From Financial Economics to Fair Valuation

Andrew J.G. Cairns
Department of Actuarial Mathematics and Statistics,
Heriot-Watt University,
Edinburgh, EH14 4AS, United Kingdom

E-mail: A.Cairns@ma.hw.ac.uk
Internet: http://www.ma.hw.ac.uk/∼andrewc/

May 17, 2001

Abstract

In this paper we address the issue of how to establish the fair value of an insurance-linked liability. This is done by considering the introduction into a simple, one-period market model of a new and quite general security (which, amongst other things, could be such a liability). We investigate the impact of this new security on the market and attempt to predict the price (the fair value) at which it will enter the market, assuming a liquid market.

The model employed is very simple for two reasons. First, it allows us to derive analytical results for equilibrium prices (often impossible for more complex models). Second, its simplicity allows us without difficulty to identify specific effects, rather than muddy the water with more general, abstract or complex features.

The model includes the following key features. First, investors have different levels of risk aversion. Second, investors are allowed to have different parameter estimates for the returns model. Third investors have different, risky future liabilities.

A principal aim of the paper is to compare the equilibrium approach with the often-more-tractable risk-minimisation approach to pricing and reserving popular in financial mathematics. We find that there is a strong link between the two approaches. However, there are differences between the two prices derived. These are caused by: different investor liabilities; different parameter estimates between investors; difficulty in estimating investors' levels of risk aversion. Of these we find that differences in estimates of the mean return on the new security causes the most significant differences.

Keywords: Fair value; Equilibrium theory; Risk-minimising prices; Utility function; Parameter risk; CAPM; Modigliani-Miller Theorem.

1 Introduction

In this paper we will discuss in reasonably general terms the concept of *fair value*. This has become an increasingly important concept within actuarial work in recent years in reaction to a number of developments internationally. There has been a general shift away from traditional actuarial methods (which have often shrouded the valuation of both assets and liabilities in mystery) to new approaches which make the valuation process more transparent. In particular, there has been increasing demand from the international accounting profession for objective and clear measurement of both liability and asset values. It is generally agreed that an objective and fair valuation of assets involves taking these at their current market values without making smoothing or other adjustments. (We will not deal in this paper the issue of bid-offer spreads, liquidity and assets, such as property, which are not readily traded.)

We deal in this paper with the more difficult issue of liability valuation. We will use the following principal to provide our definition of what is meant by fair value:

The fair value of a liability is the price at which it would trade if a liquid market existed in such an asset.

A key concept within this definition is the liquid market. This indicates that there will be both buyers and sellers of the liability. Furthermore, the appropriate price will that at which there is a balance between buyers and sellers: that is, the market will be in *equilibrium*.

Where a future insurance liability is known with certainty (for example, €1000 payable in 4 years) the fair value is straightforward to establish: that is, the value of the matching zero-coupon bond. No other price could persist in a *liquid market*: there would be an arbitrage opportunity. Certain uncertain liabilities are also simple to value: those where the risks are diversifiable. For example, basic mortality risk is diversifiable and can, in effect be replaced by deterministic mortality rates. However, the pricing of mortality risks is clouded by the existence of parameter and model uncertainty as well as the rate of future mortality improvements (none of which is a diversifiable risk). Many other risks do not offer the opportunity to diversify. This prevents the identification of a perfect matching portfolio and of a corresponding unique price.

The author's work in this field was motivated by the UK debate on the fair valuation of defined-benefit pension plans (see, for example, Head *et al.,* 1999, and Exley, Mehta & Smith, 1997). Such plans contain a mixture of liabilities which do not have perfectly-matching portfolios. For example, certain benefits are linked to salaries. Index-linked bonds provide a reasonable but imperfect match and certainly still leave considerable risk for the sponsoring employer.

As we will see the idea of identifying the portfolio of assets which best matches (albeit imperfectly) a set of liabilities forms an important component in the establishment of a fair value. This means that there is a perhaps-surprising link between a liquid market in risky assets and risk-minimising reserves. The development of such an approach is not new to actuaries. The concept of matching well-defined cashflows is a well established process. More recently Wise (1984a,b, 1987a,b, 1989), Wilkie (1985) and Keel & Müller (1995) have addressed the issue of matching when a perfect match is not possible.

Development of ideas in the fair valuation of other insurance-linked liabilities and their link with financial economics has also been ongoing (see, for example, Reitano, 1997, Babbel & Merrill, 1998, Longley-Cook, 1998, Møller 1998, 2000, Phillips, Cummins & Allen, 1998, Girard, 2000, Lane, 2000, Panjer, 2000, and Wang, 2000).

Techniques for the pricing of liabilities or contingent claims (that is, financial contracts whose payoff is contingent on the value of some other financial or economic index or price) have developed earlier and at a faster pace within the financial-economics literature. Key early developments were made by Black & Scholes (1973), Merton (1973) and Cox, Ross & Rubinstein (1979). They showed how relatively complex contingent claims can be replicated (that is, perfectly matched) using *dynamic* hedging given certain restrictions on the models for security prices. The existence of a replicating, dynamic-hedging strategy then means that we can establish a unique no arbitrage price for the contingent claim (what we would describe in the present paper as the *fair value*).

In reality, almost all financial contracts cannot be replicated: that is, dynamic hedging can reduce risk but not to zero. There are many reasons for this such as the presence of transactions costs, jumps in market prices, illiquidity in the market and the lack of a suitable, liquid asset for hedging a given contingent claim. An immediate consequence of this is that there is no longer a unique no arbitrage price. Instead there is a range of prices (possibly infinite) which are all consistent with no arbitrage.

A number of approaches have been proposed to deal with this problem. These are primarily concerned with the *hedging* of a contingent claim given the aim is to optimise a certain objective function rather than the establishment of a price. Duffie & Richardson (1991) and Schweizer (1992) looked at the application of *mean-variance* hedging as one form of the more general *variance-minimising* hedging approach. This involves minimising the variance of the surplus or deficit at the time of payment of a claim over both the initial value of the hedge portfolio and the *self-financing*, dynamic-hedging strategy. Föllmer &

Sondermann (1986) (see also, for example, Föllmer & Schweizer, 1989) had previously proposed *risk-minimising* strategies. These strategies are very similar to variance-minimising strategies but, rather than being self-financing, the value of the hedge portfolio is continually topped up or reduced to keep it equal to the perceived value of the contingent claim at any point in time.

These papers are principally concerned with the issue of how to hedge once a contract has been entered into. However, where the optimisation includes the initial value of the hedge portfolio there has been considerable discussion of whether this initial value can be used as a candidate for the price at which this contract would trade in the financial markets.

These initial papers focussed on means and variances (equivalently a quadratic utility function) partly for mathematical tractability but partly also for reasons of symmetry. In particular, an investor does not know in advance of the price being set whether or not he will go long or short in a new security. Under such circumstances it would be inappropriate to use an increasing (and concave) utility function.

Other authors have looked at hedging on the basis that an investor has a defined position in a contingent claim at some future time. The issue is then clearly one sided and the investor then aims to invest his portfolio in a way which will maximise his expected terminal utility or shortfall risk. Examples include Karatzas *et al.* (1991), Kramkov & Schachermayer (1999), Föllmer & Leukert (1999, 2000), Schachermayer (2000), Delbaen *et al.* (2000), Cvitanić, Schachermayer & Wang (2000), Schweizer (2001) and Owen (2001). The issue of pricing is related to the problem of utility maximisation. In particular, consider an investor who has a long position in financial contract with some random payoff at a given time in the future. It is proposed that this contract is exchanged immediately for a fixed amount of money P. After the exchange the investor will choose to employ a different dynamic hedging strategy in order to maximise the same terminal utility function. The expected terminal utility might be above or below that which was previously determined. There will be some threshold value for P at which the investor is indifferent to the exchange of the contract for cash. In some sense this represents a possible price for this particular contract. However, other investors with different total assets and utility functions will have different threshold prices P (all of which are consistent with no arbitrage). This means that we are only part of the way to determining at what price this financial contract will trade in the financial markets because we must find an equilibrium between these investors.

Equilibrium modelling is a key aspect of this paper. Thus, rather than consider the microeconomic impact of a security or the market on a single investor, we consider the macroeconomic point of view. In particular, we need to consider how the structure of a market and how the preferences and views of investors affect the prices of all securities in the financial markets. Equilibrium pricing of securities is, in general, a complex problem (and the present author cannot claim to be an expert!). Some general results can be found in Debreu (1982), Duffie (1996) and Starr (1997) regarding (a) the existence of equilibrium in the market and (b) *Pareto efficiency*[1].

With many models the location of a competitive equilibrium can only be found using numerical methods. This makes the problem of estimating the price at which a new security would trade, if a liquid market in such an asset existed, a difficult one to solve. One exception to this rule is the introduction of a new derivative product into a complete market where a price for the marginal impact of the new security is the uniquely-determined no-arbitrage price.

Risk-minimisation hedging as a means to establishing a candidate for the market price of a new security, on the other hand, does often yield concrete solutions (see, for example, Møller, 1998, 2000). It would be useful, therefore, if risk-minimising strategies give a price which turns out to match the price determined by the more-complex equilibrium model. Alternatively, in the event that the two prices do not coincide, how close are the prices: that is, can the variance-minimising price (or some variation on it) be used as an effective approximation?

This question is, of course, not easy to answer. However, what we attempt to do in this paper is to make some progress towards a resolution by applying the question to a very simple market model. The model has a single time period, with normally-distributed returns and investors with exponential utility. Other aspects of the model allow for small, but important, degrees of complexity (investors have different levels of risk aversion, individual, random liabilities and potentially different parameters for the investment-returns model).

For this simple model we find that there are very close links between the equilibrium and the risk-minimising prices. However, the two only coincide under special circumstances. A key issue is whether or not it is possible to determine in advance the *market price of risk*[2] for the additional source of risk (uncorrelated with the original n sources of risk) associated only with the new security. The size of this market price of risk depends upon whether or not the new security can be used to match more effectively investors' personal liabilities. It also depends upon a knowledge of investors' risk preferences and both of these

[1] An allocation of securities and a price structure is said to be *Pareto efficient* if all alternative allocations and price structures result in a strict reduction in the expected utility of at least one investor.

[2] Each uncorrelated source of risk has an associated market price of risk. These, in turn, determine the risk premium on each security.

quantities may be unobservable. Despite the fact that equality of prices will be unlikely, we find from numerical studies that risk-minimising prices do often provide a good approximation to equilibrium prices.

2 A model for the existing market

We start by considering the existing market: that is, the financial market before the introduction of any new securities. Suppose that there are initially n securities available for investment. (It may be assumed that these n securities are corporate stock.) In subsequent sections we will investigate the effect on the prices of existing securities of the introduction of a new security.

The n securities have prices $S_i(t)$ per unit at time t ($t = 0, 1$) for $i = 1, \ldots, n$. Each underlying business will exist for one year only and then will be wound up on the basis of unlimited liability.

Cash is available as an alternative risk-free investment, returning £r at time 1 for each £1 invested at time 0. It is assumed that this is offered by a bank which is distinct from the investors described below. It also assumed that the bank has no effect on the market model other than as an agent willing to borrow or lend unlimited amounts at the risk-free rate.[3]

We assume that there are no restrictions on long or short investments and that cash can be borrowed by all investors, without limit, at the risk-free rate.

In addition there are m investors. Investor k has initial wealth W_{k0} and has a utility function for wealth w at time 1 of $U_k(w) = -\exp(-\alpha_k w)$. In the general model we will assume that different investors can have different views on the distributions of the security values at time 1, $S_1(1), \ldots, S_n(1)$. This may arise because: they use different amounts of historical data; they use different models (e.g. trend chasers versus mean reverters); they use different estimation techniques; they react in different ways to the arrival of new information.

The combination of the n securities, the m investors (and their liabilities as described below) and the bank constitutes a closed market: in particular, investors' decisions and the prices of the securities are not influenced by any external factors.

Let $\hat{S}(t) = (S_1(t), \ldots, S_n(t))'$ be the vector of prices at time t. (The $\hat{\cdot}$ notation indicates a vector containing the first n elements of a potentially longer

[3]An alternative model which we do not investigate here assumes that there is no such bank, but investors can borrow or lend cash from each other at the risk-free rate. This is equivalent to the existence of a bank with net deposits equalling net loans. In both cases it is necessary to establish the risk-free rate of interest at which net deposits equal net loans.

sequence. Aee Appendix A for a full summary of the vector and matrix notation.) Investor k assumes that $\hat{S}(1) = \hat{\mu}_k + C_k Z$ where:

- $\hat{\mu}_k = (\mu_1^{(k)}, \ldots, \mu_n^{(k)})'$;
- the volatility matrix C_k is $n \times (n+m+1)$ (and of rank n) with elements $c_{ij}^{(k)} = 0$ for all $j > i$ (that is, C_k is lower triangular with zeros in the final $m+1$ columns);
- and $Z = (Z_1, \ldots, Z_{n+m+1})'$ with $Z_1, \ldots, Z_{n+m+1}$ i.i.d. standard normal random variables.

All investors, therefore, assume normality in this simple model, but each has different estimates of the means, variances and covariances. This includes $\hat{V}_k = C_k C_k'$, investor k's estimate of the covariance matrix for the $S_i(1)$. Let us now define $C_k^S = (c_{ij}^{(k)})_{i,j=1}^n$ to be the square matrix which takes the first n columns of C_k. Since C_k is lower triangular we have $V_k = C_k^S C_k^{S\prime}$. Furthermore, since C_k is of rank n, C_k^S and V_k are invertible.

For security i the risk premium estimated by investor k is $\rho_i^{(k)} = \mu_i^{(k)}/S_i(0) - r$. If we define $\hat{e} = (1, \ldots, 1)'$ (n elements) and $\hat{\Lambda}_s = \mathrm{diag}(\hat{S}(0))$ as the diagonal matrix generated by the vector $\hat{S}(0)$, we can write $\hat{\rho}_k = (\rho_1^{(k)}, \ldots, \rho_n^{(k)})' = \hat{\Lambda}_s^{-1}\hat{\mu}_k - r\hat{e}$.

At time 1 investor k has wealth:

$$\begin{aligned} W_{k1} &= W_{k0}\left[r + \hat{p}_k'\left(\hat{\Lambda}_s^{-1}(\hat{\mu}_k + C_k Z) - r\right)\right] + \gamma_k + \delta_k' Z \\ &= W_{k0}\left[r + \hat{p}_k'\hat{\rho}_k + \hat{p}_k'\hat{\Lambda}_s^{-1} C_k Z\right] + \gamma_k + \delta_k' Z \qquad (2.1) \end{aligned}$$

where:

- $\hat{p}_k = (p_1^{(k)}, \ldots, p_n^{(k)})'$ represents the proportions at time 0 invested in each security by investor k;
- γ_k is a scalar;
- and $\delta_k = (\delta_1^{(k)}, \ldots, \delta_{n+m+1}^{(k)})'$.

Part of the equation for the investor's wealth at time 1, $-(\gamma_k + \delta_k' Z)$, represents investor k's personal liabilities at time 1.

The expected utility for investor k based upon his subjective assessment of the returns distributions is then:

$$\begin{aligned} E[U_k(W_{k1})] &= -\exp\Bigg[-\alpha_k \left\{ W_{k0}(r + \hat{p}_k'(\hat{\Lambda}_s^{-1}\hat{\mu}_k - r\hat{e})) + \gamma_k \right\} \\ &\qquad + \frac{1}{2}\alpha_k^2 \left(W_{k0}\hat{p}_k'\hat{\Lambda}_s^{-1}C_k + \delta_k' \right) \left(W_{k0}C_k'\hat{\Lambda}_s^{-1}\hat{p}_k + \delta_k \right) \Bigg] \end{aligned} \tag{2.2}$$

We wish to maximise the expected utility over $\hat{p}_k$. Thus we differentiate:

$$\begin{aligned} & \frac{d}{d\hat{p}_k} \log\left(-E[U_k(W_{k1})]\right) \\ = \; & -\alpha_k W_{k0}(\hat{\Lambda}_s^{-1}\hat{\mu}_k - r\hat{e}) + \alpha_k^2 \left(W_{k0}^2 \hat{\Lambda}_s^{-1} V_k \hat{\Lambda}_s^{-1}\hat{p}_k + W_{k0}\hat{\Lambda}_s^{-1}C_k\delta_k \right) \\ = \; & 0 \end{aligned}$$

Hence we have:

$$\hat{p}_k = \frac{1}{\alpha_k W_{k0}}\hat{\Lambda}_s V_k^{-1}(\hat{\mu}_k - r\hat{S}(0)) - \frac{1}{W_{k0}}\hat{\Lambda}_s V_k^{-1} C_k \delta_k \tag{2.3}$$

Let $\hat{u}_i^{(k)}$ be the optimal number of units held in security i by investor k. Then:

$$u_i^{(k)} = \frac{p_{ki}W_{k0}}{S_i(0)} \tag{2.4}$$

$$\text{Hence } \; \hat{u}^{(k)} = (u_1^{(k)}, \ldots, u_n^{(k)})' \tag{2.5}$$

$$= W_{k0}\hat{\Lambda}_s^{-1}\hat{p}_k \tag{2.6}$$

$$= V_k^{-1}\left[\frac{1}{\alpha_k}\hat{\mu}_k - C_k\delta_k - r\frac{1}{\alpha_k}\hat{S}(0) \right] \tag{2.7}$$

Note that the component $V_k^{-1}C_k\delta_k$ gives the numbers of units for investor k to match most closely the risky component of his liability of $(-\delta_k' Z)$ using the existing n securities. This means that the liability risks associated with the n sources or risk generating security returns $(Z_1, \ldots, Z_n)$ are *eliminated* first before the investor then considers the construction of a risky portfolio to maximise his expected utility. A key point here is that all investors act in the same way with regard to their personal liability risks, rather than act in a way which takes account of their level of risk aversion.

Now let u_i be the total number of units of security i which are required. Then:

$$\begin{aligned}
\hat{u} &= (u_1, \ldots, u_n)' \\
&= \sum_{k=1}^{m} \hat{u}^{(k)} \\
&= \sum_{k=1}^{m} V_k^{-1} \left[\frac{1}{\alpha_k} \hat{\mu}_k - C_k \delta_k - r \frac{1}{\alpha_k} \hat{S}(0) \right] \\
&= \sum_{k=1}^{m} \frac{1}{\alpha_k} V_k^{-1} \left(\hat{\mu}_k - \alpha_k C_k \delta_k \right) - r \left(\sum_{k=1}^{m} \frac{1}{\alpha_k} V_k^{-1} \right) \hat{S}(0) \qquad (2.8)
\end{aligned}$$

Now the number of units of each security in issue is specified so it is left to us to derive the set of prices which will allow the market to clear. Hence we find:

$$\hat{S}(0) = \frac{1}{r} \left(\sum_{k=1}^{m} \frac{1}{\alpha_k} V_k^{-1} \right)^{-1} \left[\sum_{k=1}^{m} \frac{1}{\alpha_k} V_k^{-1} \hat{\mu}_k - \sum_{k=1}^{m} V_k^{-1} C_k \delta_k - \hat{u} \right] \qquad (2.9)$$

which, we note, is linear in $\hat{u}$. Note also that the lower-triangular form of C_k implies that $V_k^{-1} C_k \delta_k = C_k^{S'^{-1}} \hat{\delta}_k$ where $\hat{\delta}_k = (\delta_1^{(k)}, \ldots, \delta_n^{(k)})'$.

Remark 2.1

Note that these prices (equation (2.9)) do not depend on the initial wealths, W_{k0}. This because the exponential utility function has constant absolute risk aversion.

Remark 2.2

In the special case that all investors agree on the parametrisation of the returns model (that is, $\hat{\mu}_k = \hat{\mu}$ and $C_k = C$ for all k) we have:

$$\begin{aligned}
\hat{u} &= \frac{1}{\alpha} V^{-1} (\hat{\mu} - r\hat{S}(0)) - V^{-1} C \delta \\
\text{where } \frac{1}{\alpha} &= \sum_{k=1}^{m} \frac{1}{\alpha_k} \\
\text{and } \delta &= \sum_{k=1}^{m} \delta_k \\
\Rightarrow \hat{S}(0) &= \frac{1}{r} \left[\hat{\mu} - \alpha V \hat{u} - \alpha C \delta \right] \qquad (2.10)
\end{aligned}$$

Remark 2.3

Assume again that all investors agree on the parameter values and further that their liabilities are non-random (that is, $\delta_k = 0$ for all k).

Optimal portfolios are made up of cash and one efficient portfolio in which the mix of assets is proportional to $V^{-1}\hat{\rho}$, where $\hat{\rho} = \hat{\Lambda}_s^{-1}\hat{\mu} - r\hat{e}$. This defines the capital market line in traditional mean-variance portfolio theory. Since all investors have the same view of the world this risky portfolio can be interpreted as the market portfolio. In other words we are looking at the Capital Asset Pricing Model.

3 Issue of a new security

3.1 New equilibrium

Now suppose an agency issues at time 0 a new security ($i = n+1$) paying $S_{n+1}(1)$ at time 1 per unit. The introduction of this new asset might result in an immediate change in the prices of the original n securities. Let $\tilde{S}_i(0)$, for $i = 1, \ldots, n+1$, be the new prices at time 0. Then $\hat{\tilde{S}}(0) = (\tilde{S}_1(0), \ldots, \tilde{S}_n(0))'$ is the revised vector of prices of the original n securities and $\bar{\tilde{S}}(0) = (\tilde{S}_1(0), \ldots, \tilde{S}_n(0), \tilde{S}_{n+1}(0))'$ is its extension to include the new security.

Suppose the issuing agency is an existing component of the market model (one of the existing companies or one of the investors). Then the issuer takes the opposite position to each investor in the new security. This means that the net quantity of new security issued is zero. It follows that if the net quantity to be issued is *non*-zero then the security must come from an external agency (for example, the government or an unquoted company) which we assume has no other interactions with the existing agencies in the model.

At time 1 we now have the extended price vector (in the view of investor k):

$$
\begin{aligned}
\bar{S}(1) &= \bar{\mu}_k + \bar{C}_k Z \\
\text{where } \bar{\mu}_k &= (\mu_1^{(k)}, \ldots, \mu_{n+1}^{(k)})' \\
\bar{C}_k &= \begin{pmatrix} C_k \\ \\ -- \\ \sigma_k' \end{pmatrix} \\
\sigma_k &= (\sigma_1^{(k)}, \ldots, \sigma_n^{(k)}, \sigma_{n+1}^{(k)}, 0, \ldots, 0)'.
\end{aligned}
$$

It follows that the extended covariance matrix for $\bar{S}(1)$ is:

$$\bar{V}_k = \left(\begin{array}{c|c} V_k & C_k\sigma_k \\ \hline \sigma_k' C_k' & \sigma_k'\sigma_k \end{array} \right).$$

Suppose that the total number of units, u_i, in each of the existing securities, $i = 1, \ldots, n$, is unchanged. It is intended that u_{n+1} units of the new security are to be issued at time 0. The prices of securities 1 to n plus the price of the new security must be set at a level which will ensure that the market clears: that is, the demand for each security exactly meets the availability, u_i, of each security, $i = 1, \ldots, n+1$.

Since prices can change immediately at time 0 in reaction to the announcement that u_{n+1} units of a new security will be issued, investor k has the revised wealth:

$$\tilde{W}_{k0} = W_{k0}\left[1 - \hat{p}_k'\hat{e} + \hat{p}_k'\hat{\Lambda}_s^{-1}\hat{\tilde{S}}(0)\right] \tag{3.1}$$

Following the same argument as before (equations (2.8) and (2.9)) we find that:

$$\bar{\tilde{u}} = \sum_{k=1}^{m} \bar{\tilde{u}}^{(k)} \tag{3.2}$$

$$\text{where } \bar{\tilde{u}}^{(k)} = \bar{V}_k^{-1}\left[\frac{1}{\alpha_k}\left(\bar{\mu}_k - r\bar{\tilde{S}}(0)\right) - \bar{C}_k\delta_k\right] \tag{3.3}$$

$$\Rightarrow \bar{\tilde{u}} = \sum_{k=1}^{m} \bar{V}_k^{-1}\left[\frac{1}{\alpha_k}\left(\bar{\mu}_k - r\bar{\tilde{S}}(0)\right) - \bar{C}_k\delta_k\right] \tag{3.4}$$

$$\text{and } \bar{\tilde{S}}(0) = \frac{1}{r}\left(\sum_{k=1}^{m}\frac{1}{\alpha_k}\bar{V}_k^{-1}\right)^{-1}\left[\sum_{k=1}^{m}\frac{1}{\alpha_k}\bar{V}_k^{-1}\bar{\mu}_k - \sum_{k=1}^{m}\bar{V}_k^{-1}\bar{C}_k\delta_k - \bar{\tilde{u}}\right] \tag{3.5}$$

where $\bar{\tilde{u}} = (\tilde{u}_1, \ldots, \tilde{u}_{n+1})'$.

Now:

$$V_k\hat{u}^{(k)} = \frac{1}{\alpha_k}\hat{\mu}_k - \frac{r}{\alpha_k}\hat{S}(0) - C_k\delta_k$$

from equation (2.7). Hence:

$$\bar{\tilde{u}}^{(k)} = \bar{V}_k^{-1}\left(\begin{array}{c} V_k\hat{u}^{(k)} + \frac{r}{\alpha_k}\left(\hat{S}(0) - \hat{\tilde{S}}(0)\right) \\ \hline \frac{1}{\alpha_k}\mu_{n+1}^{(k)} - \frac{r}{\alpha_k}\tilde{S}_{n+1}(0) - \sigma_k'\delta_k \end{array} \right). \tag{3.6}$$

Now for each k there exists a vector $\bar{b}_k$ for which:

$$\bar{V}_k^{-1} = \left(\begin{array}{c|c} V_k^{-1} & 0 \\ \hline 0 & 0 \end{array} \right) + \bar{b}_k \bar{b}_k'$$

(see Appendix B).

We therefore have:

$$\bar{\tilde{u}}^{(k)} = \left(\begin{array}{c} \hat{u}^{(k)} + \frac{r}{\alpha_k} V_k^{-1} \left(\hat{S}(0) - \hat{\tilde{S}}(0) \right) \\ \hline 0 \end{array} \right)$$
$$+ \ \bar{b}_k \left\{ \hat{b}_k' V_k \hat{u}^{(k)} + \frac{r}{\alpha_k} \hat{b}_k' \left(\hat{S}(0) - \hat{\tilde{S}}(0) \right) + b_{n+1}^{(k)} \left(\frac{1}{\alpha_k} \mu_{n+1}^{(k)} - \frac{r}{\alpha_k} \tilde{S}_{n+1}(0) - \sigma_k' \delta_k \right) \right\} \tag{3.7}$$

where the $n \times 1$ vector $\hat{b}_k$ contains the first n elements of $\bar{b}_k$.

Finally:

$$\begin{aligned} \bar{\tilde{u}} = \sum_{k=1}^{m} \bar{\tilde{u}}^{(k)} \ &= \left(\begin{array}{c} \hat{u} \\ \hline 0 \end{array} \right) + r \left(\begin{array}{c} \left(\sum_{k=1}^{m} \frac{1}{\alpha_k} V_k^{-1} \right) \left(\hat{S}(0) - \hat{\tilde{S}}(0) \right) \\ \hline 0 \end{array} \right) \\ &+ \sum_{k=1}^{m} \bar{b}_k \left\{ \hat{b}_k' V_k \hat{u}^{(k)} + \frac{b_{n+1}^{(k)} \mu_{n+1}^{(k)}}{\alpha_k} - b_{n+1}^{(k)} \sigma_k' \delta_k \right\} \\ &+ \left(\sum_{k=1}^{m} \frac{r}{\alpha_k} \bar{b}_k \hat{b}_k' \right) \left(\hat{S}(0) - \hat{\tilde{S}}(0) \right) \\ &- \left(\sum_{k=1}^{m} \frac{r b_{n+1}^{(k)}}{\alpha_k} \bar{b}_k \right) \tilde{S}_{n+1}(0). \end{aligned} \tag{3.8}$$

The next step is to specify $\bar{\tilde{u}}$. We will assume there are no new units in securities 1 to n so $\hat{\tilde{u}} = \hat{u}$ while $\tilde{u}_{n+1}$ may or may not be zero. An important question to ask is what is the impact (if any) of the mere existence of the new security (that is, taking $\tilde{u}_{n+1} = 0$)? In particular, can we find a price, $\tilde{S}_{n+1}(0)$, for the new security which gives rise to an equilibrium where there is no impact on the prices of the existing securities (thst is, where $\tilde{u}_i = u_i$ for $i = 1, \ldots, n$, $u_{n+1} = 0$ and $\tilde{S}_i(0) = S_i(0)$ for $i = 1, \ldots, n$)? If such an equilibrium exists then the introduction of the new security at $\tilde{S}_{n+1}(0)$ will have no impact on market prices.

It follows that we must attempt to solve equation (3.8) subject to the given constraints. Thus take $\hat{\tilde{S}}(0) = \hat{S}(0)$ and $\hat{\tilde{u}} = \hat{u}$. The desired equilibrium exists if and only if:

$$\begin{aligned}\begin{pmatrix} \hat{u} \\ -- \\ 0 \end{pmatrix} = \bar{\tilde{u}} &= \begin{pmatrix} \hat{u} \\ -- \\ 0 \end{pmatrix} + \sum_{k=1}^{m} \bar{b}_k \left\{ \hat{b}_k' V_k \hat{u}^{(k)} + \frac{b_{n+1}^{(k)} \mu_{n+1}^{(k)}}{\alpha_k} - b_{n+1}^{(k)} \sigma_k' \delta_k \right\} \\ &\quad - \left(\sum_{k=1}^{m} \frac{r b_{n+1}^{(k)}}{\alpha_k} \bar{b}_k \right) \tilde{S}_{n+1}(0) \end{aligned}$$

$$\Leftrightarrow \left(\sum_{k=1}^{m} \frac{r b_{n+1}^{(k)}}{\alpha_k} \bar{b}_k \right) \tilde{S}_{n+1}(0) = \sum_{k=1}^{m} \bar{b}_k \left\{ \hat{b}_k' V_k \hat{u}^{(k)} + \frac{b_{n+1}^{(k)} \mu_{n+1}^{(k)}}{\alpha_k} - b_{n+1}^{(k)} \sigma_k' \delta_k \right\} \tag{3.9}$$

Now equation (3.9) consists of $n + 1$ linear equations but only one unknown, $\tilde{S}_{n+1}(0)$. In general, therefore, there is no solution and no equilibrium with $\tilde{u}_i = u_i$ and $\tilde{S}_i(0) = S_i(0)$ for $i = 1, \ldots, n$ and $\tilde{u}_{n+1} = 0$. On the other hand, equation (3.8) is linear in $\bar{\tilde{S}}(0)$ and of full rank, so an equilibrium set of prices $\bar{\tilde{S}}(0)$ can be found.

Remark 3.1

Suppose that $\hat{\tilde{u}} = \hat{u}$ and $\tilde{u}_{n+1} = 0$.

In general the new equilibrium prices $\hat{\tilde{S}}(0)$ for the existing securities are not equal to their prices $\hat{S}(0)$ before the introduction of the new security.

Thus, in a market where investors use different model parameters for investment returns, the mere existence of the new security, even though zero net units are issued, has an impact on all market prices. One might argue that this is not surprising. If the δ_k (the investors' liability risks) are non-zero then, although $\tilde{u}_{n+1} = 0$ some investors will have long and others short positions in the new security. This depends upon how the well the new security can be used to reduce each investor's liability risks. This trading might then affect other prices. However, the following corollary suggests that this is not the case.

Corollary 3.2

Suppose that all investors agree on the returns model parameters (that is, $\bar{\mu}_k = \bar{\mu}$ and $\bar{C}_k = \bar{C}$ for all k).

(a) The equilibrium with $\hat{\tilde{u}} = \hat{u}$ and $\tilde{u}_{n+1} = 0$ exists when:

$$\hat{\tilde{S}}(0) = \hat{S}(0) \qquad \text{and} \qquad \tilde{S}_{n+1}(0) = \frac{1}{r}\left(\mu_{n+1} - \alpha\sigma'\delta - \alpha\sigma' C'\hat{u}\right).$$

where α and δ are defined in equation (2.10).

(b) The equilibrium with $\hat{\tilde{u}} = \hat{u}$ and $\tilde{u}_{n+1} \neq 0$ exists when:

$$\begin{aligned}
\bar{\tilde{S}}(0) &= \frac{1}{r}\left[\bar{\mu} - \alpha\bar{V}\bar{u} - \alpha\bar{C}\delta\right] && (3.10)\\
\text{that is:}\ \hat{\tilde{S}}(0) &= \frac{1}{r}\left(\hat{\mu} - \alpha V\hat{u} - u_{n+1}\alpha C\sigma - \alpha C\delta\right) \\
&= \hat{S}(0) - \frac{1}{r}u_{n+1}\alpha C\sigma && (3.11)\\
\text{while}\ \tilde{S}_{n+1}(0) &= \frac{1}{r}\left(\mu_{n+1} - \alpha\sigma' C'\hat{u} - \alpha\sigma'\delta - \alpha u_{n+1}\sigma'\sigma\right). && (3.12)
\end{aligned}$$

Proof:

(a) Since $\bar{C}_k = \bar{C}$ for all k we have $\bar{b}_k = \bar{b}$ for some $\bar{b}$ for all k. Equation (3.9) then holds if and only if:

$$\begin{aligned}
\frac{rb_{n+1}}{\alpha}\bar{b}\tilde{S}_{n+1}(0) &= \left(\hat{b}'V\hat{u} + \frac{b_{n+1}}{\alpha} - b_{n+1}\sigma'\delta\right)\bar{b} \\
\Leftrightarrow \tilde{S}_{n+1}(0) &= \frac{1}{r}\left(\mu_{n+1} - \alpha\sigma'\delta + \frac{\alpha}{b_{n+1}}\hat{b}'V\hat{u}\right). && (3.13)
\end{aligned}$$

But $b_{n+1} = \sigma_{n+1}^{-1}$ and $\hat{b} = -\sigma_{n+1}^{-1}(C^S)'^{-1}\hat{\sigma}$ (Appendix B) so (3.13) holds if and only if:

$$\begin{aligned}
\tilde{S}_{n+1}(0) &= \frac{1}{r}\left(\mu_{n+1} - \alpha\sigma'\delta - \alpha\hat{\sigma}'(C^S)^{-1}V\hat{u}\right) \\
&= \frac{1}{r}\left(\mu_{n+1} - \alpha\sigma'\delta - \alpha\hat{\sigma}'(C^S)'\hat{u}\right) \\
&= \frac{1}{r}\left(\mu_{n+1} - \alpha\sigma'\delta - \alpha\sigma' C'\hat{u}\right) && (3.14)
\end{aligned}$$

since $\sigma_j = 0$ for $j = n+2, \ldots, n+m+1$.

The proof of (b) follows in a similar fashion.

Remark 3.3

It follows that the prices of securities 1 to n remain unaffected by the introduction of the new security if (and usually only if) both $\tilde{u}_{n+1} = 0$ and investors agree on the parameter values in the returns model.

If we consider the case $\delta_k = 0$ for all k then we can see that this result is not surprising. All investors will still hold a mixture of cash and the same market portfolio meaning that all investors will have positive holdings in the new security, all negative or all zero. Since the net amount issued is zero, no investor will have a long or a short holding. This leaves us to optimise over the original securities which we have already done.

Remark 3.4

Suppose $\tilde{u}_{n+1} = 0$ and that investors agree on the parameters. Even if the δ_k are non-zero we find that the prices of the existing securities remain unchanged.

This observation clarifies the comments following Remark 2.1. Thus, even though there will be some trading in the new security when $\tilde{u}_{n+1} = 0$, this will *not* have an effect on the prices of the existing securities.

This is, perhaps, a counterintuitive result. However, we can make two further observations. First, although prices are not affected, the expected utilities of the investors will increase as a result of the introduction of the new security. Second, the lack of an effect on prices is related to the Modigliani-Miller theorem (see, for example, Bodie & Merton, 1998).

Remark 3.5

Suppose $\tilde{u}_{n+1} = 0$ and that investors agree on the parameters. There is no allowance in the price of the new security (equation (3.12)) for its volatility, σ, except:
(i) where it is matched by volatility in the value of existing securities (the $\alpha\sigma' C' \hat{u}$ term); and by
(ii) the volatility in the investors' liabilities (the $\alpha\sigma'\delta$ term).

In particular, if $\delta_{n+1} = 0$, the price of the new security does not take into account the σ_{n+1} term. If, on the other hand, $\delta_{n+1} \neq 0$ then the price of the new security will be affected by the size of σ_{n+1}.

The main effect of σ_{n+1} occurs when $u_{n+1} > 0$: the larger u_{n+1} is the lower the price at which the new security must be issued. Furthermore, the prices of the existing securities will be affected by the size of the new issue.

3.2 Risk minimisation

Risk minimisation entails establishing the portfolio of existing securities which matches most closely the payoff on the new security. It is then proposed that the value at time zero of this matching portfolio be a candidate for the market price of the new security (that is, our best estimate of the price at which the new security would trade).

We will concentrate first on the special case where investors all agree on the returns model: that is, $\bar{\mu}_k = \bar{\mu}$ and $\bar{C}_k = \bar{C}$ for all k.

Let P represent the subjective probability measure used by each investor for this model and let Q be an equivalent risk-neutral probability measure. Under Q we have $E_Q[S_j(1)] = rS_j(0)$ for $j = 1, \ldots, n$ and for $j = n+1$ once the price of the new security has been established.

Let x_0 be the amount of money we invest in cash and x_i be the number of units of security i ($i = 1, \ldots, n$) we purchase at time 0 (with $x = (x_1, \ldots, x_n)'$). The portfolio, $(x_0, \ldots, x_n)$, which provides the best match is defined as the solution to the following optimisation problem:

$$\text{Minimise } \; E_P\left[\left(S_{n+1}(1) - x_0 r - \sum_{i=1}^{n} x_i S_i(1)\right)^2\right]$$

where E_P represents expectation under the measure P. This is equivalent to minimisation first of the variance (under P) of $S_{n+1}(1) - x'S(1)$ and then choosing x_0 so that $E_P\left[S_{n+1}(1) - x_0 r - x'\hat{S}(1)\right] = 0$.

Now:

$$\begin{aligned}
S_{n+1}(1) - x'\hat{S}(1) &= \mu_{n+1} - x'\hat{\mu} + \sigma' Z - x'CZ \\
\Rightarrow Var_P\left[S_{n+1}(1) - x'\hat{S}(1)\right] &= (\sigma' - x'C)(\sigma' - x'C)' \\
&= \sigma'\sigma - 2x'C\sigma + x'Vx \\
\Rightarrow \frac{d}{dx} Var\left[S_{n+1}(1) - x'\hat{S}(1)\right] &= 2Vx - 2C\sigma = 0 \\
\Rightarrow x &= V^{-1}C\sigma \qquad (3.15) \\
E_P\left[S_{n+1}(1) - x'\hat{S}(1)\right] &= 0 \\
\Rightarrow x_0 &= \frac{1}{r}(\mu_{n+1} - x'\hat{\mu}) \\
&= \frac{1}{r}\left(\mu_{n+1} - \sigma C' V^{-1}\hat{\mu}\right) \qquad (3.16)
\end{aligned}$$

The value at time 0 of this best matching portfolio, and our candidate for the price of the new security, is thus:

$$
\begin{aligned}
\pi &= x_0 + x'\hat{\tilde{S}}(0) \\
&= \frac{1}{r}\left(\mu_{n+1} - \sigma' C' V^{-1}\hat{\mu}\right) + \sigma' C' V^{-1}\left(\hat{S}(0) - \frac{1}{r}\alpha u_{n+1} C\sigma\right) \\
&\qquad \text{(by equation (3.11))} \\
&= \frac{1}{r}\left(\mu_{n+1} - \sigma' C' V^{-1}\hat{\mu}\right) + \frac{1}{r}\sigma' C' V^{-1}\left(\hat{\mu} - \alpha V\hat{u} - \alpha C\delta - \alpha u_{n+1} C\sigma\right) \\
&\qquad \text{(by equation (2.10))} \\
&= \frac{1}{r}\left(\mu_{n+1} - \alpha\sigma' C'\hat{u} - \alpha\hat{\sigma}'\hat{\delta} - \alpha u_{n+1}\hat{\sigma}'\hat{\sigma}\right) \qquad (3.17)
\end{aligned}
$$

since the form of C (rank n and only the first n columns are non-zero) ensures that:

$$
C'V^{-1}C = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & 0 \end{array}\right).
$$

Equation (3.17) gives us the candidate for the price of the new security. This should be compared with the equilibrium price in equation (3.12). In both cases the price has been calculated taking into account the effect (under the equilibrium model) of the introduction of the new security on the prices of the new securities. We aim to see then if the price of the new security under the equilibrium model is consistent with the risk minimisation approach. We can see that equations (3.12) and (3.17) almost coincide. In particular, we have:

$$
\pi - \tilde{S}_{n+1}(0) = \alpha\sigma_{n+1}(\delta_{n+1} + \tilde{u}_{n+1}\sigma_{n+1}).
$$

The question then arises: can this gap be closed?

First we should note that if $\sigma_{n+1} = 0$ the two prices coincide (we have a complete market). Furthermore, if $\sigma_{n+1} = 0$ it is straightforward to show that we will get the same price, π, if risk is calculated relative to the risk-neutral measure Q (described below) rather than the subjective measure P (because the same portfolio (x_0, x) results in *zero* risk under both P and Q). Suppose, instead, that $\sigma_{n+1} \neq 0$. Is there a risk-neutral measure Q under which the risk-minimising price equals the equilibrium price?

Under P we have:

$$
\bar{S}(1) = \bar{\mu} + \bar{C}Z = \bar{\mu} + \bar{C}^S\bar{Z}.
$$

For an arbitrary vector $\bar{\lambda}$ we can write this as:

$$
\begin{aligned}
\bar{S}(1) &= \bar{\mu} - \bar{C}^S\bar{\lambda} + \bar{C}^S(\bar{Z} + \bar{\lambda}) \\
&= \bar{\mu} - \bar{C}^S\bar{\lambda} + \bar{C}^S\dot{\bar{Z}} \\
\text{where } \dot{\bar{Z}} &= \bar{Z} + \bar{\lambda}.
\end{aligned}
$$

Let Q be the measure equavalent to P under which the $\dot{Z}_i$ are i.i.d. $\sim N(0,1)$. In particular, choose $\lambda_1, \ldots, \lambda_n$ such that, for $i = 1, \ldots, n$:

$$
\begin{aligned}
E_Q[S_i(1)] &= \mu_i - \sum_{j=1}^{n+1} c_{ij}\lambda_j \\
&= r\tilde{S}_i(0) \\
\Rightarrow \hat{\mu} - \hat{C}^S\hat{\lambda} &= r\hat{\tilde{S}}(0) \\
\Rightarrow \hat{\lambda} &= (C^S)^{-1}(\hat{\mu} - r\hat{\tilde{S}}(0)) \\
&= (C^S)^{-1}\left(\hat{\mu} - (\hat{\mu} - \alpha V\hat{u} - \alpha C\delta - \alpha u_{n+1}C\sigma)\right) \\
&= \alpha\left((C^S)'\hat{u} + \hat{\delta} + \hat{\sigma}u_{n+1}\right). \qquad (3.18)
\end{aligned}
$$

Since $E_Q[S_i(1)] = r\tilde{S}_i(0)$ for $i = 1, \ldots, n$, it follows that Q is a risk-neutral measure (with respect to the existing securities). We call λ_i the *market price of risk* with respect to risk i. It can be seen that the market prices of risk depend upon:

- investors' risk preferences (more risk averse implies higher absolute market price of risk);
- uncertainty in individual security returns (more risky implies higher absolute market price of risk);
- liability risks (the more useful an asset is for hedging investors' liabilities the lower is the absolute market price of risk);
- the amount of new security to be issued (more units to clear in the market pushes down prices implying higher absolute market prices of risk).

The concept of the market price of risk is somewhat abstract. A related and well understood quantity is the risk premium on each security: that is, the excess expected return over the risk-free rate. In the present case this is:

$$\hat{\rho} = \hat{\Lambda}_S^{-1}\left(\hat{\mu} - r\hat{S}(0)\right) = \hat{\Lambda}_S^{-1}C^S\hat{\lambda} \qquad (3.19)$$

so there is an explicit link between risk premia and the market prices of risk. In particular, the market price of risk represents the excess expected return per unit of risk for each source of risk $Z_1, Z_2, \ldots$.

For the moment let λ_{n+1} be a general market price of risk for risk $n+1$. Let us revisit the risk-minimisation problem, but this time measure risk relative to the risk-neutral measure Q:

$$\text{Minimise } E_Q\left[\left(S_{n+1}(1) - x_0 r - \sum_{i=1}^{n} x_i S_i(1)\right)^2\right]$$

where E_Q represents expectation under the measure Q. This is equivalent to minimisation first of the variance under Q of $S_{n+1}(1) - x'S(1)$ and then choosing x_0 so that $E_Q\left[S_{n+1}(1) - x_0 r - x'\hat{S}(1)\right] = 0$.

Now:

$$\begin{aligned} S_{n+1}(1) - x'\hat{S}(1) &= \mu_{n+1} + \bar{\sigma}'\bar{Z} - x'\left(\hat{\mu} + C^S\hat{Z}\right) \\ &= \mu_{n+1} - \bar{\sigma}'\bar{\lambda} + \bar{\sigma}'\bar{\tilde{Z}} - x'\left(r\hat{\tilde{S}}(0) + C^S\hat{\tilde{Z}}\right) \qquad (3.20) \end{aligned}$$

$$\begin{aligned} \Rightarrow Var_Q\left[S_{n+1}(1) - x'\hat{S}(1)\right] &= (\sigma' - x'C)(\sigma' - x'C)' \\ \Rightarrow \frac{d}{dx}Var\left[S_{n+1}(1) - x'\hat{S}(1)\right] &= 2Vx - 2C\sigma = 0 \\ \Rightarrow x &= V^{-1}C\sigma \quad \text{as before} \qquad (3.21) \\ E_Q\left[S_{n+1}(1) - x'\hat{S}(1)\right] &= 0 \\ \Rightarrow x_0 &= \frac{1}{r}\left(\mu_{n+1} - \bar{\sigma}'\bar{\lambda} - x'(\hat{\mu} - C^S\hat{\lambda})\right) \\ &= \frac{1}{r}\left(\mu_{n+1} - \sigma'C'V^{-1}\hat{\mu} - \sigma_{n+1}\lambda_{n+1}\right) \qquad (3.22) \end{aligned}$$

$$\begin{aligned} \text{Hence } \pi &= x_0 + x'\hat{\tilde{S}}(0) \\ &= \frac{1}{r}\left(\mu_{n+1} - \sigma'C'V^{-1}\hat{\mu} - \sigma_{n+1}\lambda_{n+1}\right) \\ &\qquad + \sigma'C'V^{-1}\left(\hat{\mu} - \alpha V\hat{u} - \alpha C\delta - \alpha u_{n+1}C\sigma\right) \\ &= \frac{1}{r}\left[\mu_{n+1} - \sigma_{n+1}\lambda_{n+1} - \alpha\hat{\sigma}'(C^S)'\hat{u} - \alpha\hat{\sigma}'\hat{\delta} - \alpha u_{n+1}\hat{\sigma}'\hat{\sigma}\right]. \qquad (3.23) \end{aligned}$$

In order for equation (3.23) to match the equilibrium price (see equations (3.11) and (3.12)) we therefore require:

$$\lambda_{n+1} = \alpha(\delta_{n+1} + u_{n+1}\sigma_{n+1}). \qquad (3.24)$$

It follows that the equilibrium and risk-minimising prices do not coincide unless one of the following holds:

1. Minimise under P and $\alpha = 0$.

   An uninteresting case!

2. Minimise under P and $\delta_{n+1} = \sigma_{n+1} = 0$.

   Thus the new security does not help with liability hedging and it can be hedged perfectly using existing securities. Again this is a rather uninteresting case.

3. Minimise under P and $\delta_{n+1} = u_{n+1} = 0$.

   As u_{n+1} increases, prices must fall in order for the market to clear. This result tells us that if $u_{n+1} = 0$ (and $\delta_{n+1} = 0$) then no risk premium is required other than that which derives from consistency with existing securities.

4. Minimise under Q with:

$$\begin{aligned} \hat{\lambda} &= \alpha\left((C^S)'\hat{u} + \hat{\delta} + \hat{\sigma}u_{n+1}\right) \\ \text{and } \lambda_{n+1} &= \alpha\left(\delta_{n+1} + u_{n+1}\sigma_{n+1}\right) \end{aligned}$$

Case 4 offers the best solution. However, in practice it will be difficult for investors to identify both the total liability risk δ_{n+1} and the aggregate risk-aversion parameter α.

3.3 Different investor viewpoints

We conclude this section with a brief treatment of risk minimisation when investors have different points of view.

Consider investor k. From his point of view the risk minimising portfolio under his subjective probability measure P_k takes:

$$x^{(k)} = V_k^{-1}C_k\sigma_k \tag{3.25}$$

$$x^{(k)} = \frac{1}{r}\left(\mu_{n+1}^{(k)} - \sigma_k C_k' V_k^{-1}\hat{\mu}_k\right) \tag{3.26}$$

$$\Rightarrow \pi^{(k)} = x_0^{(k)} + x^{(k)\prime}\hat{S}(0) \tag{3.27}$$

$$\text{and } \tilde{\pi}^{(k)} = x_0^{(k)} + x^{(k)\prime}\tilde{\hat{S}}(0) \tag{3.28}$$

where $\pi^{(k)}$ and $\tilde{\pi}^{(k)}$ represent investor k's risk-minimising estimates of the price of the new security using the prices of the existing securities before and after respectively. Investors may find it difficult to estimate the new prices $\tilde{\hat{S}}(0)$ so we will concentrate in later sections of this paper on $\pi^{(k)}$ rather than $\tilde{\pi}^{(k)}$. (In any event $\hat{\tilde{S}}(0)$ turns out to be quite close, in general, to $\hat{S}(0)$.)

These prices are subject to all of the causes described in Section 3.2. However, the differences in individual parameter estimates causes further strains. In particular, different investors will come up with different estimates of the price of the new security. We can see (and this is verified in Section 5) that differences between investors' estimates for μ_{n+1} can potentially cause significant differences between the $\pi^{(k)}$.

4 Company holdings in the new security

4.1 Companies as additional investors

In the previous development we assumed that the new security would be held only by the investors. Suppose that the companies are allowed to borrow at the risk-free rate to invest in the new security, with company i holding $v_i u_i$ units. Then $S_i(1)$ becomes $\dot{S}_i(1) = S_i(1) + v_i(S_{n+1}(1) - r\tilde{S}_{n+1}(0))$. If investor k invests proportion $\dot{p}_{ki}$ in asset i (for $i = 1, \ldots, n+1$) his wealth at time 1 is:

$$\begin{aligned}
\tilde{W}_{k,1} &= \tilde{W}_{k0}\left[r + \sum_{k=1}^{n} \dot{p}_{ki}\left(\frac{\hat{S}_i(1)}{\dot{S}_i(0)} - r\right) + \dot{p}_{k,n+1}\left(\frac{S_{n+1}(1)}{\tilde{S}_{n+1}(0)} - r\right)\right] \\
&= \tilde{W}_{k0}\left[r + \sum_{k=1}^{n+1} \hat{p}_{ki}\left(\frac{S_i(1)}{\tilde{S}_i(0)} - r\right)\right]
\end{aligned}$$

where $\hat{p}_{ki} = \dot{p}_{ki}$ for $i = 1, \ldots, n$, and $\hat{p}_{k,n+1} = \tilde{p}_{k,n+1} + \sum_{k=1}^{n} \tilde{p}_{ki} v_i \tilde{S}_{n+1}(0)/\tilde{S}_i(0)$.

We can therefore equally well optimise over $\dot{p}_k$ or $\hat{p}_k$ with the same optimal utility for a given set of prices $\tilde{\tilde{S}}(0)$. Furthermore, the total demand for units of the new security held by companies and investors at the optimal solution will not be affected by the change in rules. It follows that, *from the investors' point of view*, it is irrelevant whether or not the company holds the new security (a point of view expressed by Exley, Mehta & Smith, 1997.)

4.2 Fair value of insurance-related liabilities: companies as the issuing agency

A special case of this is where the new security is issued by one of the existing companies (or securities). This is equivalent to issue by an external agency with

$u_{n+1} = 0$ as the issuing company takes on the opposite holding to investors in the new security.

This then is what is of critical interest to actuarial advisers. For example, suppose a quoted insurance company wishes to determine the fair value of part or all of its liabilities. These liabilities could be divided into units and placed as securities in the market with the insurer taking the opposite holding (that is, acting as counter party). The key point to note is that the fair valuation problem is simplified in most cases (that is, quoted companies) by the fact that the net number of units issued, u_{n+1}, is zero. Furthermore, the arguments in Section 4.1 indicate that the company itself can take any long or short position it wishes without affecting the resulting prices: in particular, it can choose to eliminate precisely a particular risk.

Consider the special case where all investors agree on the parametrisation of the returns model. We have seen that $u_{n+1} = 0$ implies that the introduction of the new insurance-linked security has no impact on the prices of the existing securities. This means that the capital structure of the company has no effect on the value of the company. (This is just a statement of the Modigliani-Miller Theorem.)

On the other hand, we have seen that if investors do not agree on the parametrisation then, even if $u_{n+1} = 0$:

- the securitisation of the insurance liability will have an impact on the prices of existing securities;
- in particular, a change in the capital structure of the company will have an effect on its value.

A key problem which remains in either case is the establishment of the market price of risk for risk $n + 1$.

4.3 Does splitting a company add value?

Another question that can be asked is does splitting a company into two parts add value? As with previous questions it may be argues that the split will allow investors to improve their expected utility and that the ensuing reorganisation of portfolios might push up prices. However, again the arguments in Section 4.1 tell us that splitting a company into two parts (which is equivalent to selling off a liability) will only have an effect on prices if investors do not agree on the parametrisation of the returns model.

5 Numerical examples

We will now consider some numerical examples to illustrate these results and investigate what the impact is of different investor viewpoints and the accuracy of risk-minimising prices.

Suppose that there are initially two risky assets with one further asset to be added. There are five investors each initially with wealth €100,000 at time 0. We will consider first the case where all investors agree on the parametrisation of the returns model. Thus:

- $\mu_i = 100$ for $i = 1, 2, 3$;
- $(\bar{V})_{ij} = 400$ if $i = j$ and $= 100$ if $i =\neq j$;
- $u_1 = 2000$, $u_2 = 1000$;
- $r = 1.05$ (and is observable).

The risk-aversion parameters for the five investors are $\alpha_1 = 2 \times 10^{-5}$, $\alpha_2 = 4 \times 10^{-5}$, $\alpha_3 = 6 \times 10^{-5}$, $\alpha_4 = 8 \times 10^{-5}$, $\alpha_5 = 10 \times 10^{-5}$. These very small values are better understood when we note that the corresponding values for the relative risk aversion factor $(wU_k''(w)/U_k'(w))$ evaluated at the initial wealth are 2, 4, 6, 8 and 10 respectively (reasonable values for private investors).

Experiment 1: no personal liabilities Suppose first that all investors have $\gamma_k = 0$ and $\delta_k = 0$.

The results are given in Table 1.

We can make the following observations:

- Security 1 has a higher risk premium because of the larger number of units in issue.
- All investors hold twice as many units of security 1 as security 2. This is in proportion to the market portfolio (reflecting the standard Capital Asset Pricing Model (CAPM) result).
- No investors go long or short in the new security. However, if the market price was less than 92.74 all investors would want to go long in security 3.
- Since no investors take a position in the new security their expected utilities are unaffected by its introduction.

Table 1

	Price	Risk prem.	Investor k $k=1$	2	3	4	5
Before			Holding (units)				
Security 1	87.73	8.99%	875.91	437.96	291.97	218.98	175.18
Security 2	90.23	5.82%	437.96	218.98	145.99	109.49	87.59
Expected utility $\times 10^6$			-113943	-14235	-1778	-222	-28
After			Holding (units)				
Security 1	87.73	8.99%	875.91	437.96	291.97	218.98	175.18
Security 2	90.23	5.82%	437.96	218.98	145.99	109.49	87.59
Security 3	92.74	2.83%	0.00	0.00	0.00	0.00	0.00
Risk-min est. of Sec. 3 price			92.74	92.74	92.74	92.74	92.74
Expected utility $\times 10^6$			-113943	-14235	-1778	-222	-28

- The investors' estimates of the risk minimising price for the new security are all accurate. This reflects the fact that $\delta_k = 0$ for all k and $u_3 = 0$.

Experiment 2: non-zero, risky personal liabilities Suppose now that the investors have personal liabilities which are risky and correlated with the securities, but also have independent risks.

The liability risks for the five investors are:

The results are given in Table 3.

We can make the following observations:

- The risk premia are higher than before. This reflects the fact that, everything else being equal, investors want to go short in both securities in order to match their liabilities.

- Investors no longer hold exactly twice as many units of security 1 as security 2. The balance has shifted away from this ratio to reflect the additional use of the two securities to match liabilities. This is one example of how the standard CAPM result breaks down.

- A variety of positions are taken in the new security. These reflect the sign and magnitude of $\delta_3^{(k)}$.

Table 2

	Investor, k				
	1	2	3	4	5
γ_k	-1000	-1000	-1000	-1000	-1000
	$\delta_i^{(k)}$				
Risk $i = 1$	2000	2000	2000	1400	200
2	200	200	200	1400	2000
3	2000	500	1000	2000	-2500
4	2000	2000	0	0	0
5	0	0	2000	0	0
6	0	0	0	2000	0
7	0	0	0	0	2000
8	0	0	0	0	0

Table 3

	Price	Risk prem.	Investor k: $k = 1$	2	3	4	5
Before			Holding (units)				
Security 1	86.46	10.66%	922.30	412.44	242.49	203.01	219.77
Security 2	89.27	7.02%	518.11	253.89	165.82	59.80	2.38
Expected utility $\times 10^6$			-112340	-14333	-1836	-237	-30
After			Holding (units)				
Security 1	86.46	10.66%	929.53	410.78	248.41	220.63	190.64
Security 2	89.27	7.02%	525.34	252.24	171.74	77.41	-26.73
Security 3	91.82	3.91%	-36.17	8.27	-29.63	-88.11	145.64
Risk-min est. of Sec. 3 price			92.29	92.29	92.29	92.29	92.29
Expected utility $\times 10^6$			-112330	-14333	-1835	-235	-29
Extra cash			4.48	0.47	9.03	106.44	363.47

- All investors have been able to use the new security to increase their expected utility. The final row of the table gives the extra money required and invested in cash under the "before" scenario to raise the expected utility to the "after" scenario expected utility. It represents the costs to the investor of the non-existence of security 3. We can see that size of the extra cost reflects the reletive holding of security 3 relative to securities 1 and 2. In this experiment, although expected utilities improve, the extra costs are relatively small relative to the investor's total wealth.

- The investors have overestimated the likely price of the new security using risk minimisation. This reflects the fact that $\sum_{k=1}^{m} \delta_3^{(k)} > 0$, meaning that investors prefer in aggregate to sell the new security to improve their liability matching.

Experiment 3: different investor parameters We next assume that investors have different estimates of the covariance matrix $\bar{V}$ for the security returns. (We will look at the additional effect of different μ_i in the next experiment.)

The investors' estimates of this matrix are:[4]

$$\text{Investor 1:} \quad \bar{V}_1 = \begin{pmatrix} 477.8596 & 104.7094 & 118.044 \\ 104.7094 & 339.4282 & 95.7807 \\ 118.044 & 95.7807 & 356.4805 \end{pmatrix} \tag{5.1}$$

$$\text{Investor 2:} \quad \bar{V}_2 = \begin{pmatrix} 384.16 & 88.788 & 88.984 \\ 88.788 & 333.8109 & 88.7112 \\ 88.984 & 88.7112 & 410.631 \end{pmatrix} \tag{5.2}$$

$$\text{Investor 3:} \quad \bar{V}_3 = \begin{pmatrix} 325.4416 & 100.4828 & 90.3804 \\ 100.4828 & 468.253 & 122.0007 \\ 90.3804 & 122.0007 & 438.9757 \end{pmatrix} \tag{5.3}$$

$$\text{Investor 4:} \quad \bar{V}_4 = \begin{pmatrix} 509.8564 & 109.9646 & 109.0614 \\ 109.9646 & 497.6498 & 104.9419 \\ 109.0614 & 104.9419 & 410.9654 \end{pmatrix} \tag{5.4}$$

$$\text{Investor 5:} \quad \bar{V}_5 = \begin{pmatrix} 410.0625 & 84.8475 & 106.11 \\ 84.8475 & 556.7245 & 111.1204 \\ 106.11 & 111.1204 & 416.6257 \end{pmatrix} \tag{5.5}$$

The results are given in Table 4.

We can make the following observations:

[4] These estimates were generated using a relative standard deviation of 10% around the true values in the Cholesky decomposition of $\bar{V}$.

Table 4

	Price	Risk prem.	Investor k $k=1$	2	3	4	5
Before			Holding (units)				
Security 1	86.03	11.23%	790.48	461.48	352.46	166.98	228.59
Security 2	89.59	6.61%	590.40	283.33	102.57	37.01	-13.31
Expected utility $\times 10^6$			-112983	-14139	-1803	-240	-29
After			Holding (units)				
Security 1	86.02	11.25%	801.65	460.01	361.06	180.27	197.01
Security 2	89.60	6.60%	598.75	279.21	109.93	49.19	-37.08
Security 3	91.77	3.97%	-46.98	14.20	-36.66	-72.87	142.31
Risk-min est. of Sec. 3 price			92.16	92.34	92.08	92.64	92.24
Expected utility $\times 10^6$			-112967	-14138	-1801	-239	-28
Extra cash			6.58	1.45	15.12	75.59	361.10

- We immediately note that the prices of securities 1 and 2 are both affected by the introduction of the new security. However, the size of this effect is tiny.
- Different investors now have different risk-minimising price estimates for the new security. This reflects the fact that they will have different risk-minimising portfolios as a consequence of different covariance matrices. The range of these risk-minimising prices is not too great. This is a promising suggestion that risk-minimisation might provide a good approximation to the fair value.

  These price estimates are all higher than the equilibrium price. This reflects the desire of investors to match their liability risks, which is resulting in a net sale of security 3.
- The extra cash required for each investor is similar in magnitude to Experiment 2.
- The optimal holdings in the various securities is significantly altered by the introduction of different investor covariance estimates. This is easier to observe when the $\delta_k = 0$ (not given here). This is another example of a break down of the standard CAPM result that all investors have risky investments in proportion to the market portfolio.

Table 5

	Price	Risk prem.	Investor k: $k=1$	2	3	4	5
Before			Holding (units)				
Security 1	85.22	–	699.25	434.90	430.41	184.23	251.20
Security 2	90.58	–	549.16	362.22	97.56	35.69	-44.62
Expected utility $\times 10^6$			-115224	-13997	-1737	-238	-29
After			Holding (units)				
Security 1	85.20	–	676.07	446.00	443.44	205.63	228.85
Security 2	90.57	–	521.42	374.13	109.78	56.06	-61.40
Security 3	93.02	–	127.20	-53.97	-56.72	-118.24	101.73
Risk prem.							
Security 1			10.22%	11.19%	13.09%	12.15%	12.12%
Security 2			6.02%	7.60%	6.91%	6.65%	4.83%
Risk-min est. of Sec. 3 price			94.45	92.62	92.88	92.60	92.05
Expected utility $\times 10^6$			-115108	-13985	-1733	-234	-28
Extra cash			48.08	20.83	36.19	199.02	184.54

Experiment 4: different investor parameters In this experiment we allow investors to have, in addition to experiment 3, different estimates of the mean returns. These are:

$$\text{Investor 1:} \quad \mu_1 = (98.19, 100.56, 102.11)' \tag{5.6}$$
$$\text{Investor 2:} \quad \mu_2 = (99.02, 101.99, 100.48)' \tag{5.7}$$
$$\text{Investor 3:} \quad \mu_3 = (100.64, 101.36, 101.23)' \tag{5.8}$$
$$\text{Investor 4:} \quad \mu_4 = (99.84, 101.13, 100.10)' \tag{5.9}$$
$$\text{Investor 5:} \quad \mu_5 = (99.81, 99.48, 99.69)' \tag{5.10}$$

The results are given in Table 5.

We can make the following observations:

- The investors' estimates of the risk-minimising price for the new security now vary quite significantly. However, this is primarily a consequence of differences between investors' estimates of the expected return on the new security.

  However, this is an important consideration in the establishment of fair value. It does not invalidate the use of risk-minimising prices. However, it does indicate that any estimate for the price for security 3 is subject to significant parameter risk, regardless of whether or not a sound and objective method is being used to assess fair value. In particular, no investor can anticipate what estimates of μ_3 will be used by other investors.

- The risk premia on assets 1 and 2 now vary between investors. As a consequence these have been presented differently in the table.
- The extra cash required for each investor has changed significantly from the previous experiment. This reflects the changes in risk premia estimated by each investor.

6 Conclusions

We have shown how we can use equilibrium theory as a means of establishing the fair value of a liability. We have also discussed the use of risk-minimisation as a method for deriving a market price.

For the simple model investigated we have discussed how the equilibrium price relates to the risk-minimising price. We have seen that there is a strong link between the two, but difference exist which are difficult to overcome. The main factors causing differences between the two prices are:

- different estimates of security-returns parameters between investors, especially the mean return on the new security;
- the existence of personal, risky liabilities for each investor (which affects the risk premium on the new security);
- the aggregate risk aversion factor (which is difficult to estimate).

Of these the mean return on the new security is the critical reason for differences between risk-minimising prices and the equilibrium (or fair) value.

As side results we have reminded ourselves of two significant flaws in the central result in the basic form of the Capital Asset Pricing Model. We have also seen that, although the Modigliani Theorem is not true in general, its central result seems to be quite robust.

This paper just scratches the surface of the issue of fair valuation and it would seem that there is a considerable amount of work still to be done.

Acknowledgements

This research was partially facilitated by a grant from the Faculty and Institute of Actuaries on the fair valuation of pension liabilities. The development of this paper has also been influenced by discussions with Andrew Wise, Andrew Wilson, Peter Lofthouse and participants in a series of financial economics seminars.

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Appendix A: Notation

Matrices

Consider the $n \times (n+m+1)$ matrix $A = (a_{ij})_{i=1,\ j=1}^{n\ \ (n+m+1)}$. Then:

$$
\begin{aligned}
A^S &= (a_{ij})_{i=1,\ j=1}^{n\ \ n} && \text{that is, a square matrix deleting the final } m+1 \text{ columns} \\
\bar{A} &= (a_{ij})_{i=1,\ j=1}^{(n+1)\ (n+m+1)} && \text{the extension of } A \text{ by one extra row} \\
\bar{A}^S &= (a_{ij})_{i=1,\ j=1}^{(n+1)\ (n+1)} && \text{that is, a square matrix deleting the final } m \text{ columns}
\end{aligned}
$$

In the main text A is usually the volatility matrix C or C_k. In addition we have the $n \times n$ covariance matrix $V = C_k C_k'$ and its extension $\bar{V} = \bar{C}_k \bar{C}_k'$.

Vectors

Let n, m be constants and $x' = (x_1, \ldots, x_{n+m+1})$. Then:

$$
\begin{aligned}
\hat{x}' &= (x_1, \ldots, x_n) \\
\bar{x}' &= (x_1, \ldots, x_{n+1}) \\
\hat{\Lambda}_x &= \operatorname{diag}(\hat{x}) \quad \text{the diagonal matrix constructed from } \hat{x} \\
\bar{\Lambda}_x &= \operatorname{diag}(\bar{x})
\end{aligned}
$$

In the main text x is usually $S(t)$, u, δ, σ, μ, ρ or γ.

Revised values after introduction of new security

Certain vectors or matrices change their value after the introduction of a new security. The revised value is indicated by the placing of a tilde over the vector or matrix: for example, $\tilde{\hat{S}}(0)$, $\tilde{\hat{\Lambda}}_S = \operatorname{diag}\big(\tilde{\hat{S}}(0)\big)$.

Appendix B: Matrix results

Suppose that C is an $n \times (n+m+1)$ lower triangular matrix of rank n. Let $\bar{C}$ be the lower-triangular extension of C to $n+1$ rows and of rank $n+1$.

Lemma B.1

(a) $C^{S^{-1}} = A$ *is lower triangular.*

(b) $\bar{C}^{S^{-1}} = \bar{A}$ *is lower triangular with* $(\bar{A})_{ij} = (A)_{ij}$ *for* $i, j \leq n$.

Proof by calculation of A and $\bar{A}$:

(a) Let the columns of C^S be denoted by the column vectors $c_1, \ldots, c_n$ with $c_i' = (0, \ldots, 0, c_{ii}, \ldots, c_{ni})$.

Let the rows of A be denoted by the row vectors $a_1, \ldots, a_n$ with $a_i = (a_{i1}, \ldots, a_{ii}, 0, \ldots, 0)$.

The lower triangular forms of A and C^S ensure that $a_i c_j = 0$ for all $j > i$.

Also we require:

$$\begin{aligned} a_i c_i &= a_{ii} c_{ii} = 1 \quad \text{for all } i \\ \text{and} \quad a_i c_j &= \sum_{k=j}^{i} a_{ik} c_{kj} = 0 \quad \text{for all } j < i \end{aligned}$$

Thus, for each i we calculate $a_{ii}, a_{i,i-1}, \ldots, a_{i1}$ recursively.

(b) Let the columns of $\bar{C}^S$ be denoted $\bar{c}_1, \ldots, \bar{c}_n, \bar{c}_{n+1}$ with the first n elements of $\bar{c}_i$ matching c_i for $i = 1, \ldots, n$.

Let the rows of $\bar{A}$ be denoted $\bar{a}_1, \ldots, \bar{a}_n, \bar{a}_{n+1}$. where $\bar{a}_i = (a_{i1}, \ldots, a_{ii}, 0, \ldots, 0)$ extends the row vector a_i with the addition of a 0. It is straightforward to see that $\bar{a}_i \bar{c}_i = 1$ for $i = 1, \ldots, n$ and $\bar{a}_i \bar{c}_j = 0$ for $i, j = 1, \ldots, n$ and $i \neq j$. It is also straightforward to see that $\bar{a}_i \bar{c}_{n+1} = 0$ for $i = 1, \ldots, n$.

Finally the elements of $\bar{a}_{n+1}$ are chosen using a backwards recursion as in (a) to ensure that $\bar{a}_{n+1} \bar{c}_{n+1} = 1$ and $\bar{a}_{n+1} \bar{c}_i = 0$ for $i = 1, \ldots, n$.

Corollary B.2

Let $\bar{b} = \bar{a}_{n+1}'$ and let $\hat{b}$ be its first n elements. Suppose $V = CC'$ and $\bar{V} = \bar{C}\bar{C}'$.

(a)

$$\bar{V}^{-1} = \left(\begin{array}{c|c} V^{-1} + \hat{b}\hat{b}' & b_{n+1}\hat{b} \\ \hline b_{n+1}\hat{b}' & b_{n+1}^2 \end{array} \right) = \left(\begin{array}{c|c} V^{-1} & 0 \\ \hline 0 & 0 \end{array} \right) + \bar{b}\bar{b}'$$

(b) Let $\sigma = (c_{n+1,1}, \ldots, c_{n+1,n+1}, 0, \ldots, 0)'$ be the transpose of the $(n+1)$th *row of $\bar{C}$. Then:*

$$b_{n+1} = \frac{1}{\sigma_{n+1}}, \qquad \text{and} \qquad \hat{b} = -\frac{1}{\sigma_{n+1}} (C^S)'^{-1} \hat{\sigma}$$