Pricing death

or

Modelling the Mortality Term Structure

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Background

- Life insurers and pension funds
- exposed to many risks

  A: investment risk

  B: interest-rate risk

  C: longevity risk

  D: others

- A, B $\longrightarrow$ can hedge to reduce risk; C?

Longevity risk or stochastic mortality risk

the risk that future mortality risk is

different from that anticipated

What is stochastic mortality?

n lives, probability p of survival, N survivors

- Unsystematic mortality risk:

  $\Rightarrow N|p \sim$ Binomial(n, p)

  $\Rightarrow$ risk is diversifiable, $N/n \longrightarrow p$ as $n \to \infty$

- Systematic mortality risk:

  $\Rightarrow p$ is uncertain

  $\Rightarrow$ risk associated with p is not diversifiable

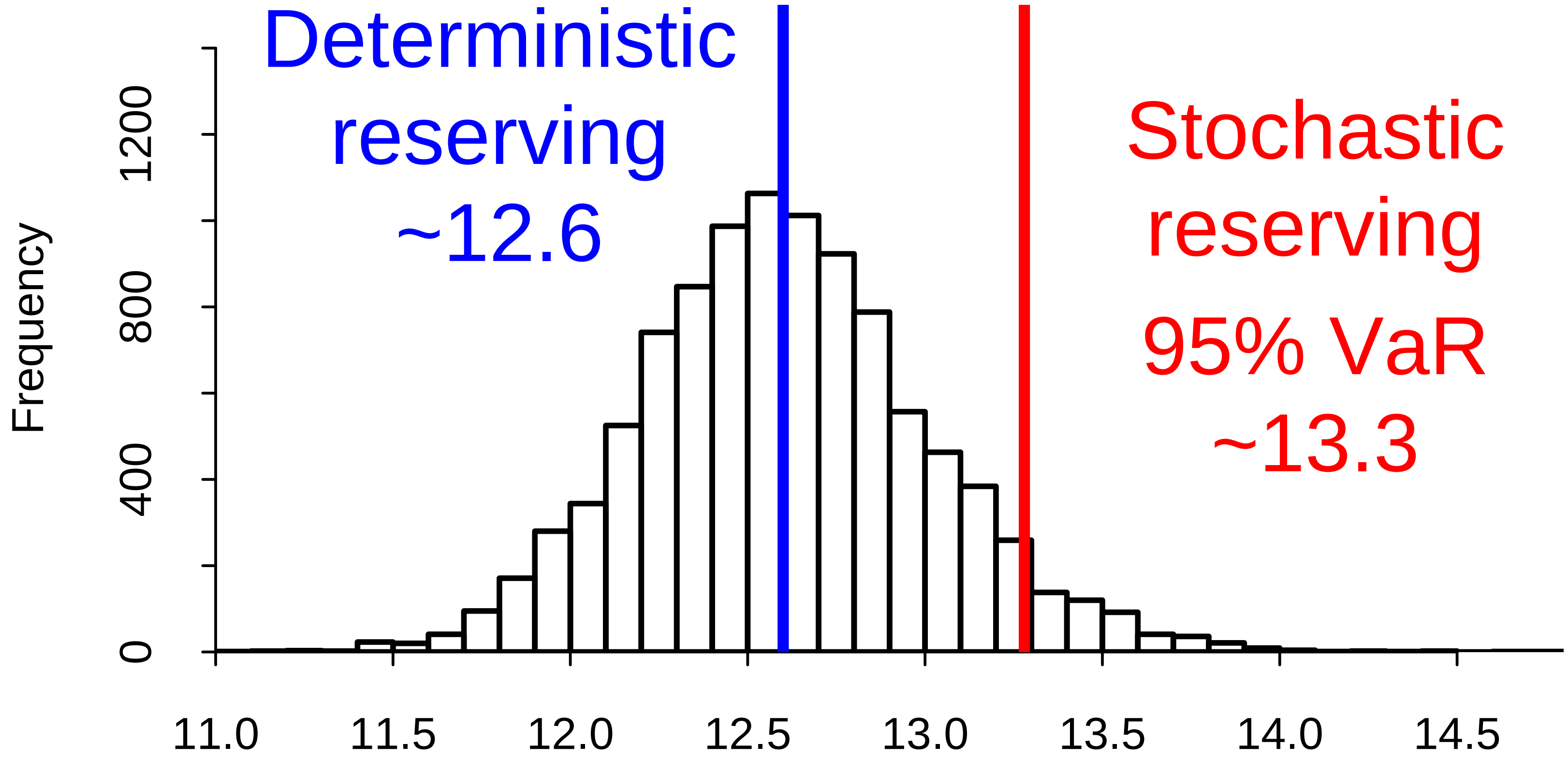
Longevity problem: Life Insurance

Example 1: Annuity portfolio

- Cohort males aged 65
- Level annuity for life
- Interest rates fixed at 4%
- Large cohort $\Rightarrow$ individual risk diversified
- Still exposed to systematic mortality risk
- What reserves do we need *per unit of annuity*?

Statistically: how significant is systematic mortality risk?

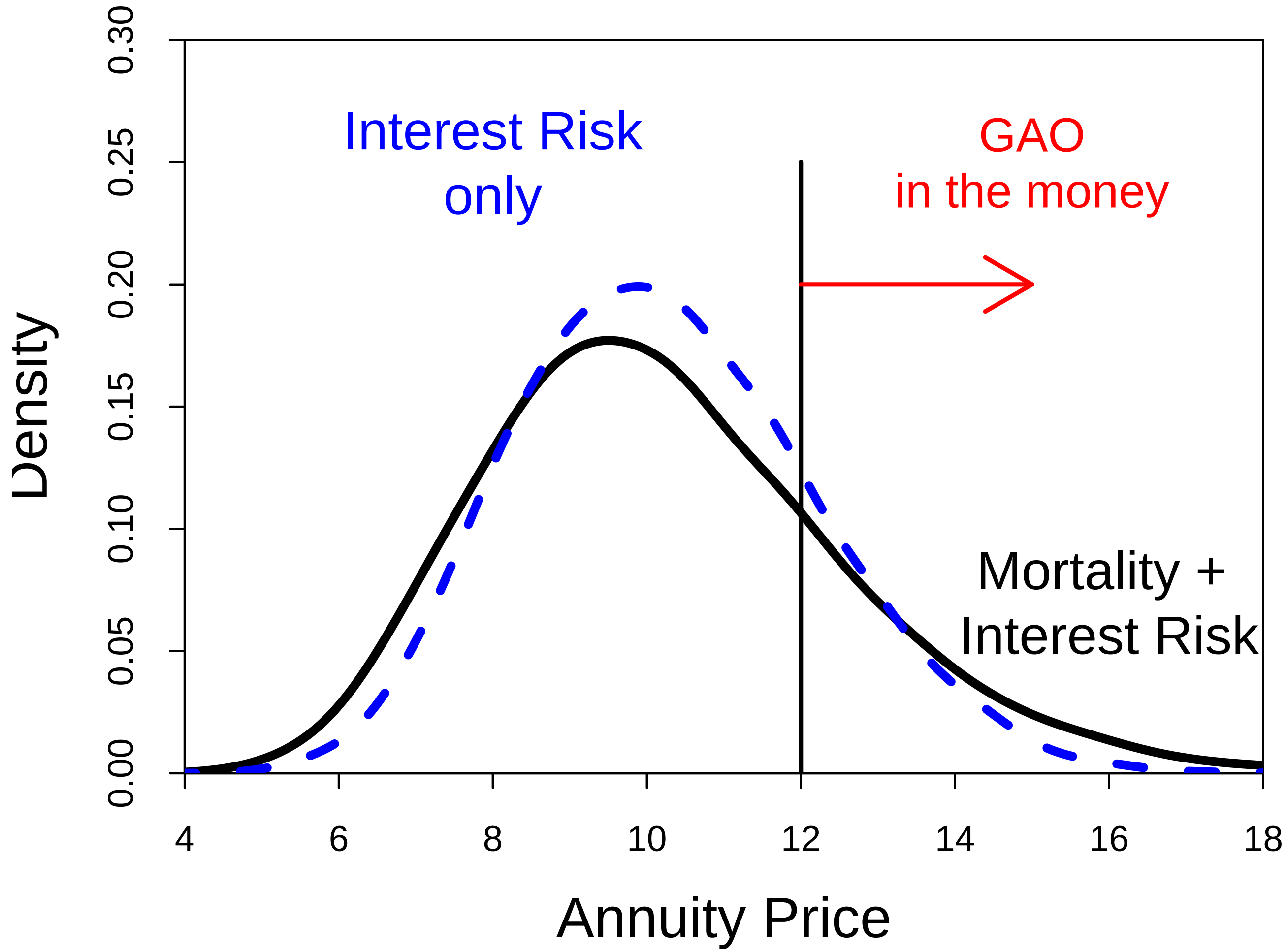
Cohort aged 65: undiversifiable mortality risk:



Example 2: Guaranteed Annuity Option

- Pension savings contract
- Male now aged 35
- Accumlated pension wealth converted to annuity at 65
- Option to convert at a *guaranteed rate*
- Option value depends on
  - Interest rates in 30 years
  - Mortality table in use in 30 years

Male, 35 – Guaranteed Annuity Option in 30 years



Mortality risk increases value of GAO

Market solutions for life insurers and pension plans

- Short-term catastrophe bonds (Swiss Re, Dec. 2003)
- Long-term longevity bonds (EIB/BNP, Nov. 2004)

  cashflows linked to survivorship index
- Survivor swaps (some OTC contracts???)

  swap fixed for mortality-linked cashflows
- Annuity futures

  traded contract; underlying=market annuity rates; many exercise dates

EIB/BNP Paribas Longevity Bond

- How do we price this bond?
- In an arbitrage-free market

  how might the price of this bond evolve through time?

We need:

(a) a stochastic mortality model;

(b) a stochastic interest-rate model.

Aims of this work

- There are many possible stochastic mortality models.
- Interest-rate theory $\Rightarrow$
  - ready-made frameworks for stochastic mortality
  - new stochastic mortality models
  - consistent pricing frameworks

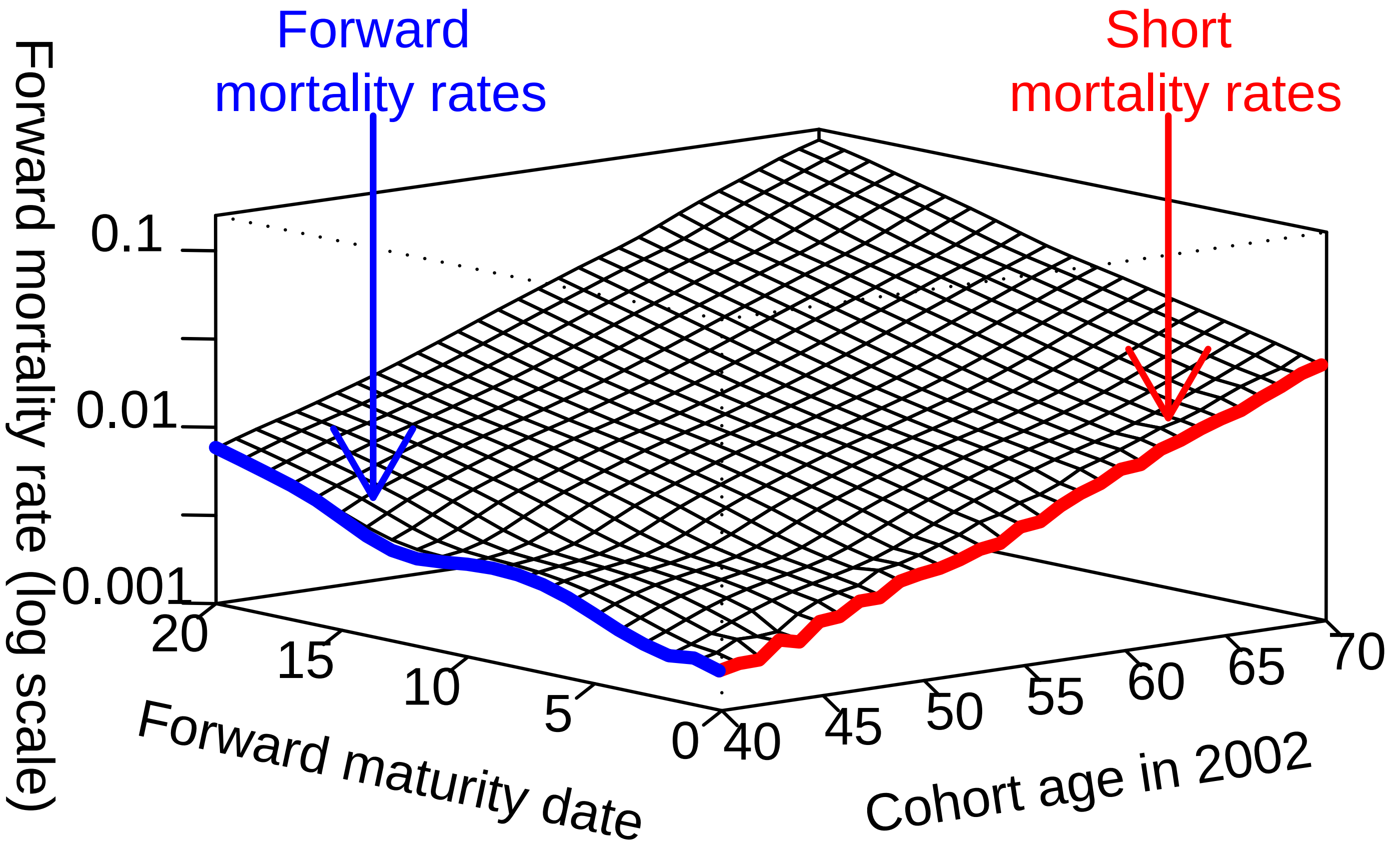
PLAN FOR TALK

- Background
- Why do we need to model stochastic mortality?
- Modelling: basic ingredients for arbitrage-free markets
- Modelling: different frameworks
- Case study: a 2-factor short-rate model for pricing longevity bonds

Why do we need a stochastic model for mortality?

- To calculate quantile reserves (VaR)
- To calculate fair values
  - – especially contracts with embedded options
- To price mortality-linked securities

The term-structure of mortality



STOCHASTIC MODELLING

$\mu(t, x)$ = force (instantaneous rate) of mortality at t for individuals aged x at time 0

$r(t)$ = risk-free rate of interest

$r(t)$, $\mu(t, x)$ represent very different quantities

Mathematically we can treat $r(t)$, $\mu(t, x)$ as equivalent

Analogy between mortality and interest rates

1: Deterministic interest and mortality (no improvements)

Force of mortality	Force of interest
μ_{x+t}	$r(t)$
${}_tp_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$	$P(0,t) = \exp\left(-\int_0^t r(s) ds\right)$
SCOR (Survivor Credit Offer Rate) $\frac{q_x}{p_x} = \frac{1-p_x}{p_x}$	LIBOR $\frac{1-P(0,1)}{P(0,1)}$

Analogy between mortality and interest rates

2: Stochastic interest and mortality

$x =$ age at time 0

$\mu(t,x)$	$r(t)$
$p(0,t,x) = E_{?}\left[\exp\left(-\int_0^t \mu(s,x)ds\right)\right]$	$P(0,t) = E_{Q}\left[\exp\left(-\int_0^t r(s)ds\right)\right]$
Forward SCOR	Forward LIBOR

$E_{?}$: Choice of measure depends on application.

Cash account, $C(t)$:

$$\begin{aligned} C(t)^{-1} &= \exp\left[-\int_0^t r(s)ds\right] \\ &= \text{risk-free discount factor} \end{aligned}$$

Survivor Index:

$$\begin{aligned} S(u,y) &= \exp\left[-\int_0^u \mu(t, y+t)dt\right] \\ &= \text{Prob. of survival of } (y) \text{ from time 0 to time } u \\ &\qquad \text{given knowledge of evolution of } \mu(t,x) \end{aligned}$$

FUNDAMENTAL SECURITIES

1. Fixed-interest zero-coupon bonds

$$P(t, T) = \text{Price at } t \text{ for \$1 at time T}$$

2. Zero-coupon survivor (longevity) bond

$$\tilde{B}(t, T, x) = \text{Price at } t \text{ for \$ } S(T, x) \text{ at time } T$$

Approximately: BNP Paribas $= \sum_{T=1}^{25} \tilde{B}(t, T, 65)$

Pricing

What can we learn from interest-rate modelling?

Present time:

- Almost no market
- Replcation arguments not appropriate
- Assumption:

  What market there is, is arbitrage free.

  $\Rightarrow$ there exists a (∞ many) martingale measure Q.

RISK-NEUTRAL PRICING

We postulate the existence of a risk-neutral pricing measure Q.

$$P(t,T) \;=\; E_Q\left[e^{-\int_t^T r(s)ds} \,\middle|\, \mathcal{H}_t\right]$$

$$\tilde{B}(t,T,x) \;=\; E_Q\left[e^{-\int_t^T r(s)ds} S(T,x) \,\middle|\, \mathcal{H}_t\right]$$

Pricing under $Q \Rightarrow$ dynamics under P are arbitrage free

NO *requirement for liquidity, or zero transaction costs*

Historical mortality data $\Rightarrow P$ dynamics

(but beware of model and parameter risk!)

No price data: $\Rightarrow$

- choice of Q is a matter of faith

Limited price data: $\Rightarrow$

- Constraints on choice of Q
- Can limit Q further by making explicit *modelling* assumptions about the market price of risk

Assumption: $\mu(t,y)$ is independent of $r(t)$

- Not okay for Swiss Re catastrophe bond
- Reasonable assumption for longevity bonds

$$
\begin{aligned}
\Rightarrow \tilde{B}(t,T,x) &= E_Q\left[e^{-\int_t^T r(s)ds} \,\middle|\, \mathcal{F}_t\right] E_Q\left[S(T,x) \mid \mathcal{M}_t\right] \\
&= P(t,T)B(t,T,x) \\
B(t,T,x) &= E_Q\left[S(T,x) \mid \mathcal{M}_t\right]
\end{aligned}
$$

$$\frac{B(t,T,x)}{B(t,t,x)} = \frac{\tilde{B}(t,T,x)}{P(t,T)B(t,t,x)}$$

$$= E_Q\left[e^{-\int_t^T \mu(s,x+s)ds} \,\Big|\, \mathcal{M}_t\right]$$

= risk-neutral probability at t that $(x+t)$

survives from time t to time T

= spot survival probability, $p_Q(t,T,x)$

TYPES OF STOCHASTIC MORTALITY MODEL

We can use the same frameworks as interest-rate modelling: $\mu(t, x+t)$ is equivalent to $r(t)$

(but we might not use the same models!!!)

- Short-rate modelling framework (e.g. CIR)
- Forward-rate modelling framework (e.g. HJM)
- Positive-interest framework (e.g. Flesaker-Hughston)
- Market Models (e.g. BGM)

SHORT-RATE MODELLING FRAMEWORK

$\Rightarrow$ model for the evolution of $\mu(t, x)$

or model for the evolution of $q(t, x)$

Examples:

- Lee & Carter (1992) and followers (discrete time)
- Cairns, Blake, Dawson and Dowd (2005) (discrete time) model for assessing risk in longevity bond
- Milevsky & Promislow (2001), Dahl (2004) (cont. time)

FORWARD-RATE MODELS

- Begin with spot survival probabilities:

$$p_Q(t,T,x) = E_Q\left[e^{-\int_t^T \mu(s,x+s)ds} \,\middle|\, \mathcal{M}_t \right]$$

  for $T = t+1, t+2, \ldots$ and current ages $x + t = 20, \ldots, 90$

- Framework $\Rightarrow$ constraints on how dynamics of the $p_Q(t,T,x)$ interact

- Smith and Olivier (slides at http://www.actuaries.ie)

MARKET MODELS

Interest rates:

- Forward swap rates are log-normal (Jamshidian, 1997)
- Forward LIBOR rates are L-N (Brace-Gatarek-Musiela, 1997)

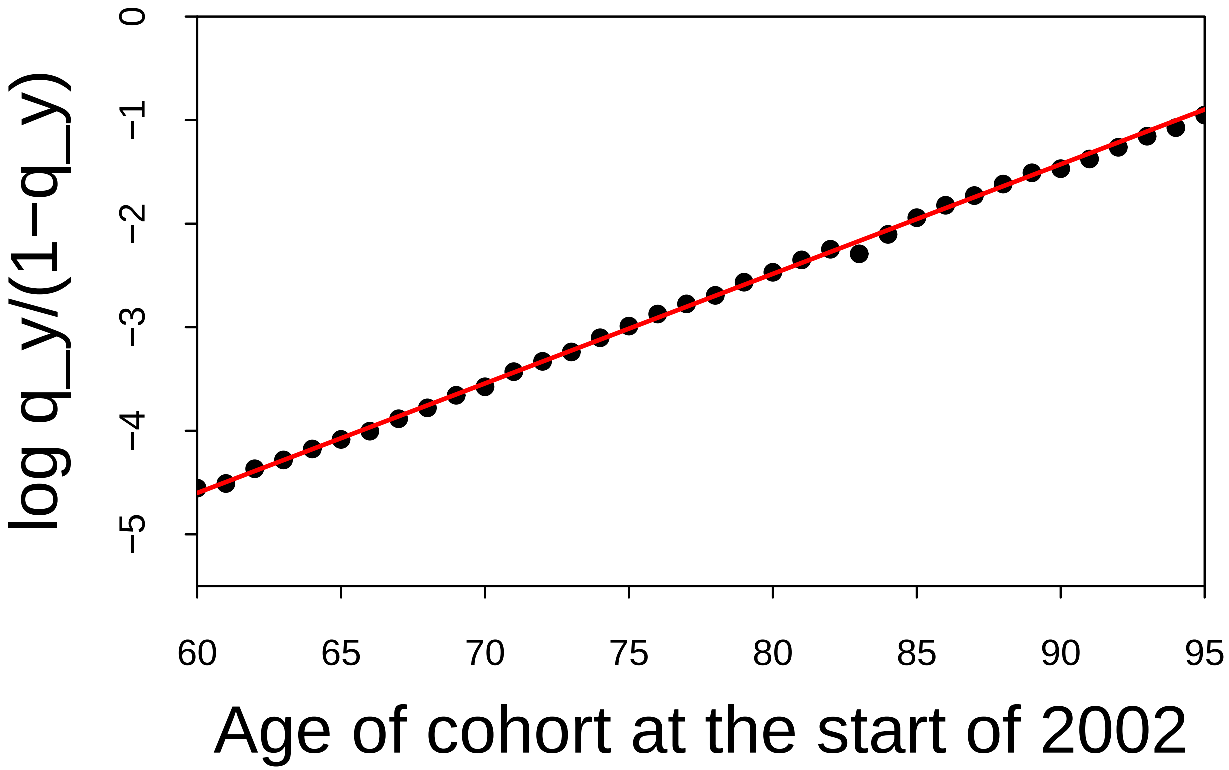
Mortality (Cairns, Blake and Dowd, 2004):

- Forward life annuity rates are L-N
- Forward Survivor Credit Offer Rates (SCOR) are L-N

POSSIBLE CRITERIA FOR STOCHASTIC MORTALITY MODELS

- $\mu(t, x) > 0$ for all $t, \ x$
- Model consistent with historical data
- Future dynamics should be *biologically* reasonable
- Complexity of model appropriate for task in hand
- Model allows fast numerical computation
- Avoid mean reversion

Case study: England and Wales males, age 60-95



Data suggests $\log q_y/(1-q_y)$ is linear

$$\Rightarrow q_y = e^{\alpha+\beta y}/\left(1+e^{\alpha+\beta y}\right)$$

A TWO-FACTOR ‘SHORT-RATE’ MODEL

Application: pricing of longevity bonds

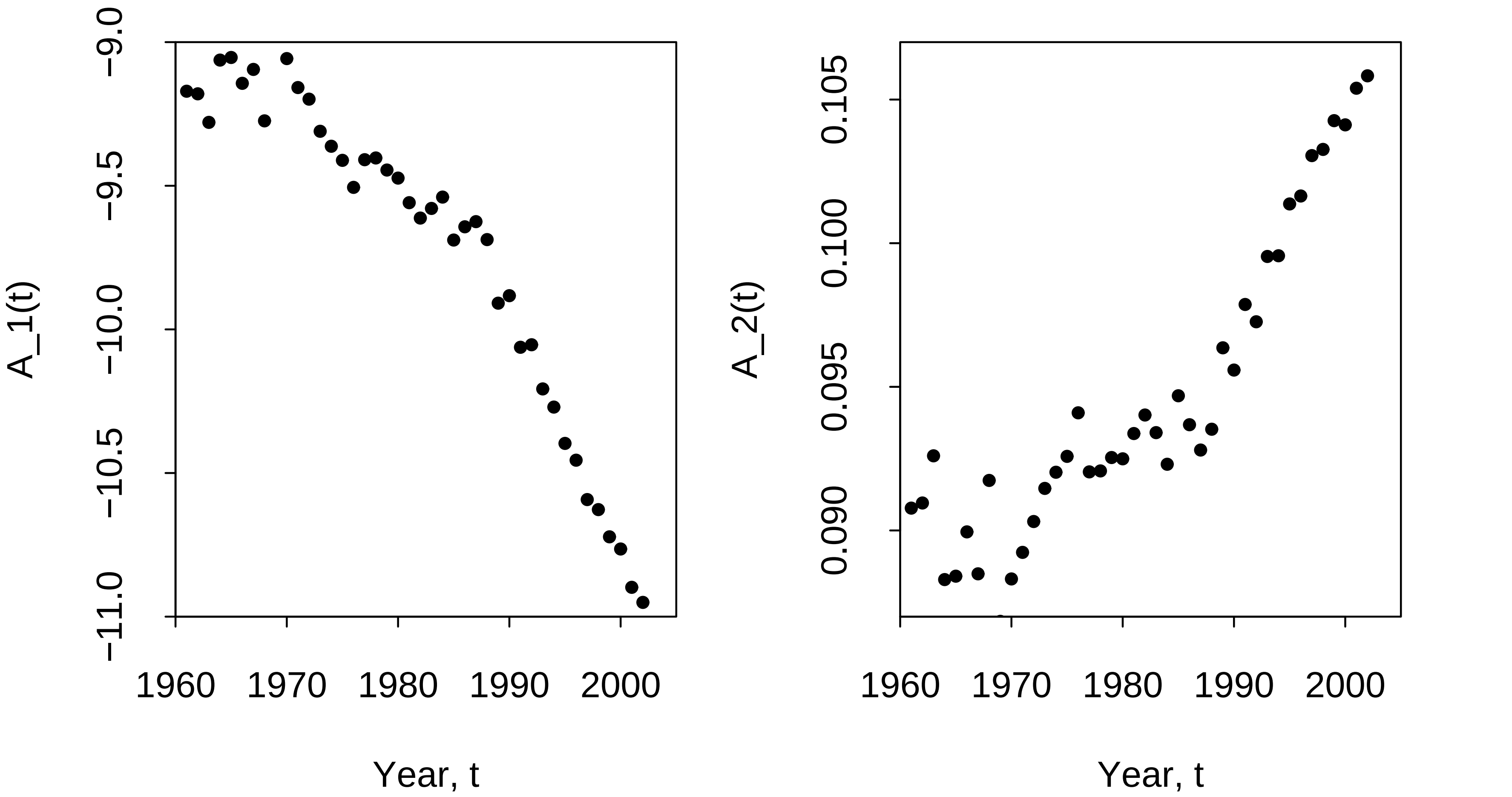
Mortality rates for the year t to $t+1$:

$$q(t,x) = \frac{e^{A_1(t)+A_2(t)(x+t)}}{1+e^{A_1(t)+A_2(t)(x+t)}}$$

We model $A(t) = (A_1(t), A_2(t))'$ as a random-walk

with drift

$A_1(t)$: level $\quad$ $A_2(t)$: slope



Comment on modelling approach:

- Here: For each t, $A(t)$ estimated without reference to other years
- Lee-Carter approach: all data used simultaneously, but no time series model assumed in Stage 1.
  Stage 2 fits a time series model to the $A(t)$.
- Next step here: combine Stages 1 and 2 into one.

STATISTICAL ISSUES

- Amount of historical data to use
- Parameter risk
- Model risk

Model: Random walk with drift

$$A(t+1) - A(t) = \mu + CZ(t+1)$$

$V = CC' =$ variance-covariance matrix

We incorporate parameter uncertainty

Bayesian approach $\Rightarrow$

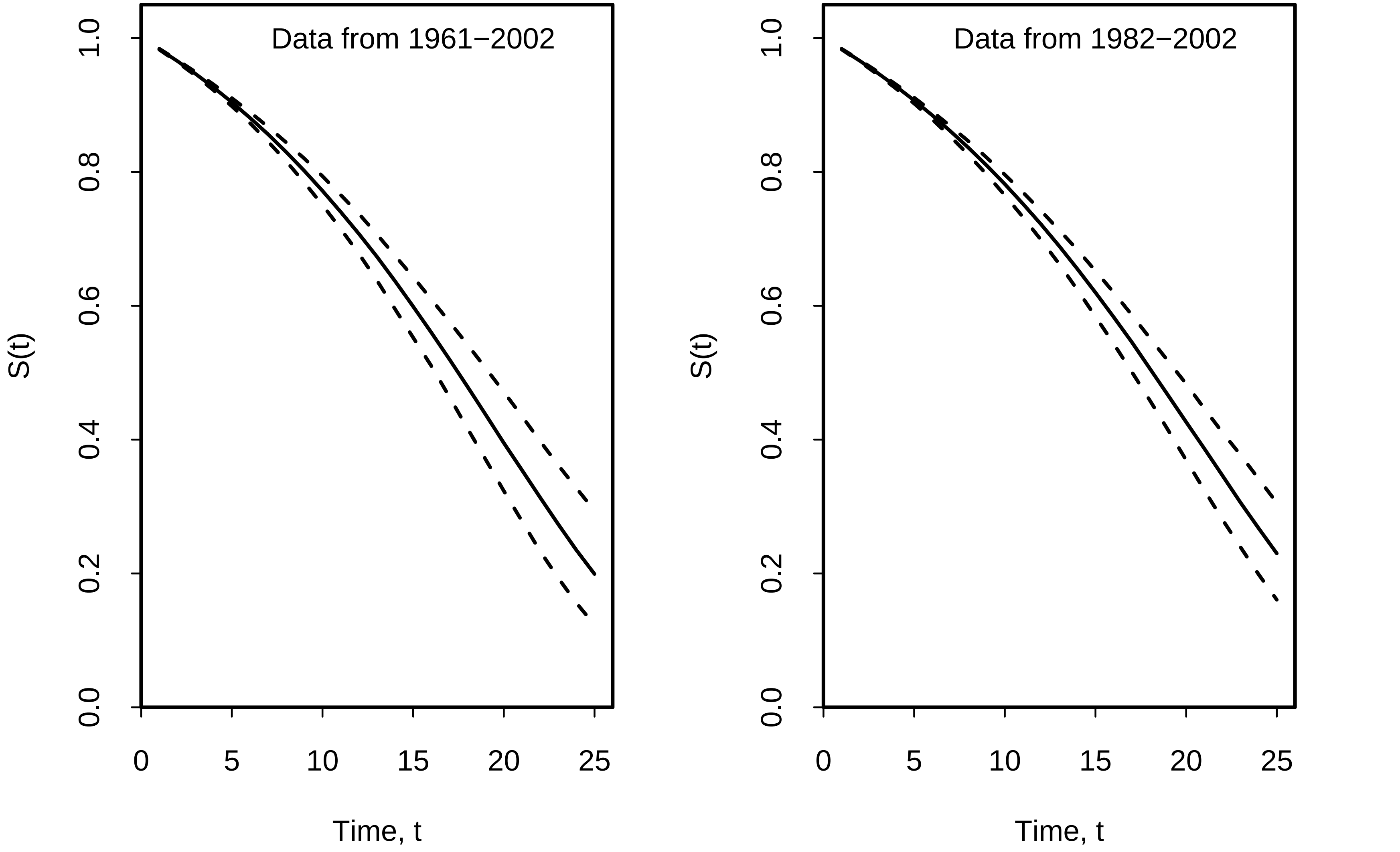
$(\mu, V)\Big|$data $\sim$ Normal-Inverse-Wishart distribution

Recap: Longevity Bond

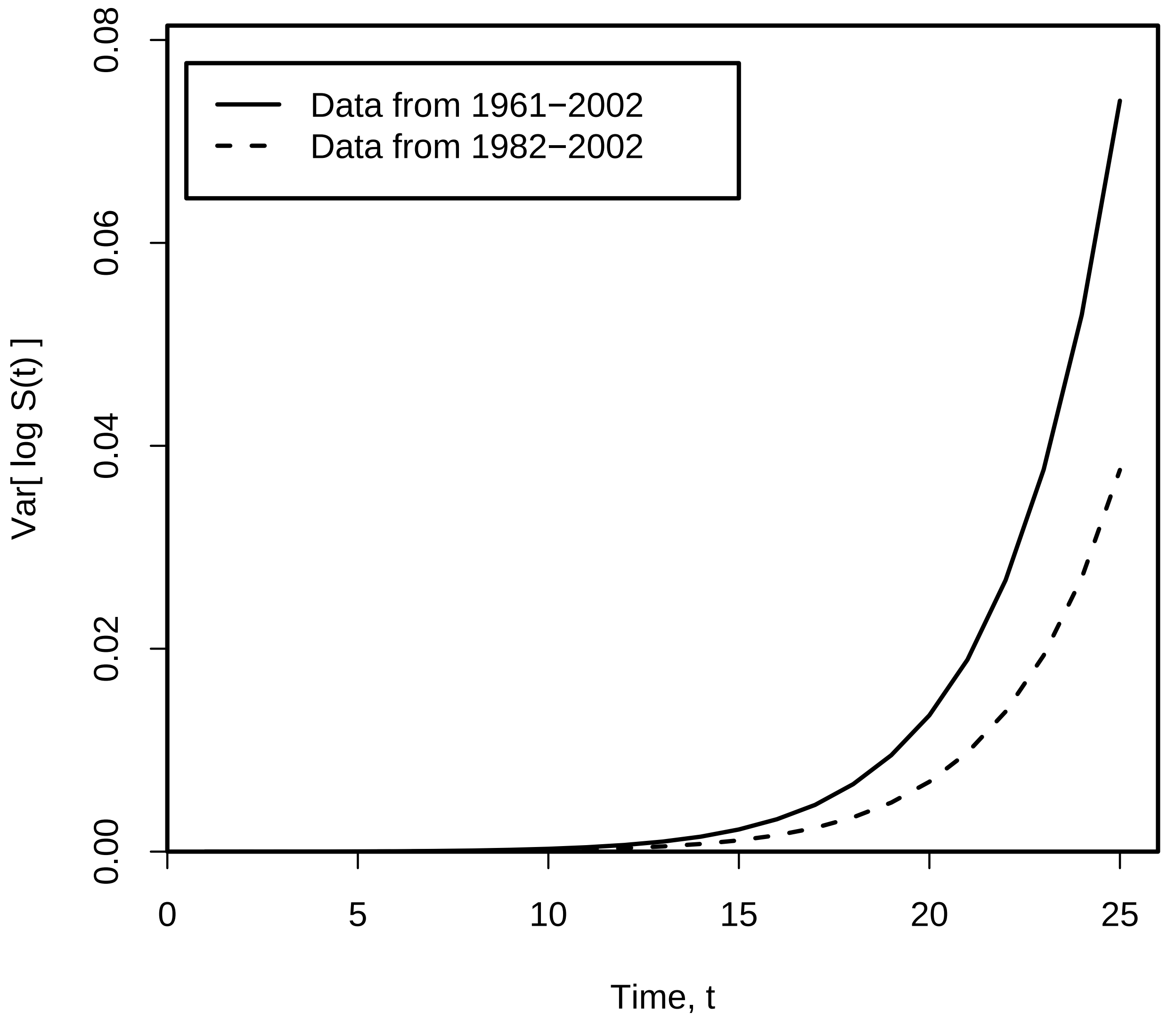
- $S(t,x) =$ survivor index at t
- Age x at time 0

$$\begin{aligned} S(t,x) &= \text{retrospective probability of survival} \\ &= (1-q(0,x)) \times \ldots \times (1-q(t-1,x)) \end{aligned}$$

Different sample periods; no parameter uncertainty

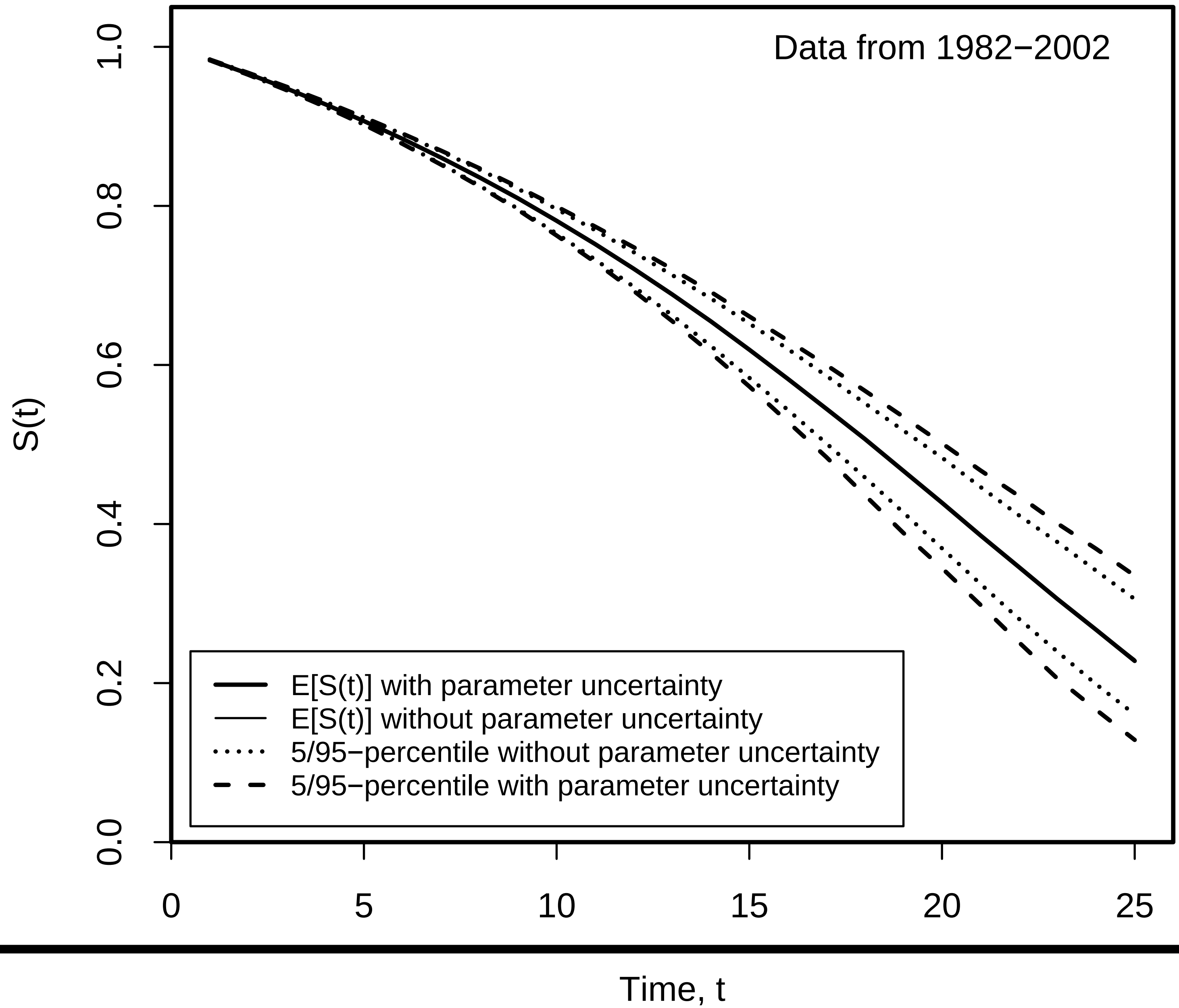


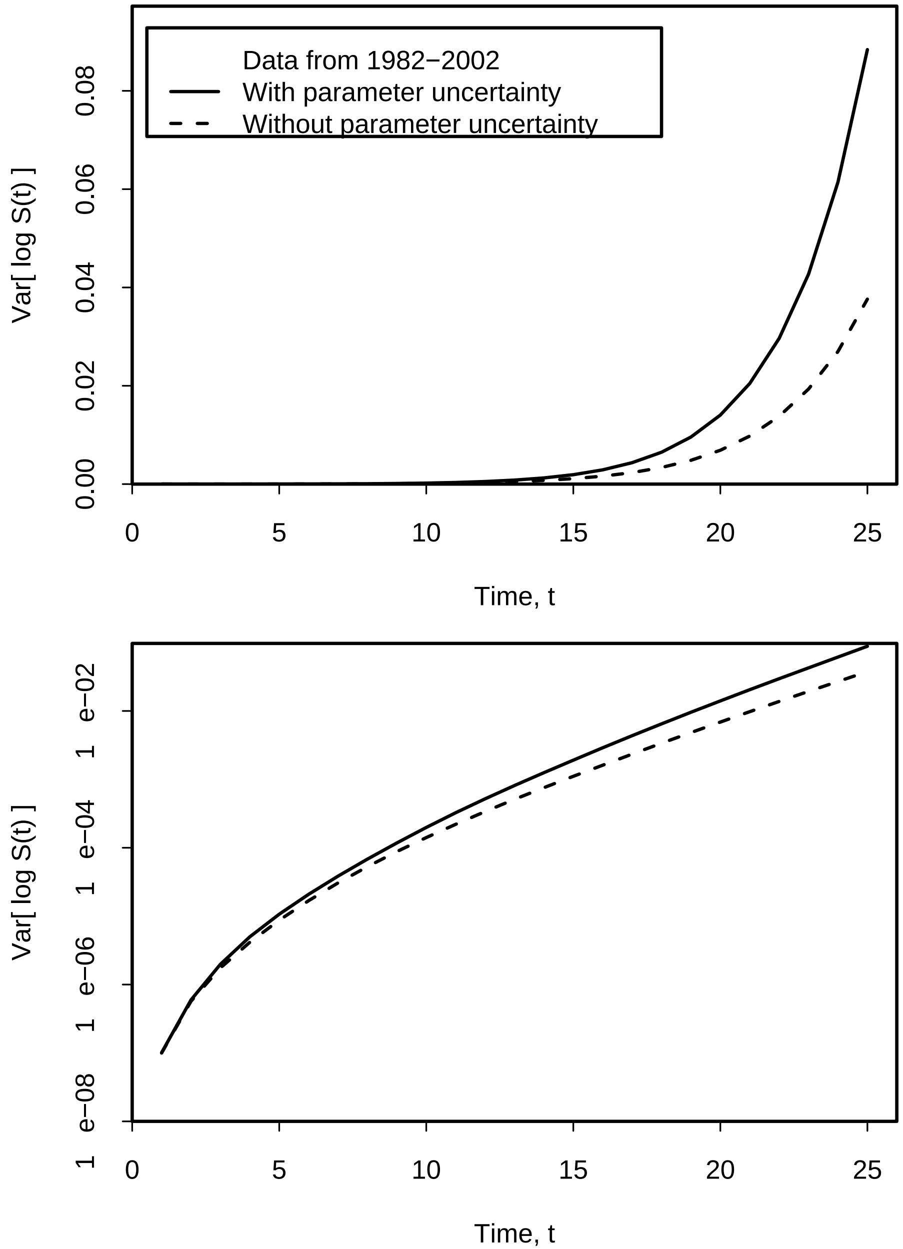
$Var[\log S(t)]$



Data from 1961−2002
Data from 1982−2002
Var[log S(t)]
0.00
0.02
0.04
0.06
0.08
0
5
10
15
20
25
Time, t

Impact of parameter uncertainty




Data from 1982−2002
With parameter uncertainty
Without parameter uncertainty
Var[log S(t)]
0.00
0.02
0.04
0.06
0.08
0
5
10
15
20
25
Time, t
Var[log S(t)]
1 e−08
1 e−06
1 e−04
1 e−02
Time, t

One item of market information

EIB longevity bond:

Expected cashflows under P are priced at 20 basis points (0.2%) *below* fixed-interest EIB bonds

$\Rightarrow$ purchasers of the bond are prepared to pay a premium

Market Prices of Risk

risk premium may arise in response to

- stochastic development of $A(t)$
- parameter risk in μ
- parameter risk in V

Market Price of Risk 1

$$A(t+1) - A(t) = \mu + C\left[\tilde{Z}(t+1) - \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}\right]$$

$\tilde{Z}(t+1) \sim$ i.i.d. $MVN(0, I)$ under Q

Modelling assumption: λ is constant

What values of λ_1, λ_2 are consistent with the 20b.p.'s risk premium?

Answer: 20 b.p. spread equates to

$$\lambda_1 = 0.375, \;\; \lambda_2 = 0$$

or

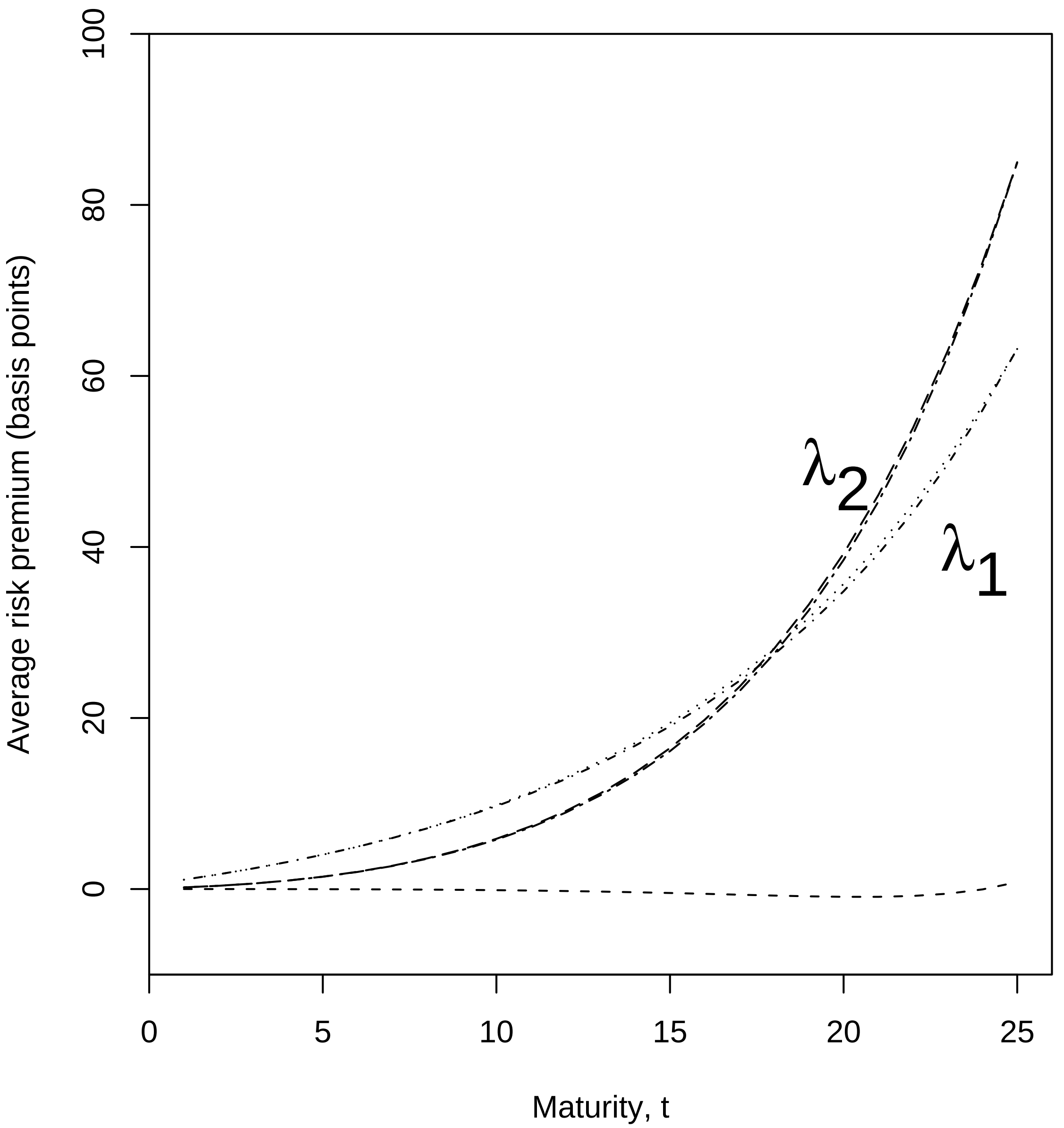
$$\lambda_1 = 0, \;\; \lambda_2 = 0.315$$

Do these values represent a *good deal*?

Why do we need to know λ_1, λ_2?

$\Rightarrow$ info. on how to price new issues in the future.

Zero-coupon Longevity bonds: avg. risk premium p.a.



Market Price of Risk 2: parameter risk

Under P: Normal-Inverse-Wishart posterior $\Rightarrow$

$$\mu = \hat{\mu} + \frac{1}{\sqrt{n}} C Z_\mu$$

Under Q

$$\mu = \hat{\mu} + \frac{1}{\sqrt{n}} C \left[\tilde{Z}_\mu - \begin{pmatrix} \lambda_3 \\ \lambda_4 \end{pmatrix} \right]$$

20 b.p.'s $\Rightarrow (\lambda_3, \lambda_4) = (1.681, 0)$ or $(0, 1.416)$

Conclusions: practical

- Life insurers and pension plans are exposed to significant systematic longevity risk
- Options:
  - bear the risk internally
  - transfer the risk to the financial markets
- Life Insurance and Pensions liabilities are huge

  ($ Trillions)
- Potential huge demand for mortality-linked securities

Conclusions: theory

- Range of frameworks possible for stochastic mortality models
- No one framework is intellectually superior to the rest

Conclusions: Challenges for the future:

- Practical: *to develop a substantial, liquid market in mortality-linked securities*
  - need to design products that are attractive for both buyers and sellers
- Theory: *Develop specific models based on these frameworks*
  - Fit each model to historical data
  - Compare different models for quality of fit