimal for low fund sizes but not for fund sizes above some threshold.

## Figure 6 here

In each graph the dots show how the plan member's fund value would change over time if he had opted at age 65 for the PLA. This gives a useful reference point for projecting the stochastic fund size under the ELID programme at different ages. Thus with an RRA of 1.58, we can see that annuitisation is likely to occur some time between the ages of 72 (if equities perform poorly) and 80 (if equities perform moderately well). However, if equities perform sufficiently well then the fund-age trajectory will lie above the shaded region and annuitisation might only take place when it is compulsory at age 85. In the adjacent plot where the RRA is 3.15, the shaded annuitisation 'hill' is somewhat lower, implying that a relatively large proportion of the stochastic trajectories of  $F(t)$  will avoid hitting the hill (i.e., and so avoid annuitisation) at ages below 85. On the other hand, if  $F(t)$  is going to hit the hill it, will probably do so within the first 3 or 4 years. We can infer from these observations that in some (i.e., low RRA) cases the dynamic stochastic element in the annuitisation decision will not add much value (the plan member will choose to annuitise at around age 80 regardless). However, in other (high RRA) cases, the shape and height of the annuitisation hill are such that the majority of stochastic fund-age trajectories cross over the hill without hitting it, suggesting that the extra timing choice captured by the dynamic stochastic element is a potentially valuable feature.

The shapes of these annuitisation regions depend upon the principal motivation for choosing to defer annuitisation, bearing in mind that members will defer the switch from ELID to PLA for two possible reasons: continued equity participation; and the desire to leave a bequest. The first of these is the primary motivation for low RRA plan members: for these there is a largely vertical annuitisation region because the decision to defer is largely unaffected by the past performance of the fund. On the other hand, for high RRA members, continued equity participation is not a reason for deferral as they would choose a very low equity mix. Instead the reason to defer is mainly based on the desire to leave a bequest. This

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<sup>25</sup> Throughout this exercise, we assume that the equity mix up to the time of annuitisation is held constant. Strictly this may, itself, not be optimal, but in the context of power utility on the pension component of the utility function this approximation will be reasonable.



motive makes the annuitisation decision dependent on the current fund size and results in the emergence of the ‘hill’ shape in Figure 6. However, if we compare the optimal utility for the fully optimal case in Figure 6 with the deterministic decision made at age 65 (Table 6), we find that the added flexibility is not an especially valuable option (almost 0% for RRA up to 8 increasing to 4% added value for RRA=25).

#### 4.7 Sensitivity analysis

We have already tested for sensitivity of our results with respect to the level of risk aversion, the equity risk premium, the weight attached to the bequest, the plan member's health status and the timing of annuitisation. We will comment briefly now on the sensitivity of our results to some remaining factors.

- We tested for sensitivity to the parameter  $d_2$  in the bequest utility function (equation (8)) by setting  $d_2 = 50000$  instead of 10000. There was little change in the results, indicating that they are robust relative to large changes in this parameter.
- We considered the exponential utility function as an alternative to power utility. This gives rise to constant *absolute* risk aversion and decreasing relative risk aversion as a function of the initial investment  $F(0)$ . Our results suggest that the value of the RRA parameter, at the specified level of  $F(0) = 100000$ , is more important than the precise shape of the utility function.<sup>26</sup>
- Our previous results were predicated on the assumption that pension income could be derived only from the initial fund  $F(0)$ . As a variant we looked at the possibility that this income could be supplemented by a level (e.g., state) pension,  $P_s$ , which cannot be commuted for cash. This we did by modifying  $J_1(P(t))$  to  $h_1(\gamma_1)((P(t) + P_s)/(P_B + P_s))^{\gamma_1}$  with  $P_s = 5000$  (or about 2/3 of  $P_B$ ). However, our results suggest that the introduction of a state pension into the model does not fundamentally alter our earlier conclusions.

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<sup>26</sup> This observation relies on the assumption that we will follow a static investment strategy up to the time of annuitisation. It is well known that *dynamic* optimal strategies would evolve differently if exponential utility were used.

## 5. Conclusions

Our results suggest that the best distribution programme does not usually involve a bequest, but rather pays regular survival credits to the plan member in return for the residual fund reverting to the life office on the plan member's death. On the other hand, the best programme does depend on the plan member's attitude to risk: if he is highly risk averse, the appropriate programme is a conventional life annuity; if he has a stronger appetite for risk, the best programme typically involves a mixture of bonds and equities, with the optimal mix depending on the plan member's degree of risk aversion; and if he has a very strong appetite for risk (i.e., an RRA less than 1.25), he will invest entirely in equities. However, if we accept Blake's (1996) estimates of the range of risk aversion parameters found for UK investors, the equity-linked annuity is likely to be chosen by relatively few plan members (only about 5% of the total), and very few of these would choose to invest 100% of their retirement fund in equities.

Our results also suggest a number of other conclusions:

- The optimal choice of distribution programme appears to be fairly insensitive to the weight attached by the plan member to making a bequest. In particular, the weight would have to be substantially higher than that used here to make programmes with a bequest optimal.
- The equity proportion chosen for the distribution programme has a considerably more important effect on the plan member's welfare than the distribution programme chosen, and a poor choice can lead to substantially reduced expected discounted utility.
- Plan members in poor health relative to the average may, depending on the severity of their ill health, still prefer the ELA programme paying standard-rate survival credits to the ELID programme paying bequests. However, those in extremely poor health and attaching some weight to a bequest are rather more likely to prefer an income drawdown programme.
- Forcing members of ELA programmes to annuitise at 75 rather than 85 can be expensive in terms of reduced expected discounted utility for those with low degrees of risk aversion: it is equivalent to 15% of the initial fund value for risk-neutral plan members. However, the costliness of this requirement declines

as the plan member becomes more risk averse, and is zero at RRAs above unity.

- The optimal annuitisation age is:
  - Very sensitive to the plan member's degree of risk aversion: Where no value is attached to bequests the optimal age ranges from 79 for a plan member with a very low RRA to immediate annuitisation for one whose RRA exceeds about 4. This suggests that any switching rule that ignores relative risk aversion as a determining factor (e.g., annuitise when the mortality drag first exceeds the equity risk premium) is likely to be suboptimal and may overestimate the optimal switching age for a risk-averse member.
  - Sensitive to the bequest motive: a bequest motive encourages plan members to defer annuitisation, other things being equal.
  - Dependent on fund size: For the HARA form chosen for the bequest utility function, a larger fund size makes it more likely it is that the plan member will delay annuitisation.

Lastly, some further extensions naturally suggest themselves. First, future work might usefully investigate the impact of a stochastic interest rate instead of the fixed-risk free rate assumed here. The analysis might also be extended to handle the important issue of mortality improvements. For example, one promising avenue is to investigate flexible unit-linked programmes where the income received and the survival credits payable fall in response to mortality improvements. Finally, we have considered only a limited range of standard programmes, and future work might investigate (fully) optimal solutions based on stochastic dynamic programming, instead of the simple programmes considered here.<sup>27</sup>

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<sup>27</sup> For an accumulation-stage study along these lines, see Cairns, Blake & Dowd (2000).

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## TABLES

**Table 1: Return Parameters**

<i>Parameter</i>	<i>Symbol</i>	<i>Nominal value</i>
Risk-free return	$r$	0.0296
Expected log return on equities	$\mu$	0.0746
Standard deviation of log return on equities	$\sigma$	0.244

**Table 2: Utility Rankings of Alternative Programmes**

Programme	Equity %	$V(0, F(0))$
PLA	0	-8.42
ELA	0	-8.42
<b>ELA</b>	<b>25</b>	<b>-6.96</b>
ELA	50	-7.35
ELA	75	-10.00
ELA	100	-19.99
ELID	0	-11.98
ELID	25	-9.42
ELID	50	-10.10
ELID	75	-14.79
ELID	100	-33.11

Notes for Table 2: Expected discounted utilities for different distribution programmes for a plan member with relative risk aversion coefficient of 3.96. Parameter values:  $k_2 = 5$ ,  $d_1 = 0.75$ ,  $d_2 = 10000$ ,  $\beta = 0.0488$ , standard mortality, and annuitization at age 75.

**Table 3: Annuitisation Model and Plan Member Characteristics**

Category	Type	
	1	2
A	Investor is risk neutral	Investor is risk averse
B	No survival credits before annuitisation	Partial survival credits before annuitisation
C	ELID+ PLA only available	ELID + ELA available
D	No bequest	Bequest payable
E	Fixed asset mix	Dynamic asset mix
F	Deterministic asset model	Stochastic asset model

**Table 4: Annuitisation Model Features and Conclusions**

<i>Paper</i>	<i>Model Features</i>
Yaari (1965)	(A1), (B1), C1, D1, E1, F1 Annuitise (PLA) immediately
Merton (1983)	A2, (B1), C2, D1, E1, F2 Purchase an annuity immediately; never opt for PLA
Milevsky (1998)	A1, B1, C1, D1, E1, F2 Annuitise (PLA) when mortality drag $\geq$ equity risk premium
Kapur & Orszag (1999)	A2, B2, C1, D1, E2, F2 Gradual annuitization (PLA) with full annuitization when mortality drag $\geq$ equity risk premium
Milevsky & Young (2002)	A2, B1, C1, D1, E2, F2 Switch to PLA at deterministic time T. ELID before T includes optimized dynamic asset mix. T depends on risk aversion and model parameters.
This paper, section 4.6	A2, B1, C1, D2, E1, F2 Switch to PLA at a stochastic stopping time T. ELID before T includes optimized static asset mix. T depends on risk aversion and bequest utility.

Notes for Table 4: Papers considering the optimal time to annuitise. A bracketed feature – e.g. (B1) – implies that a particular assumption is not essential to the conclusions.

**Table 5: Optimal Annuitisation (1)**

<i>RRA</i>	<i>Optimal equity mix</i>	<i>Optimal annuitisation age</i>	<i>Cost of annuitisation at age 75</i>
0.25	100	79	1.6%
0.31	100	79	1.3%
0.40	100	78	1.1%
0.50	100	78	0.8%
0.63	100	77	0.5%
0.79	100	77	0.2%
1.00	100	76	0%
1.25	100	74	0%
1.58	75	72	0%
1.99	75	70	0%
2.50	50	69	0%
3.15	50	66	0%
3.96	0	65	0%
4.99	0	65	0%
6.28	0	65	0%
7.91	0	65	0%
9.95	0	65	0%
12.53	0	65	0%
15.77	0	65	0%
19.86	0	65	0%
25.00	0	65	0%

Notes for Table 5: Optimal decision rules for a plan member choosing the ELID programme when annuitisation can occur at any time before a compulsory annuitization age of 85, with the absence of a bequest motive. The final column shows the cost of compulsory annuitisation at age 75 as a percentage of the initial fund. Standard mortality,  $k_2 = 0$ ,  $\beta = 0.0488$ .



**Table 6: Optimal Annuitisation (2)**

<i>RRA</i>	<i>Optimal equity mix</i>	<i>Optimal annuitisation age</i>	<i>Cost of annuitisation at age 75</i>
0.25	100	80	2.4%
0.31	100	80	2.1%
0.40	100	80	1.9%
0.50	100	80	1.5%
0.63	100	79	1.2%
0.79	100	78	0.7%
1.00	100	77	0.3%
1.25	100	76	0%
1.58	75	74	0%
1.99	75	72	0%
2.50	50	70	0%
3.15	50	68	0%
3.96	25	67	0%
4.99	0	65	0%
6.28	0	65	0%
7.91	0	65	0%
9.95	0	65	0%
12.53	0	65	0%
15.77	0	69	0%
19.86	0	71	0%
25.00	0	72	0%

Notes for Table 6: Optimal decision rules for a plan member choosing the ELID programme when annuitisation can occur at any time before a compulsory annuitization age of 85, in the presence of a bequest motive. The final column shows the cost of compulsory annuitisation at age 75 as a percentage of the initial fund. Standard mortality,  $k_2 = 0$ ,  $\beta = 0.0488$ .

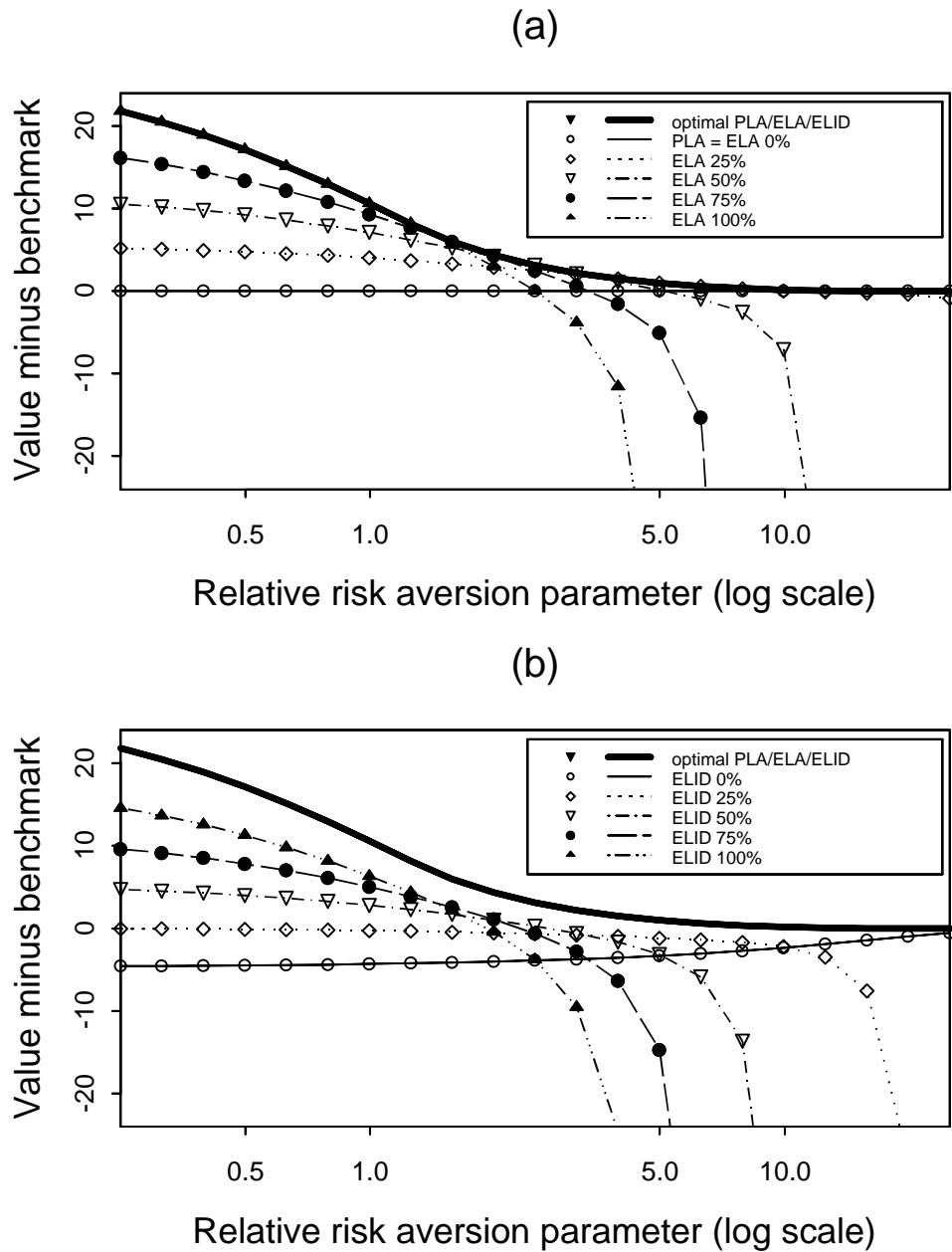


Figure 1: Expected discounted utilities for different equity-linked distribution programmes (relative to the purchased life annuity) as a function of the relative risk aversion parameter: (a) annuity programmes paying survival credits, (b) income drawdown programmes paying bequests. Standard mortality,  $k_2 = 5$ ,  $d_2 = 10000$ ,  $\beta = 0.0488$ .

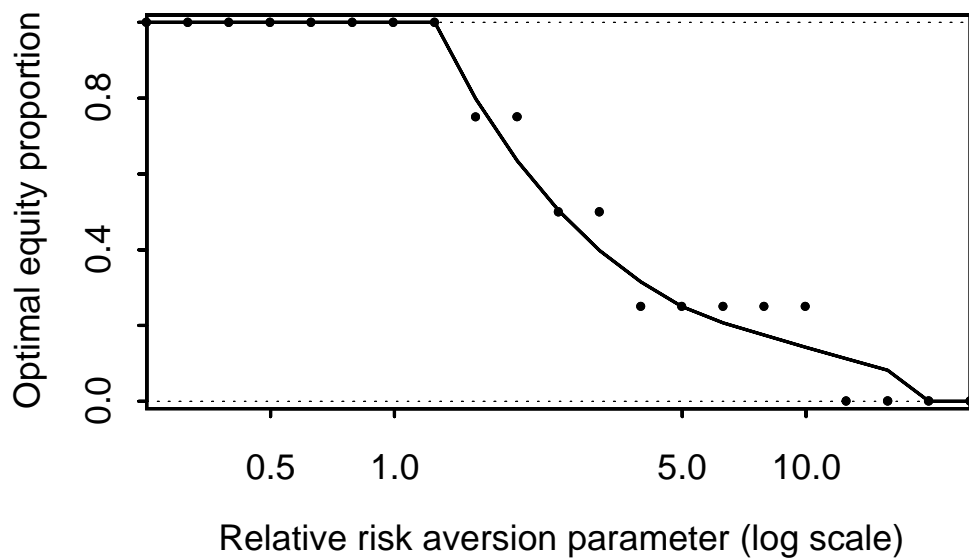


Figure 2: Optimal equity proportions in equity-linked distribution programmes as a function of the relative risk aversion parameter. Solid line: optimal relationship when the equity proportion can take any value between 0% and 100%. Dots: relationship when the equity proportions are constrained to 0%, 25%, 50%, 75% or 100%. Standard mortality,  $k_2 = 5$ ,  $d_2 = 10000$ ,  $\beta = 0.0488$ .

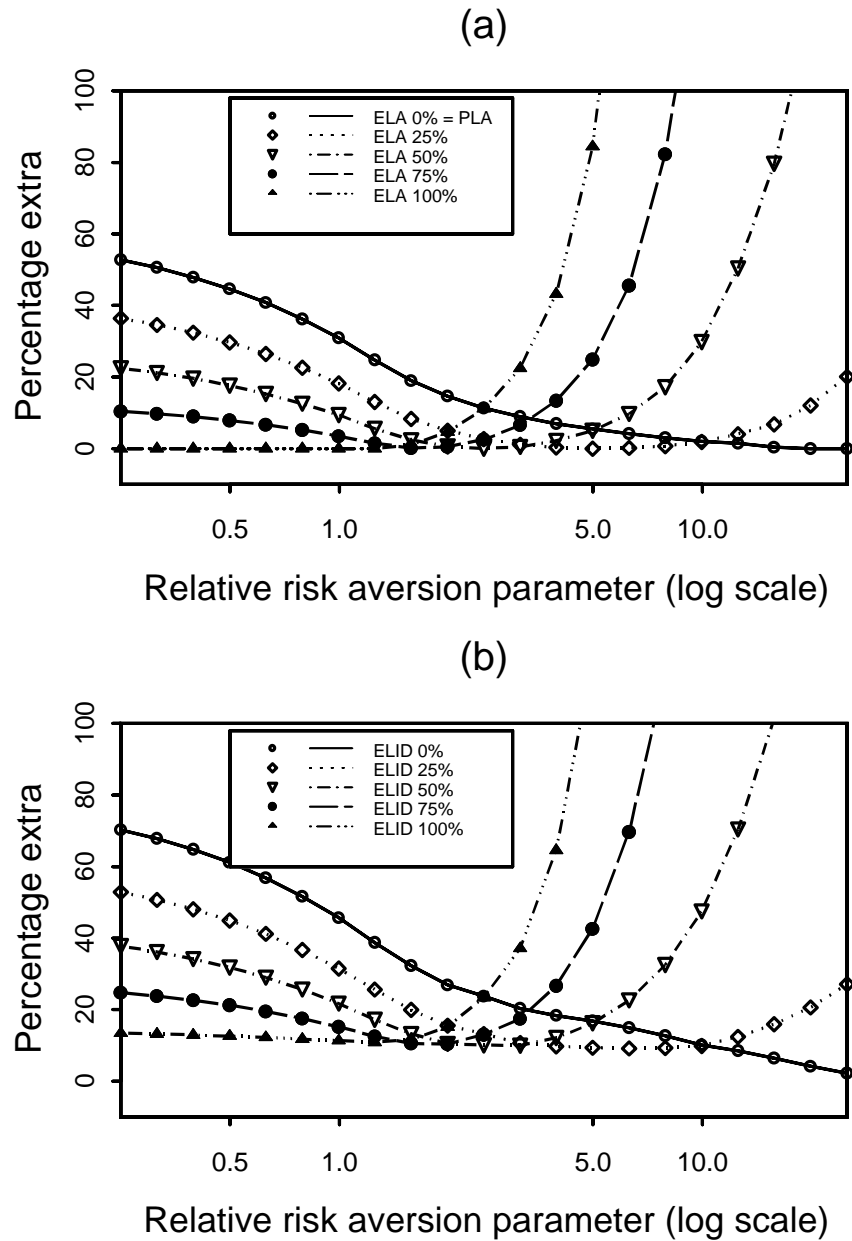


Figure 3: Extra cash required at time 0 for different distribution programmes to match the expected discounted utility of the optimal ELA programme as a function of the relative risk aversion parameter: (a) annuity (PLA and ELA) programmes paying survival credits versus the optimal ELA programme, (b) income drawdown programmes paying bequests versus the optimal ELA programme. Standard mortality,  $k_2 = 5$ ,  $d_2 = 10000$ ,  $\beta = 0.0488$ .

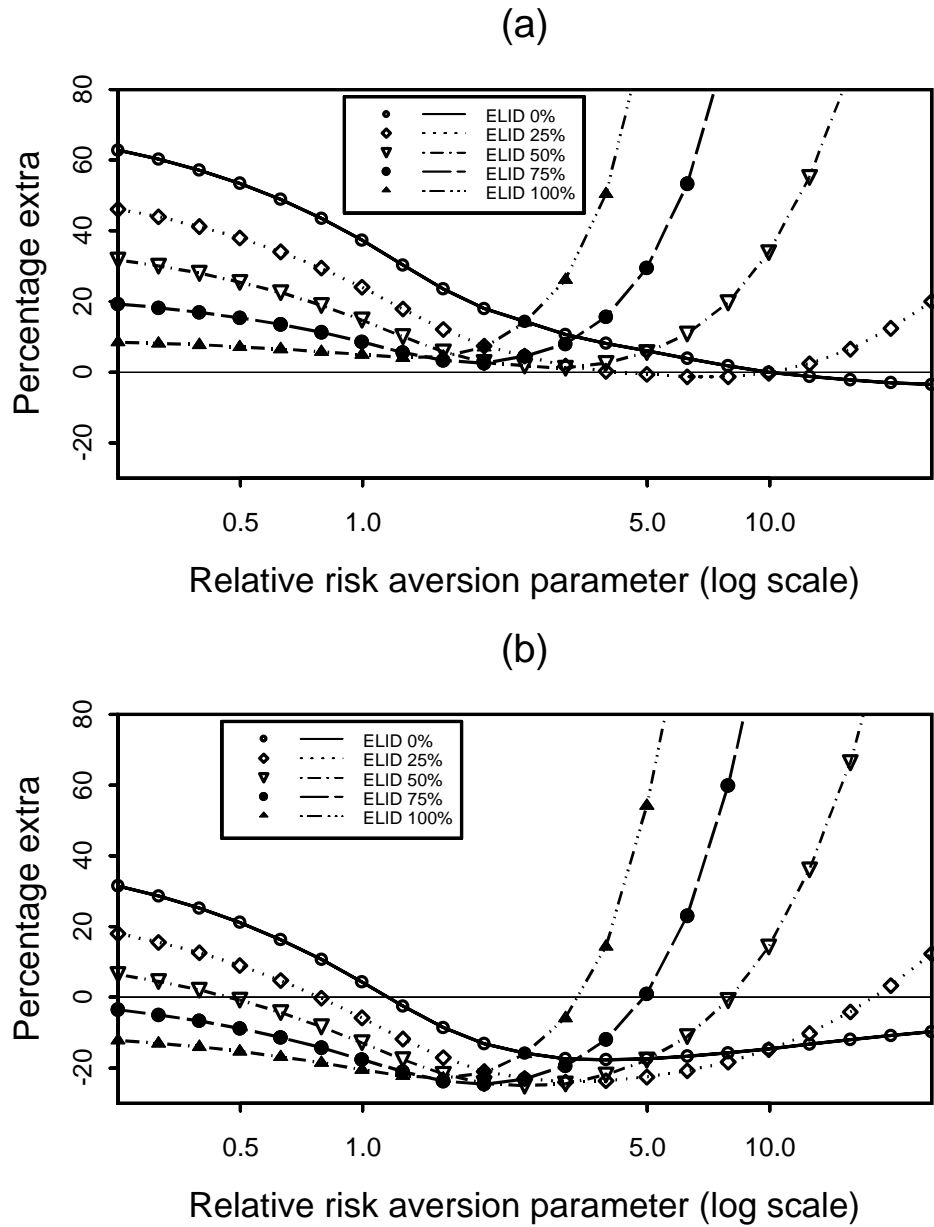


Figure 4: Extra cash required at time 0 for different income drawdown programmes to match the expected discounted utility of the optimal ELA programme as a function of the relative risk aversion parameter. Standard mortality,  $k_2 = 20$  (top) and  $k_2 = 100$  (bottom),  $d_2 = 10000$ ,  $\beta = 0.0488$ .

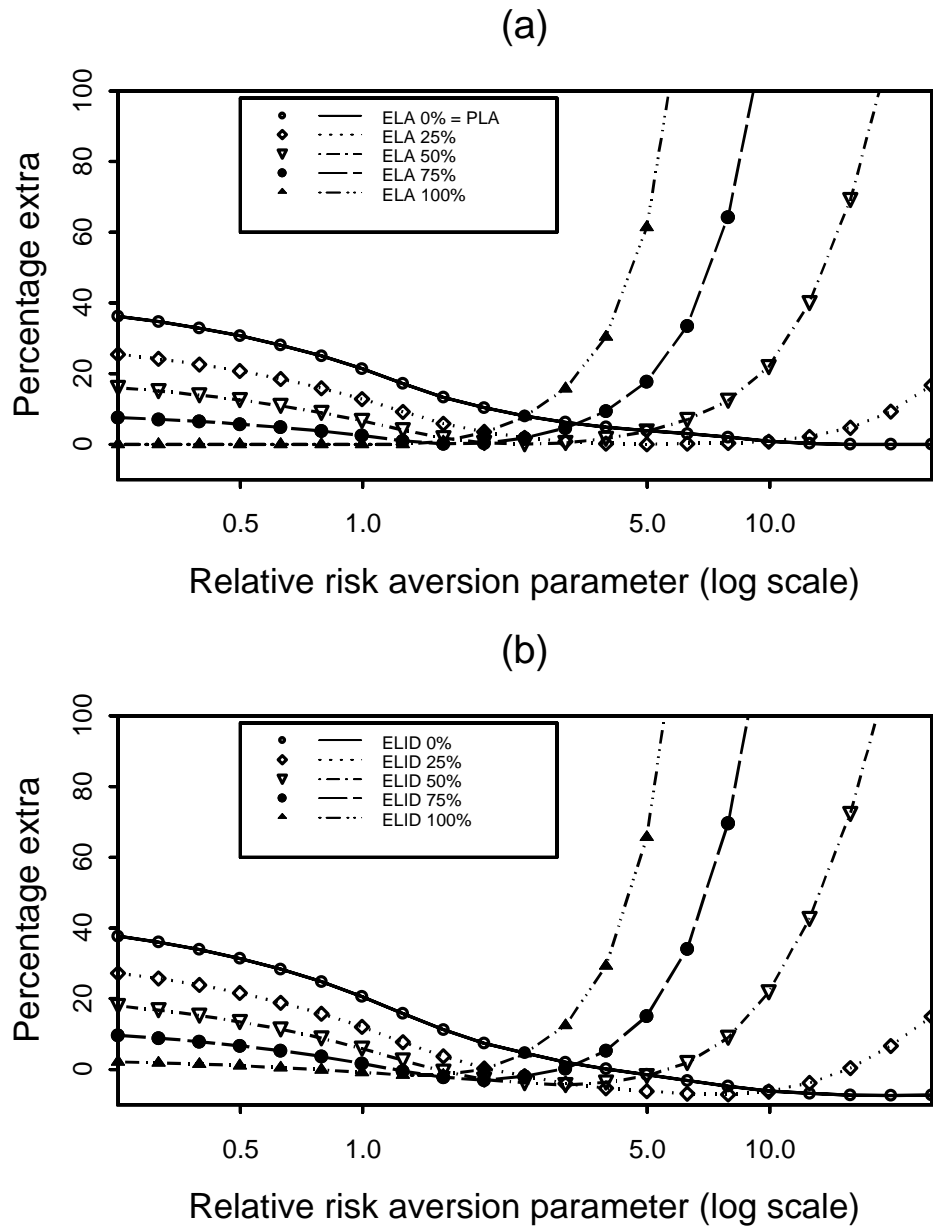


Figure 5: Extra cash required at time 0 for different impaired-life distribution programmes to match the expected discounted utility of the optimal ELA programme as a function of the relative risk aversion parameter: (a) annuity (PLA and ELA) programmes paying survival credits versus the optimal ELA programme, (b) income drawdown programmes paying bequests versus the optimal ELA programme. Impaired mortality,  $k_2 = 5$ ,  $d_2 = 10000$ ,  $\beta = 0.0488$ .

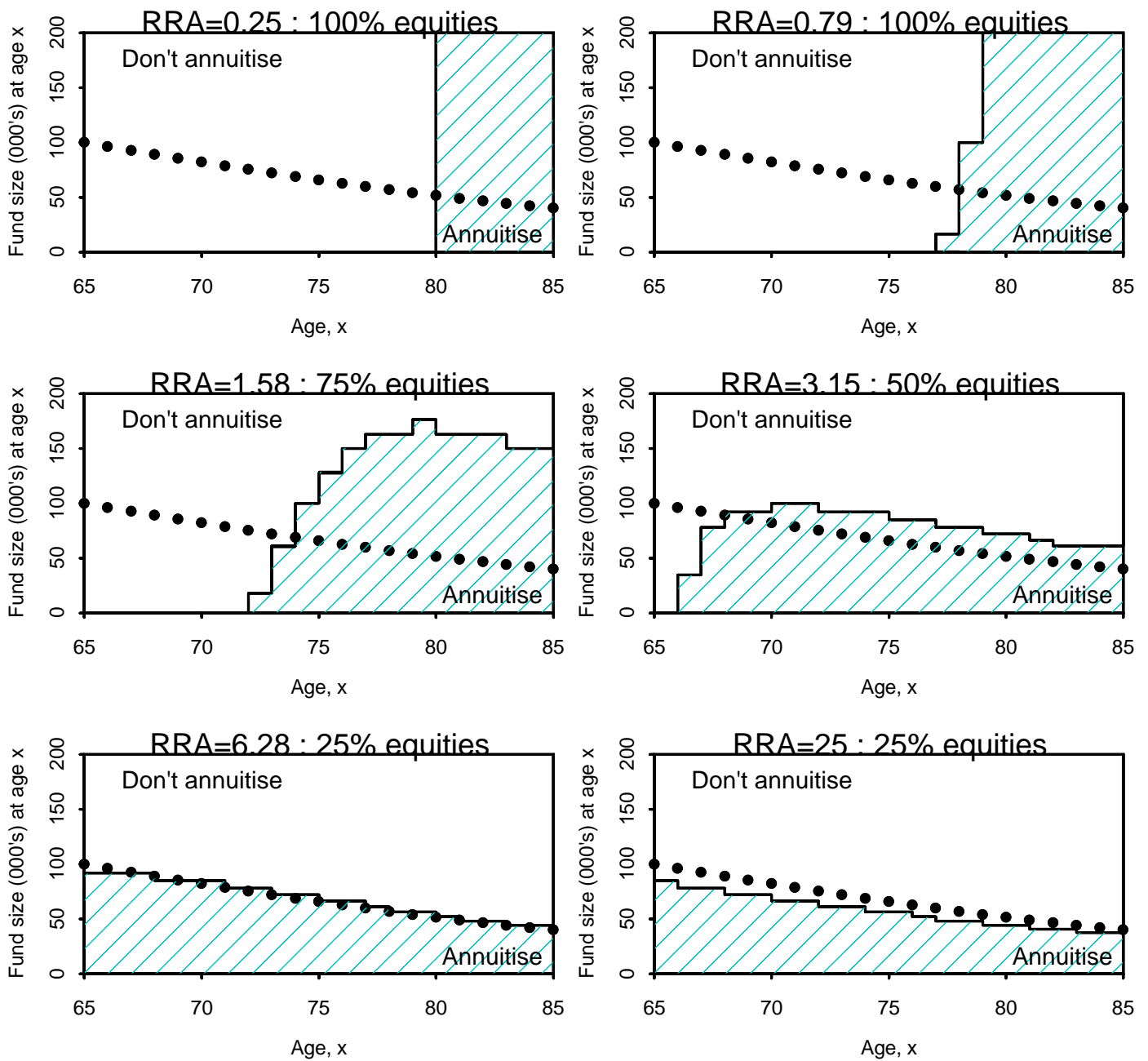


Figure 6: Relationship between the annuitisation decision and the plan member's age and fund size. If the fund-age trajectory enters the shaded area, then the plan member should annuitise immediately. The graphs are for different levels of risk aversion (RRA). Each graph assumes that before annuitisation the indicated optimal equity mix has been used. The dotted line shows how the fund size would evolve with age if the plan member had opted for the PLA programme. Standard mortality,  $k_2 = 5$ ,  $d_2 = 10000$ ,  $\beta = 0.0488$ .