Pricing Frameworks for Securitization of Mortality Risk

Andrew Cairns

Heriot-Watt University, Edinburgh

Joint work with David Blake & Kevin Dowd

Updated version at: http://www.ma.hw.ac.uk/∼andrewc

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“BNP offers hedge against longevity”

PLAN FOR TALK

- Introduction
- Frameworks for modelling stochastic mortality
- Mortality market models
- Related issues

STOCHASTIC MORTALITY

n lives, probability p of survival, N survivors

- <u>Unsystematic mortality risk:</u>

  $\Rightarrow N|p \ \sim\ $ Binomial(n,p)

  $\Rightarrow$ risk is diversifiable, $\quad N/n \longrightarrow p \quad$ as $n \to \infty$

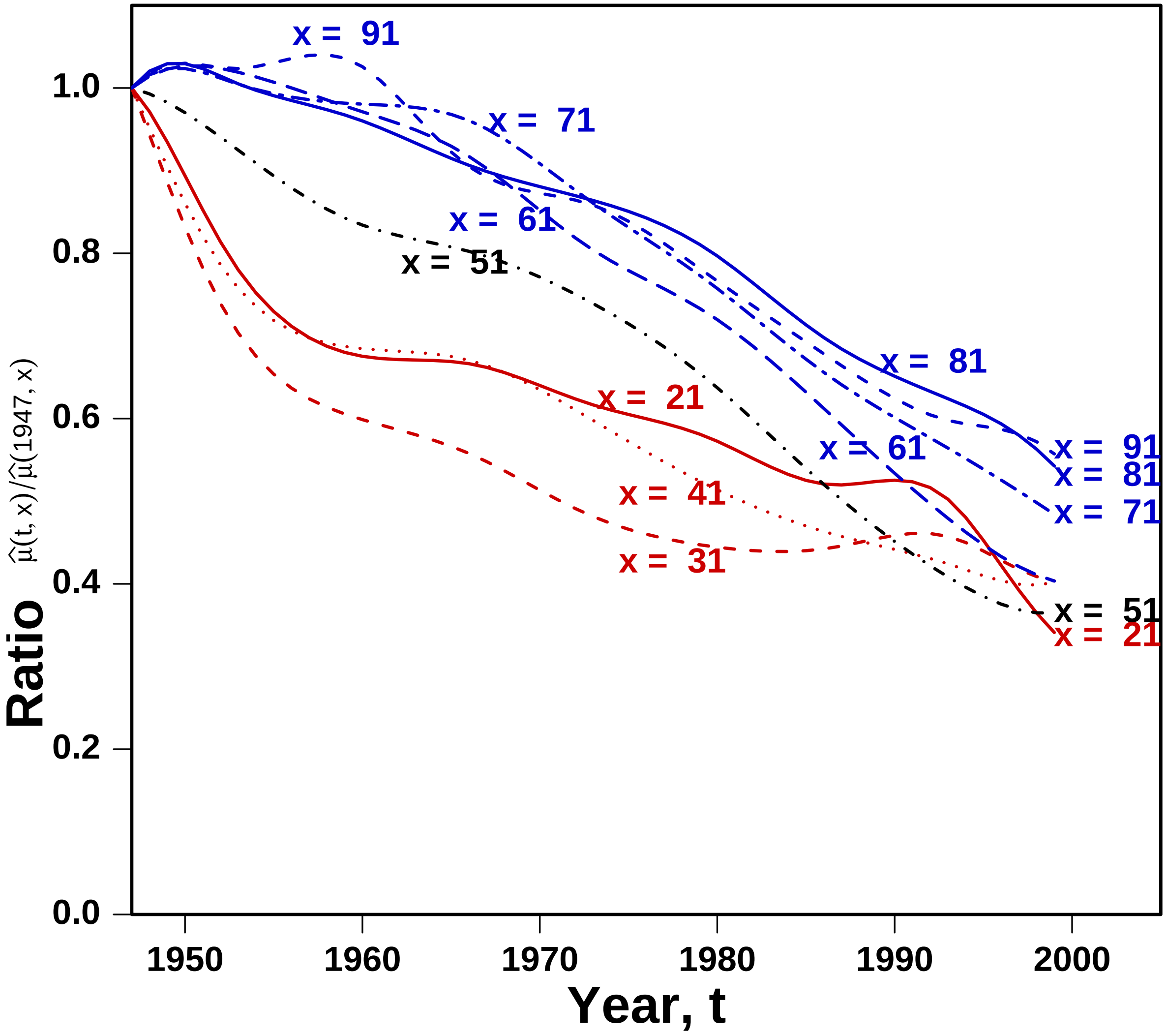
- <u>Systematic mortality risk:</u>

  $\Rightarrow p$ is uncertain

  $\Rightarrow$ risk associated with p is not diversifiable

Estimated $\hat{\mu}(t,x)/\hat{\mu}(1947,x)$

UK Males Assured Lives; 1947-1999.



STOCHASTIC MODELS FOR MORTALITY RISK

$$\begin{aligned} \mu(t,x) &= \text{force of mortality at } t \text{ for individuals aged } x \\ q(t,x) &= \text{mortality rate at } t \text{ for individuals aged } x \end{aligned}$$

Why do we need a stochastic model for $\mu(t,x)$?

- To calculate quantile reserves (VaR)
- To value mortality-linked securities
- To calculate fair values – Plain vanilla contracts

  – contracts with embedded options

MORTALITY-LINKED SECURITIES

- Short-term catastrophe bonds (Swiss Re, Dec. 2003)
- Long-term longevity bonds (BNP Paribas, Nov. 2004)

  cashflows linked to survivorship index

  (Short report available later this week.)
- Survivor swaps (some OTC contracts???)

  swap fixed for mortality-linked cashflows
- Annuity Futures

  traded contract; underlying=market annuity rates; many exercise dates

STOCHASTIC MODELLING

$$\mu(t, x) \;=\; \text{force of mortality at } t \text{ for individuals aged } x$$

$$r(t) \;=\; \text{risk-free rate of interest}$$

$r(t)$, $\mu(t, x)$ represent very different quantities

Mathematically we can treat $r(t)$, $\mu(t, x)$ as equivalent

Cash account:

$$C(t) \;=\; \exp\left[\int_0^t r(s)ds\right]$$

Survivor Index:

$$S(u,y) \;=\; \exp\left[-\int_0^u \mu(t, y+t)dt\right]$$

$=$ Prob. of survival of (y) from time 0 to time u

given knowledge of evolution of $\mu(t,x)$

FUNDAMENTAL SECURITIES

1. fixed-interest zero-coupon bonds

$$P(t,T) = \text{Price at } t \text{ for \$1 at time T}$$

2. Zero-coupon survivor (longevity) bond

$$\tilde{B}(t,T,x) = \text{Price at } t \text{ for \$ } S(T,x) \text{ at time } T$$

Approximately: BNP Paribas $= \sum_{T=1}^{25} \tilde{B}(t,T,65)$

RISK-NEUTRAL PRICING

- $P =$ real-world measure

  $\Rightarrow$ real-world dynamics for $r(t)$, $\mu(t, y)$

- $Q =$ risk-neutral pricing measure

$$P(t,T) \;=\; E_Q\left[e^{-\int_t^T r(s)ds} \,\middle|\, \mathcal{H}_t\right]$$

$$\tilde{B}(t,T,x) \;=\; E_Q\left[e^{-\int_t^T r(s)ds} S(T,x) \,\middle|\, \mathcal{H}_t\right]$$

Pricing under $Q \Rightarrow$ dynamics under P are arbitrage free

NO requirement for liquidity, or zero transaction costs

Assumption: $\mu(t,y)$ is independent of $r(t)$

$$\Rightarrow \tilde{B}(t,T,x) = E_Q\left[e^{-\int_t^T r(s)ds} \,\middle|\, \mathcal{F}_t\right] E_Q\left[S(T,x) \mid \mathcal{M}_t\right] = P(t,T)B(t,T,x)$$

where

$$B(t,T,x) = E_Q\left[S(T,x) \mid \mathcal{M}_t\right]$$

($\Rightarrow B(t,T,x)$ is a Q-martingale)

Now $\mu(s, x+s)$ is known for $0 < s < t$

$$\Rightarrow B(t,t,x) \;=\; S(t,x) = \exp\left[-\int_0^t \mu(s,x+s)ds\right]$$

and

$$\begin{aligned}
\frac{B(t,T,x)}{B(t,t,x)} &= E_Q\left[e^{-\int_t^T \mu(s,x+s)ds} \,\middle|\, \mathcal{M}_t\right] \\
&= \text{risk-neutral probability at } t \text{ that } (x+t) \\
&\qquad\qquad \text{survives from time } t \text{ to time } T \\
&= \text{spot survival probability, } p_Q(t,T,x)
\end{aligned}$$

TYPES OF STOCHASTIC MORTALITY MODEL

We can use the same frameworks as interest-rate modelling: $\mu(t, x + t)$ is equivalent to $r(t)$

(but we might not use the same models!!!)

- Short-rate modelling framework (e.g. CIR)
- Forward-rate modelling framework (e.g. HJM)
- Positive-interest framework (e.g. Flesaker-Hughston)
- Market Models (e.g. BGM)

SHORT-RATE MODELLING FRAMEWORK

n-factor models

$$d\mu(t,y) = a(t,y)dt + b_1(t,y)d\tilde{W}_1(t) + \ldots + b_n(t,y)d\tilde{W}_n(t)$$

Examples:

- Lee & Carter (1992) and followers (discrete time)
- Milevsky & Promislow (2001), Dahl (2004) (continuous time)

MARKET MODELS

Interest rates:

- Forward swap rates are log-normal (Jamshidian, 1997)
- Forward LIBOR rates are L-N (Brace-Gatarek-Musiela, 1997)

Mortality:

- Forward life annuity rates are L-N
- Forward Survivor Credit Offer Rates (SCOR) are L-N

THE ANNUITY MARKET MODEL

Forward annuity contract:

- N_t policyholders enter into forward contract at t
- No money exchanges hands at t
- N_T survivors at T out of N_t
- Survivors at T pay \$ 1 at T

  and receive an annuity of \$ F in arrears from T
- Deaths before $T \Rightarrow$ contract expires worthless

$x =$ age of policyholders at time 0.

The fair forward annuity rate at t is

$$F(t,T,x) = \frac{P(t,T)B(t,T,x)}{\sum_{s=T+1}^{\infty} P(t,s)B(t,s,x)} = \frac{\tilde{B}(t,T,x)}{X(t)}$$

where

$$\begin{aligned} X(t) &= \sum_{s=T+1}^{\infty} \tilde{B}(t,s,x) \\ &= \text{price at } t \text{ for \$ 1 at } T+1,\ T+2,\ldots \end{aligned}$$

provided still alive

$X(t) > 0$ is a tradeable asset

Therefore we can find a probability measure P_X under which $\frac{V(t)}{X(t)}$ is a martingale for any other tradeable asset $V(t)$.

i.e. under P_X, $F(t,T,x)$ is a martingale.

Martingale under $P_X \Rightarrow$

$$dF(t,T,x) \;=\; F(t,T,x)\left[\gamma_P(t)dZ^X(t) + \gamma_B(t)dW^X(t)\right]$$

where

$$Z^X(t) \;=\; \text{interest-rate risk, } P_X\text{-Brownian Motion}$$

$$W^X(t) \;=\; \text{mortality-rate risk, } P_X\text{-Brownian Motion}$$

Key assumption:

$\gamma_P(t)$ and $\gamma_B(t)$ are deterministic functions

$\Rightarrow F(T,T,x)$ is log-normal

EXAMPLE: GUARANTEED ANNUITY OPTION

Surviving policyholders at T have a guarantee:

For each \$ 1 at T they will receive an annuity of *at least* \$ K.

Total value of option component at T is

$$G(T) = N_T \max\{K - F(T,x),\ 0\}\frac{1}{F(T,x)}$$

Value of option at t, with N_t policyholders at t

$$G(t) = \frac{N_t}{S(t,x)} X(t) \left(K\Phi(-d_2) - F(t,x)\Phi(-d_1) \right)$$

$$\begin{aligned} \text{where } d_1 &= \frac{\log F(t,x)/K + \frac{1}{2}\sigma_F^2}{\sigma_F} \\ d_2 &= d_1 - \sigma_F \end{aligned}$$

$$\text{and } \sigma_F^2 = \int_t^T (|\gamma_P(u,x)|^2 + |\gamma_B(u,x)|^2) du.$$

THE *SCOR* MARKET MODEL

(*SCOR* $=$ Survivor Credit Offer Rate)

Income drawdown example:

- Pool of identical policyholders, age x at time 0
- N_t survivors at time t
- $E_Q[N_{t+1}|\mathcal{M}_t] = N_t \, p_Q(t, t+1, x)$
- Policyholder i has fund $F_i(t)$ at t^-, *draws* $P_i(t)$ at t^+
- Fund reverts to insurer if the policyholder dies

- $F_i(t+1) = (F_i(t) - P_i(t)) \times (1 + \tilde{\beta}(t+1))$

- $\tilde{\beta}(t+1)$ is a bonus rate to cover investment returns and survivorship

- $(F_i(t) - P_i(t))\beta(t+1) =$ Survivor Credit

If we set $\tilde{\beta}(t+1)$ *at* t, the economically fair value is:

$$\begin{aligned} \tilde{\beta}(t+1) &= \frac{1 - P(t, t+1)p_Q(t, t+1, x)}{P(t, t+1)p_Q(t, t+1, x)} \\ &= \text{Survivor Credit Offer Rate} \end{aligned}$$

Forward SCOR: Define

$$S(t,T,T+1,x) = \frac{\tilde{B}(t,T,x) - \tilde{B}(t,T+1,x)}{\tilde{B}(t,T+1,x)}$$

i.e. fix at time t *the bonus rate payable at* $T+1$

Then

$$0 < \frac{1 - P(T,T+1)p_Q(T,T+1,x)}{P(T,T+1)p_Q(T,T+1,x)} = S(T,T,T+1,x) < \infty$$

$\Rightarrow$ we can choose to model $S(t,T,T+1,x)$ as a

Log-Normal process.

COMPARISON OF MARKET MODELS

- Annuity Market Model
  - ‘elegant’ solution for specific valuation problem
  - BUT different models required for different problems
- SCOR Market Model
  - less elegant
  - more flexible (‘one model fits all’)

SCOR Market Model details:

$$\begin{aligned} dP(t,T) &= P(t,T)\Big[r(t)dt + V_P(t,T)d\tilde{Z}(t)\Big] \\ dB(t,T,x) &= B(t,T,x)V_B(t,T,x)d\tilde{W}(t) \end{aligned}$$

$T_0 =$ key future valuation date for cohort x

Let $T_k = T_0 + k$.

Let $P_{k,x}$ be the measure under which $S(t, T_{k-1}, T_k, x)$ is a martingale.

For compactness write $S_k(t) = S(t, T_{k-1}, T_k, x)$.

Martingale property $\Rightarrow$

$$dS_k(t) = S_k(t)\left(\phi_{Bk}(t)dW^k(t) + \phi_{Pk}(t)dZ^k(t)\right)$$

for $k = 1, 2, \ldots,$ where

$$\begin{aligned}
\frac{S_k(t)}{(1+S_k(t))}\phi_{Bk}(t) &= V_B(t, T_{k-1}, x) - V_P(t, T_k, x) \\
\frac{S_k(t)}{(1+S_k(t))}\phi_{Pk}(t) &= V_P(t, T_{k-1}) - V_P(t, T_k)
\end{aligned}$$

For valuing at t embedded options maturing at T_1 we need to simulate all rates under the same measure.

For example, simulating under $P_{1,x}$, for $k > 1$, substitute

$$dW^k(t) \;=\; dW^1(t) + \sum_{j=2}^{k} \frac{S_j(t)}{1+S_j(t)} \phi_{Bj}(t)dt$$

$$dZ^k(t) \;=\; dZ^1(t) + \sum_{j=2}^{k} \frac{S_j(t)}{1+S_j(t)} \phi_{Zj}(t)dt$$

Option pricing

$\pi(T_1) =$ option payoff at T_1

Depends on $S(T_1, T_{k-1}, T_k, x), \;\; P(t, T_k)$ for $k = 2, 3, \ldots$

$\pi(t) =$ option value at $t \Rightarrow$

$$\frac{\pi(t)}{\tilde{B}(t, T_1, x)} = E_{P_{1,x}} \left[\frac{\pi(T_1)}{\tilde{B}(T_1, T_1, x)} \;\middle|\; \mathcal{H}_t \right]$$

RELATED ISSUES

- Possible criteria for stoch. mortality models
- Types of investor
- Basis risk

POSSIBLE CRITERIA FOR STOCHASTIC MORTALITY MODELS

- $\mu(t,x) > 0$ for all $t,\ x$
- Model consistent with historical data
- Future dynamics should be *biologically* reasonable
- Complexity of model appropriate for task in hand
- Model allows fast numerical computation

TYPES OF INVESTOR

- Want to hedge mortality risk
  - Annuity providers, pension plans
  - Life offices, reinsurers
- Portfolio diversification / Return enhancement
  - Investment banks
  - Hedge funds
  - ....

BASIS RISK

How well does the underlying index match the hedger's own mortality risks?

Example: Index $=$ UK pensioner mortality

- Hedger 1 $=$ Mexican life insurer $\Rightarrow$ high basis risk
- Hedger 2 $=$ UK pension plan $\Rightarrow$ low basis risk

High basis risk $\Rightarrow$ security may not be worth buying.

BASIS RISK

Issuer of security has to make potential investors feel confident about what they are buying

- Index $=$ buyers/sellers own mortality risks

  $\Rightarrow$ moral hazard/distrust

  $\Rightarrow$ no sales

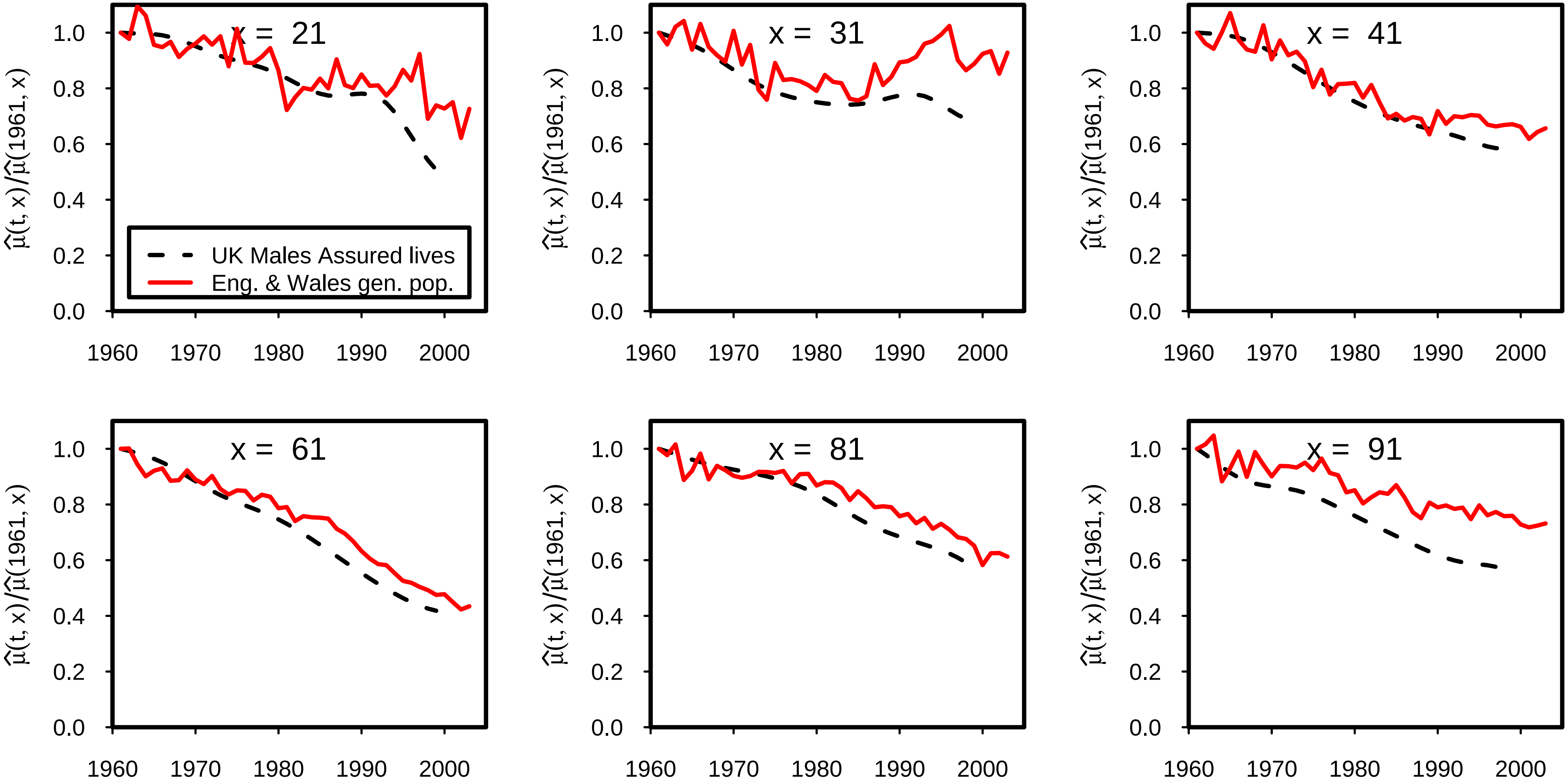
- Independent index (e.g. national mortality)

  $\Rightarrow$ greater investor confidence

  BUT more basis risk

Basis risk: UK Assured lives versus E&W population

Mortality rates relative to 1961 values



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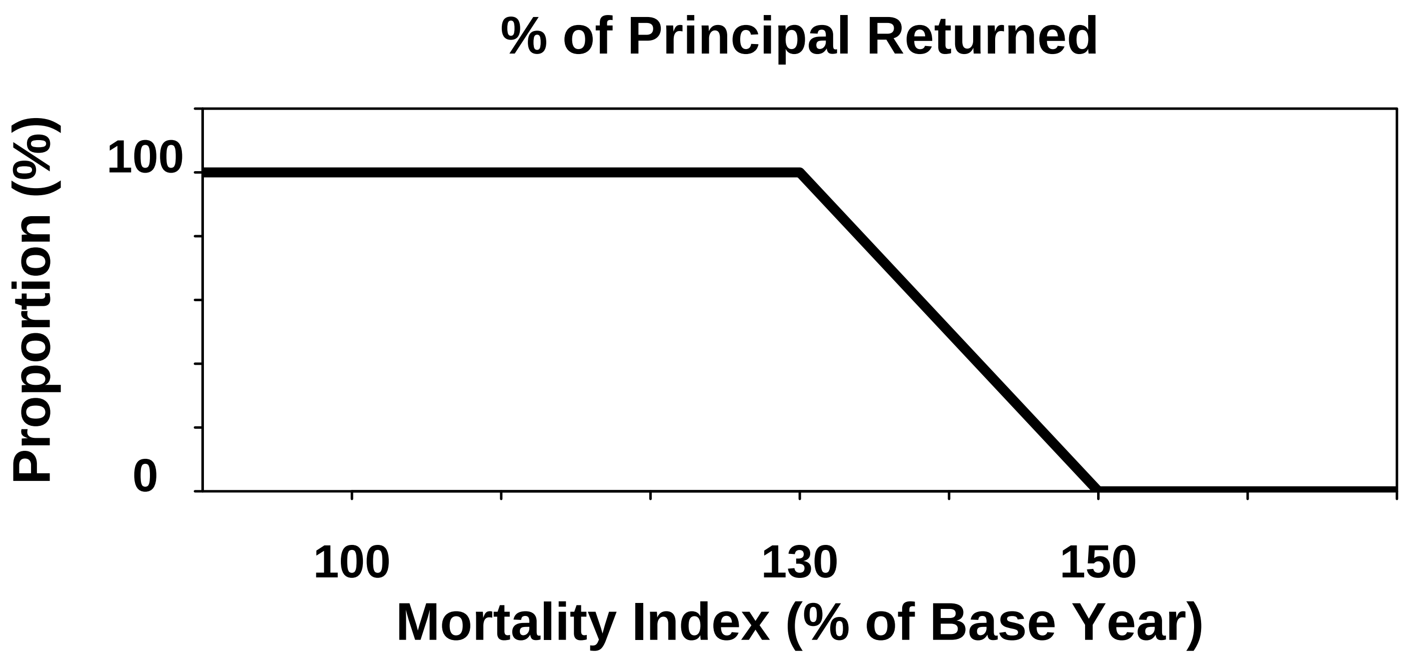
Related to basis risk:

Measurement error

$\Rightarrow$ don't issue 0-1 securities

e.g. Swiss Re catastrophe bond



TIME LAG

Mortality data takes time to report and analyse

- Swiss Re bond

  Principal reduced only if a catastrophe

  $\Rightarrow$ delay in repayment only if mortality $>$ c.130%

- BNP Paribas bond:

  Payment at T linked to mortality at $T-2$
