MODELS FOR STOCHASTIC MORTALITY WITH PARAMETER UNCERTAINTY

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Working paper: Cairns, Blake, Dowd, Coughlan, Epstein, Ong, Balevich (2007)

A quantitative comparison of stochastic mortality models using data from England & Wales and the United States

Plan

- Introduction
- Approaches to modelling mortality improvements
- A two-factor model for stochastic mortality
- Application
  - The survivor index
- Adding in a cohort effect
- Conclusions
- Pricing: $P \longrightarrow Q$

The facts about mortality:

- Life expectancy is increasing.
- Future development of life expectancy is uncertain.

"Longevity risk"

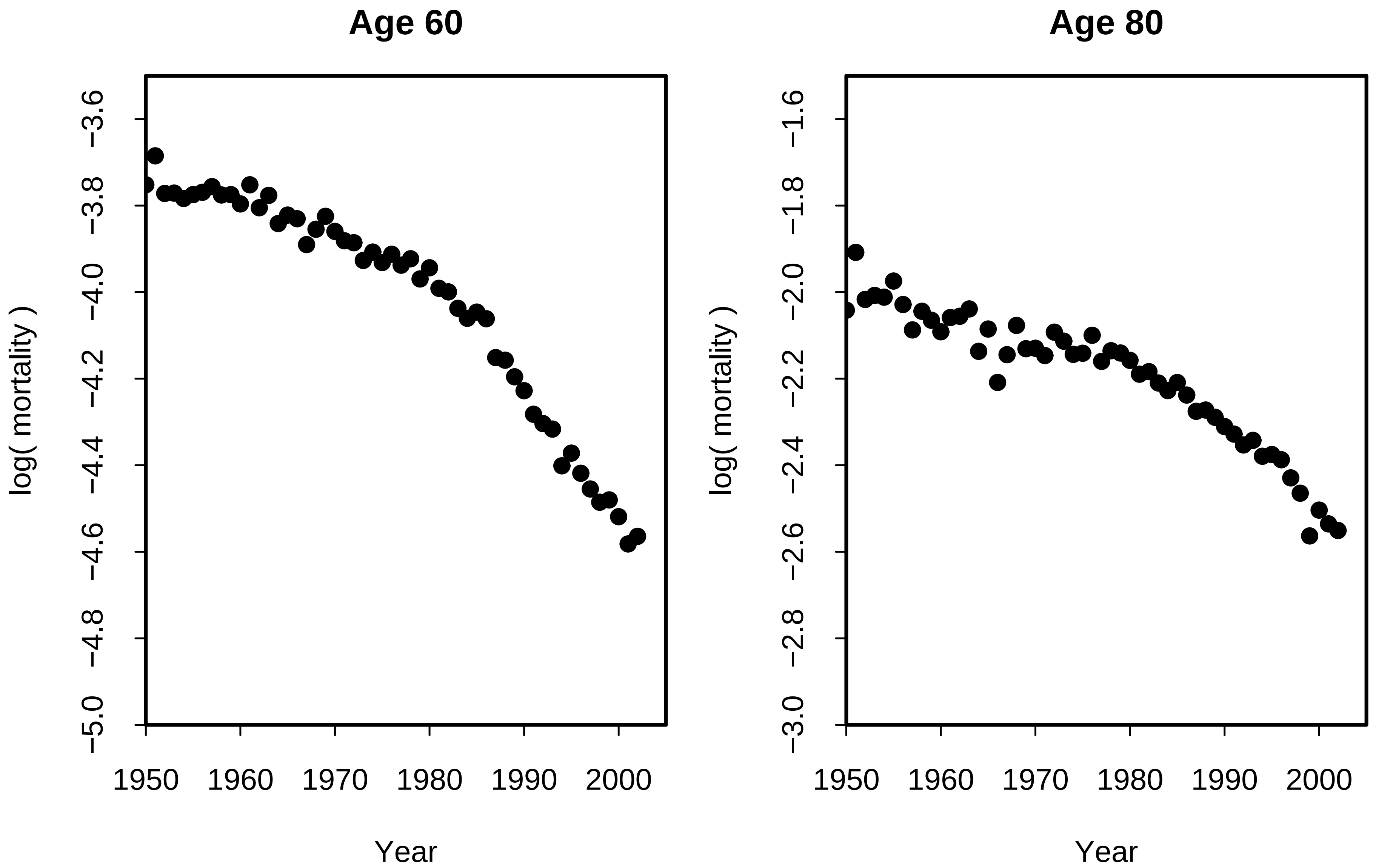
Longevity Risk $=$ the risk that aggregate future mortality rates are lower <u>than anticipated</u>

Focus here: Mortality rates above age 60

Where is stochastic mortality relevant?

- Risk management in general
- Pension plans: what level of reserves?
- Life insurance contracts with embedded options.
- Pricing and hedging longevity-linked securities.

England and Wales log mortality rates 1950-2002



Stochastic Models

Different approaches to modelling

- Lee-Carter
- P-splines
- Parametric, time-series models
- Market models
- Age-Period-Cohort extensions

Stochastic Models

Limited historical data $\Rightarrow$

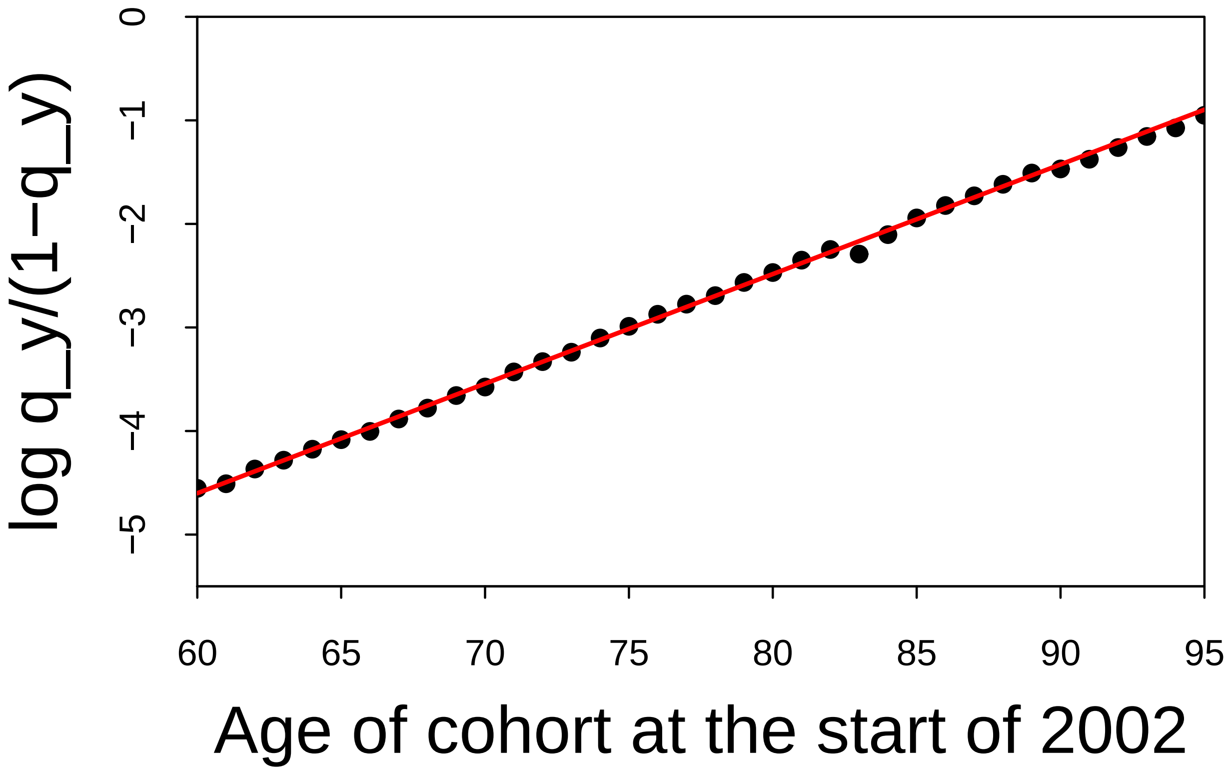
- No single model is ‘the right one’

  limited data $\Rightarrow$ Model risk

- Even with the right model

  limited data $\Rightarrow$ Parameter risk

Case study: England and Wales males, age 60-95



$q_y =$ mortality rate at age y in 2002

Data suggests $\log q_y/(1-q_y)$ is linear

"PARAMETRIC" TIME-SERIES MODELS

$q(t, x)$ Mortality rate for the year t to $t + 1$ for individuals aged x at t:

General class of models

$$\text{logit } q(t, x) = \sum_{i=1}^{N} \beta_x^{(i)} \kappa_t^{(i)} \gamma_{t-x}^{(i)}$$

"Parametric" $\Rightarrow$ $\beta_x^{(i)}$ is a simple function of x

Estimation

- Data: Deaths $D(t,x)$, Exposures $E(t,x)$

  $\Rightarrow$ Crude death rates $\hat{m}(t,x) = D(t,x)/E(t,x)$

- Underlying $m(t,x) = -\log[1-q(t,x)]$

  (by assumption)

- $D(t,x) \sim$ independent Poisson$\Big(m(t,x)E(t,x)\Big)$

- Maximum likelihood $\Rightarrow \hat{\beta}_x^{(i)}$, $\hat{\kappa}_t^{(i)}$ and $\hat{\gamma}_{t-x}^{(i)}$

Excursion: Data issues

- $D(t,x)$ Deaths reasonably accurate
- $E(t,x)$ Exposures are estimates
  - even in census years
  - US data $\Rightarrow$ generally very unreliable!
  - E+W data $\Rightarrow$ better, except 1886 cohort
  - impact on Poisson assumption needs further study

TWO PARAMETRIC TIME-SERIES MODELS

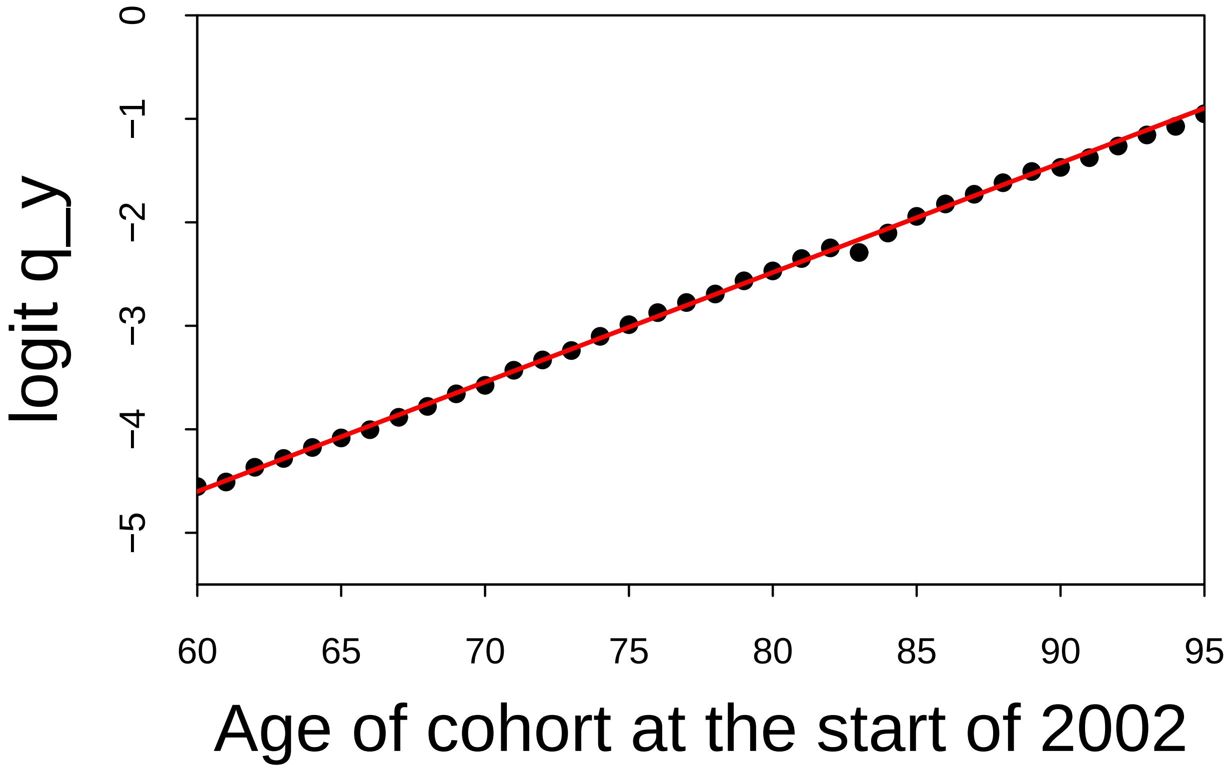
Model 1 (Age-Period model):

$$\text{logit}\, q(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x})$$

Model 2 (Age-Period-Cohort model):

$$\begin{aligned}\text{logit}\, q(t,x) &= \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) \\ &\quad +\kappa_t^{(3)}[(x - \bar{x})^2 - \sigma_x^2] \\ &\quad +\gamma_{t-x}^{(4)}\end{aligned}$$

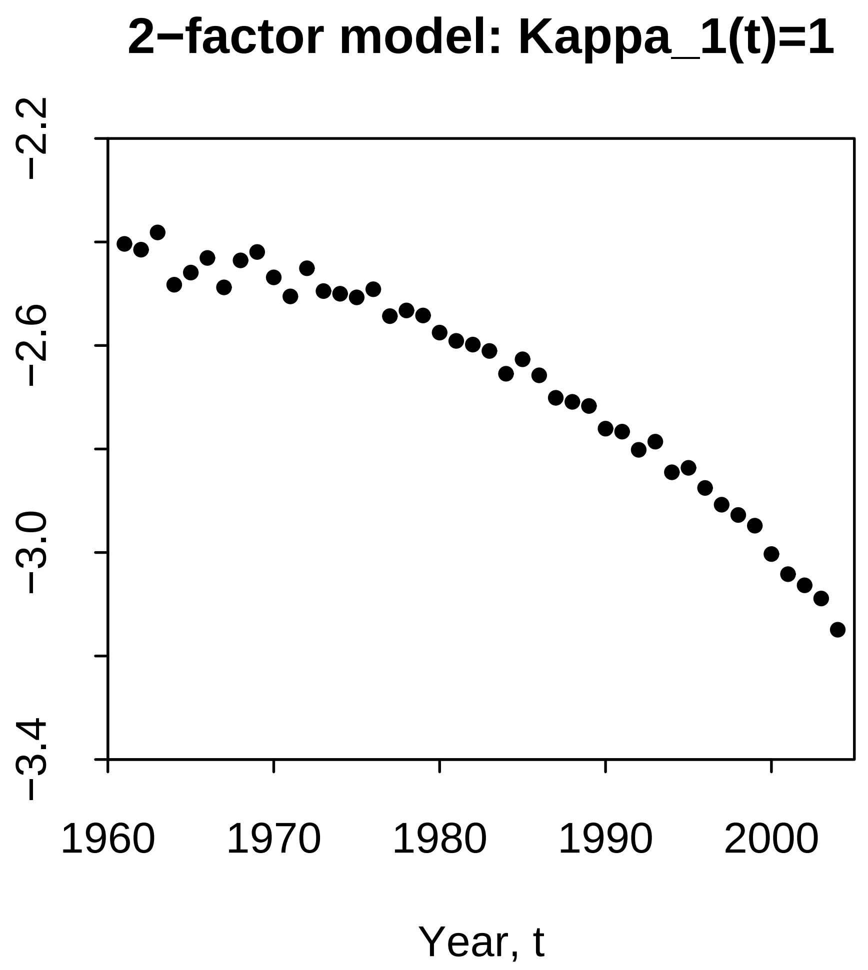
Model 1: Case study – England and Wales males

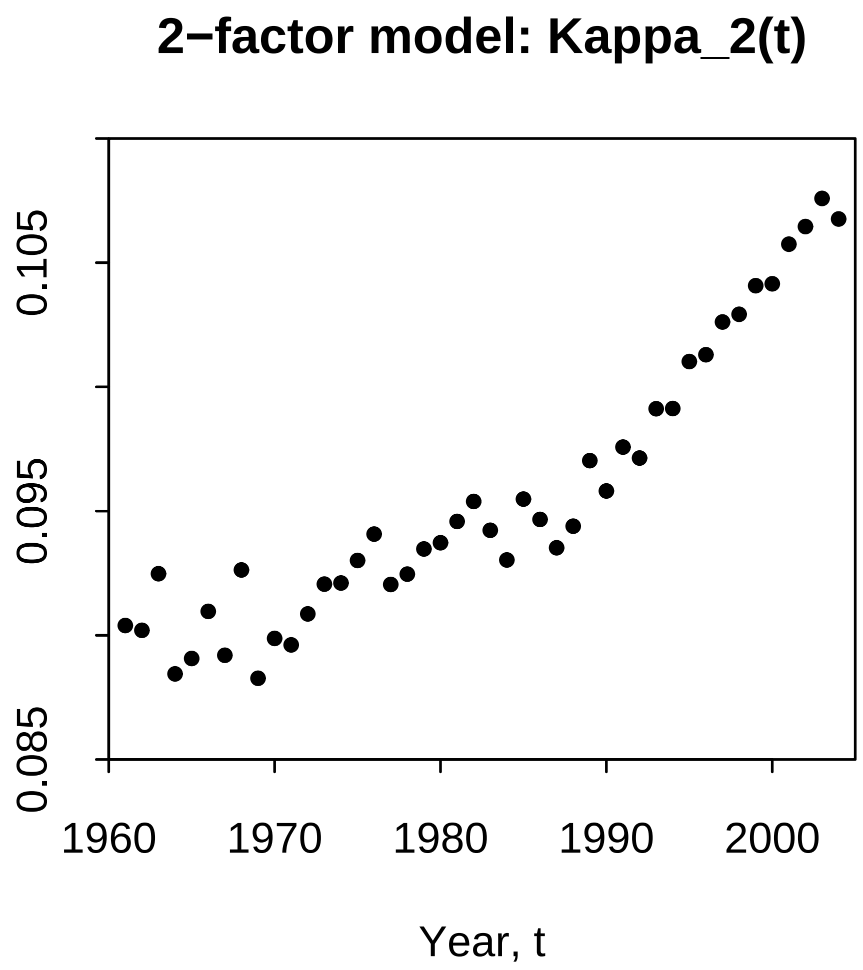


$\kappa_t^{(1)} \Rightarrow$ level

$\kappa_t^{(2)} \Rightarrow$ slope

Model 1


2−factor model: Kappa_1(t)=1
−2.2
−2.6
−3.0
−3.4
1960
1970
1980
1990
2000
Year, t


2−factor model: Kappa_2(t)
0.105
0.095
0.085
1960
1970
1980
1990
2000
Year, t

$\kappa_t = (\kappa_t^{(1)}, \kappa_t^{(2)})'$

Model: Random walk with drift

$$\kappa_{t+1} - \kappa_t = \mu + CZ(t+1)$$

- $\mu = (\mu_1, \mu_2)' =$ drift
- $V = CC' =$ variance-covariance matrix
- Estimate μ and V
- Quantify parameter uncertainty in μ and V

WHY 2 FACTORS? (i.e. $\kappa_t^{(1)}$ and $\kappa_t^{(2)}$)

Data suggest changes in *underlying* mortality rates are not perfectly correlated across ages.

1 factor (e.g. most Lee-Carter-based models)
$\Rightarrow$ changes over time in the $q(t,x)$ are perfectly correlated.

Bayesian approach to parameter uncertainty

- Jeffreys prior $p(\mu, V) \propto |V|^{-3/2}$.
- Data: vector $D(t) = \kappa_t - \kappa_{t-1}$ for $t = 1, \ldots, n$
- MLE's: $\hat{\mu}$ and $\hat{V}$.
- Posterior:

$$\begin{aligned} V^{-1}|D &\sim \text{Wishart}(n-1, n^{-1}\hat{V}^{-1}) \\ \mu|V, D &\sim MVN(\hat{\mu}, n^{-1}V) \end{aligned}$$

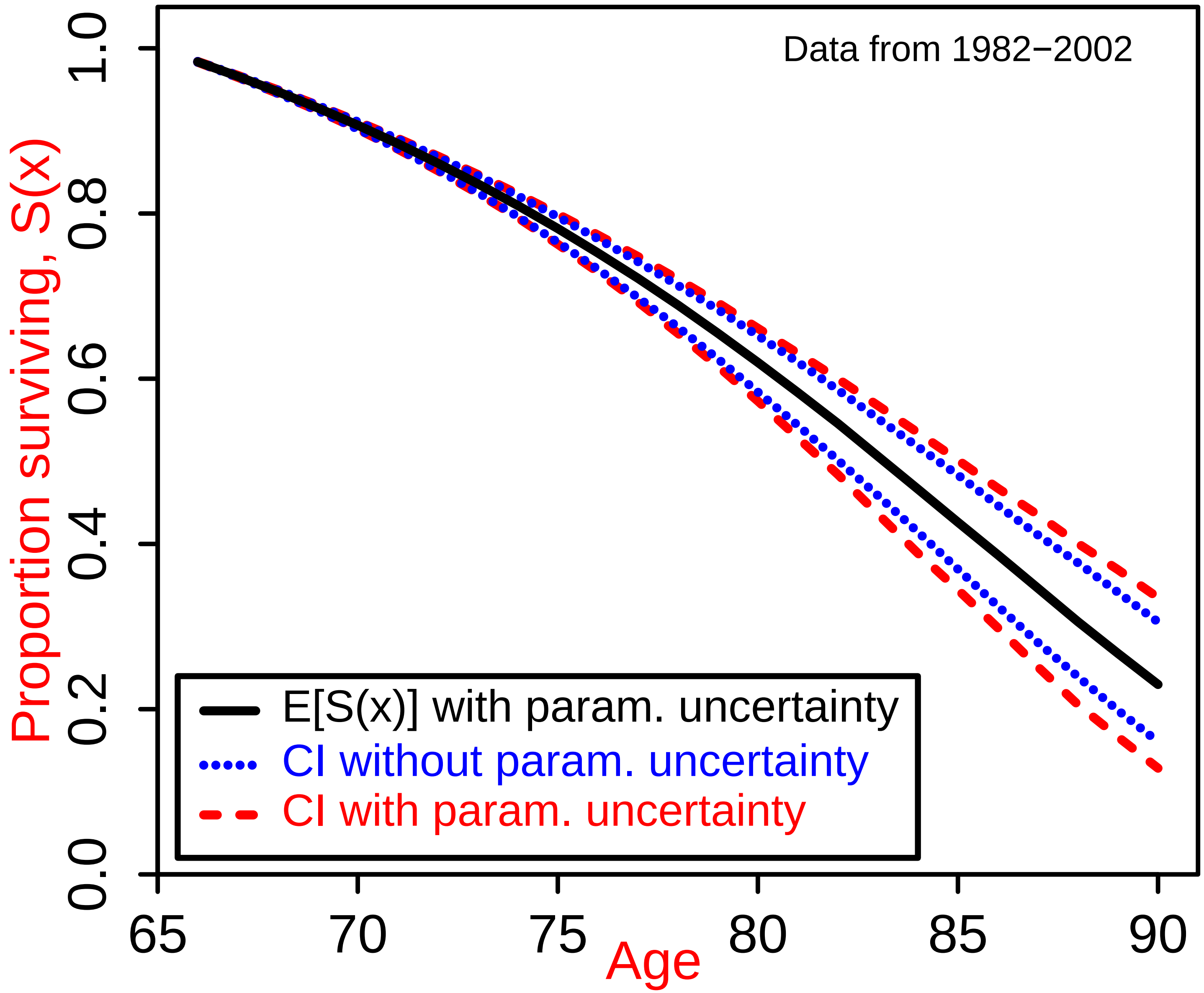
Application: cohort survivorship

- Cohort: Age x at time $t = 0$
- $S(t, x) =$ survivor index at t

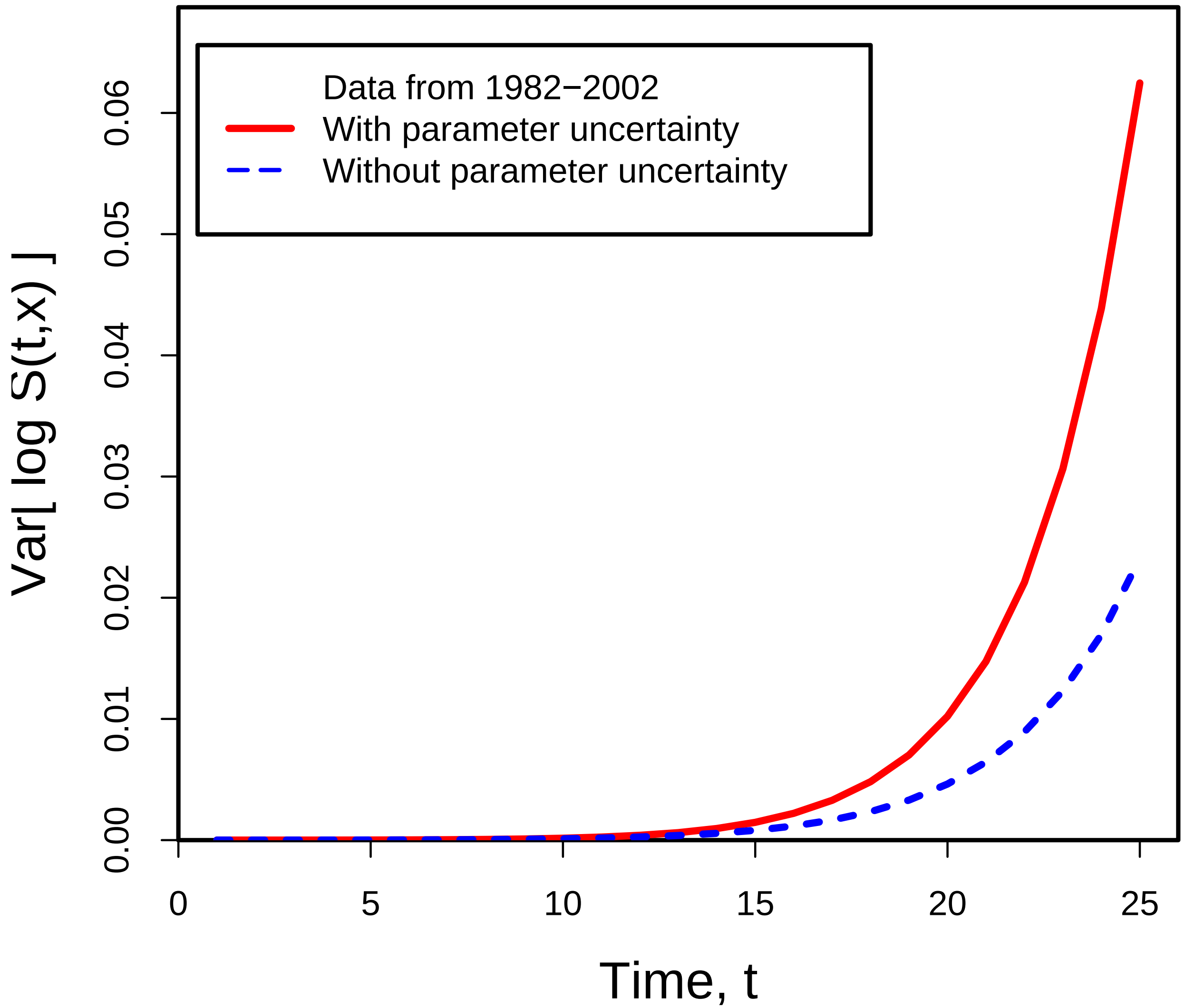
  proportion surviving from time 0 to time t

$$
\begin{aligned}
S(t, x) \;&=\; (1 - q(0, x)) \times (1 - q(1, x + 1) \times \ldots \\
&\qquad \ldots \times (1 - q(t - 1, x + t - 1))
\end{aligned}
$$

90% Confidence Interval (CI) for Cohort Survivorship



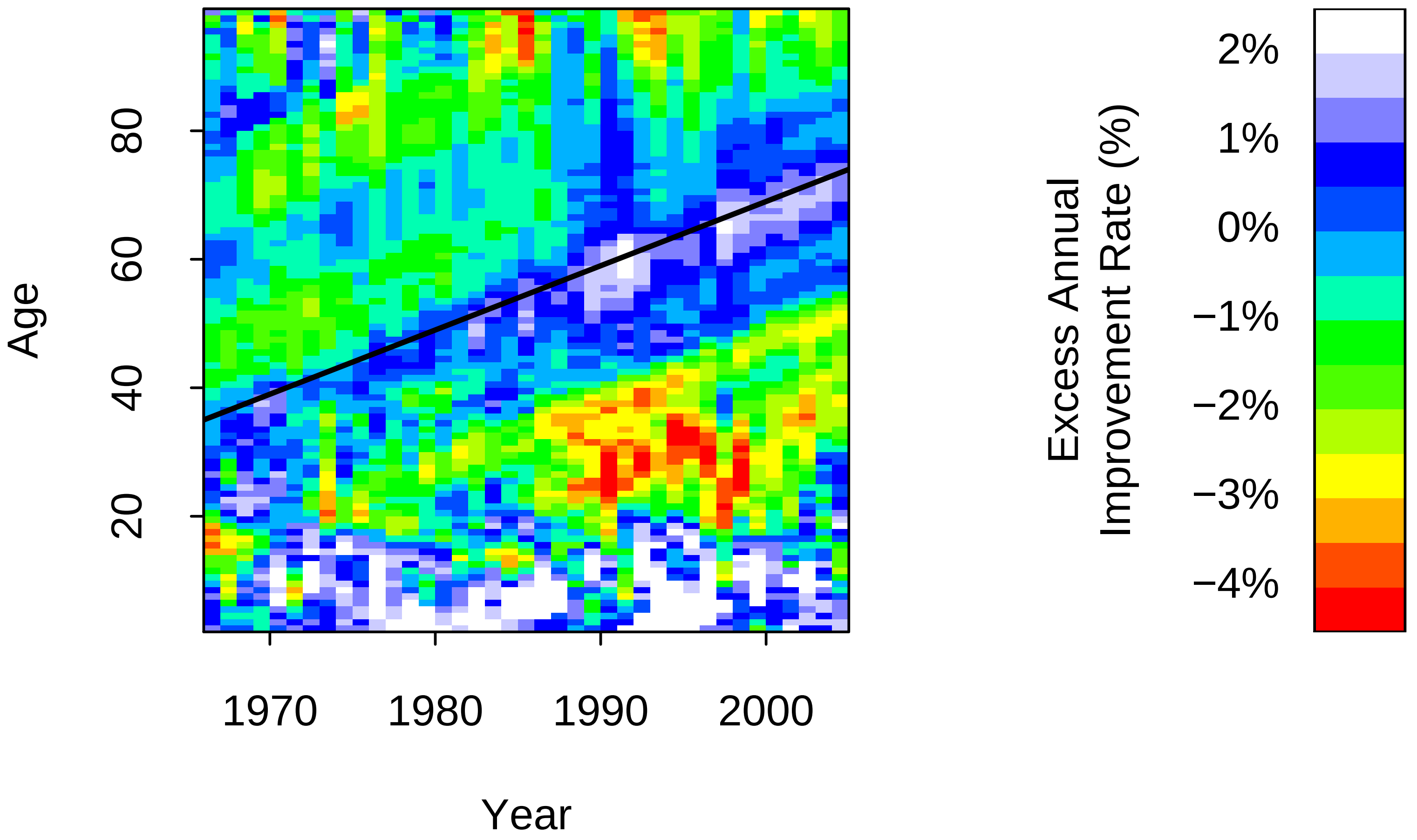
$Var[\log S(t,x)]$ for $x = 65$



Cohort Survivorship: General Conclusions

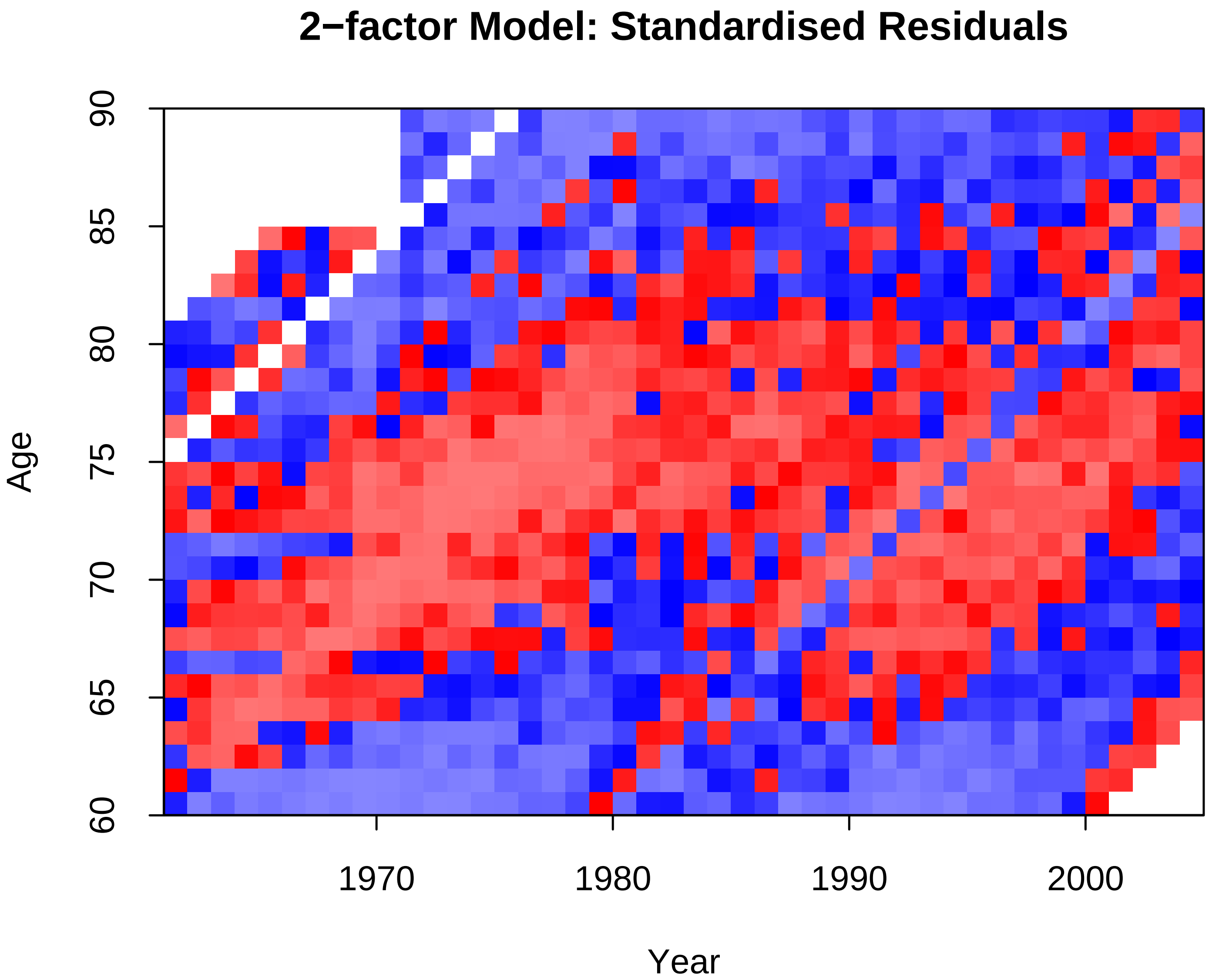
- Less than 10 years:
  - Systematic risk not significant
- Over 10 years
  - Systematic risk becomes more and more significant over time
- Over 20 years
  - Model and parameter risk begin to dominate

The cohort effect: England and Wales



Mortality improvement relative to calendar year average.

The Cohort Effect



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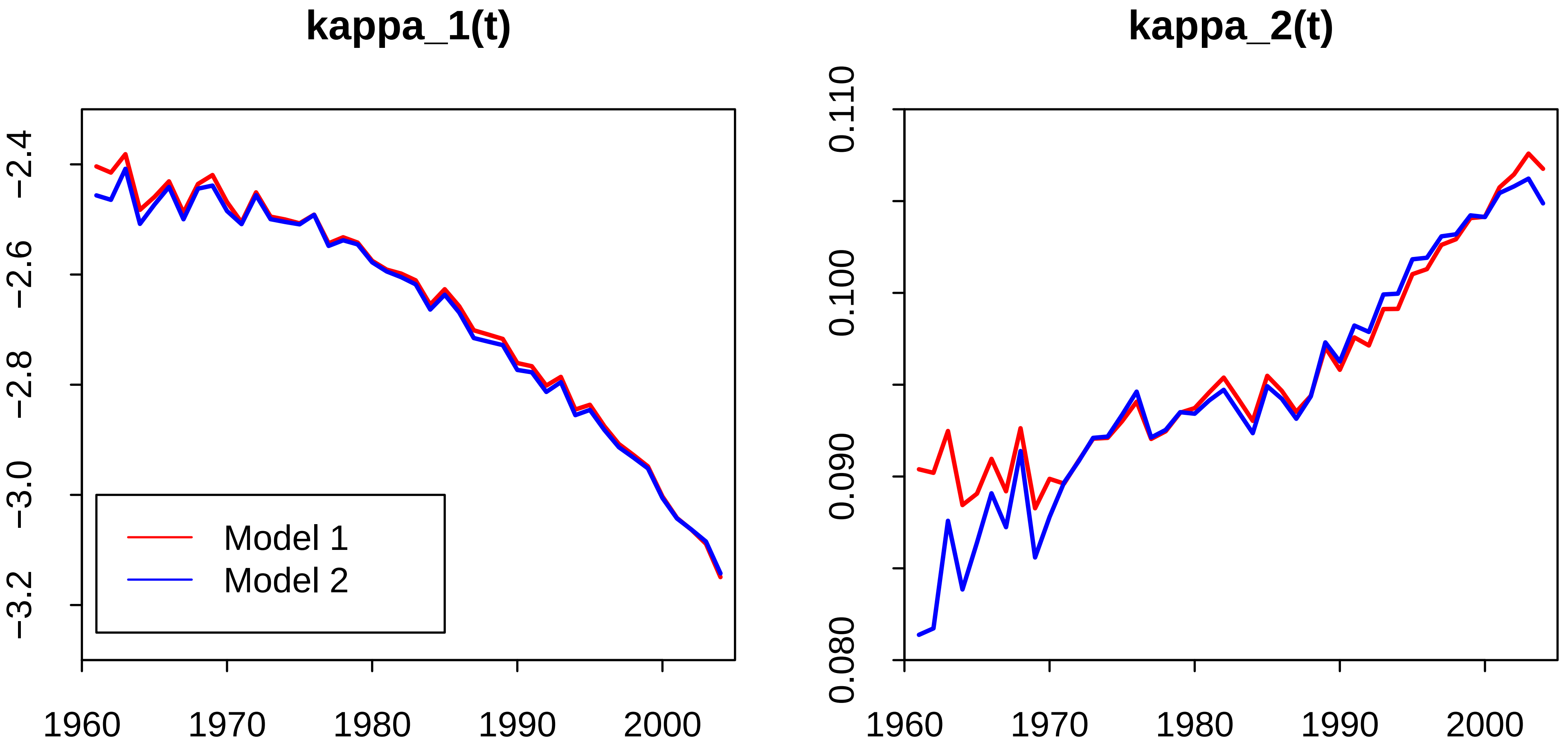
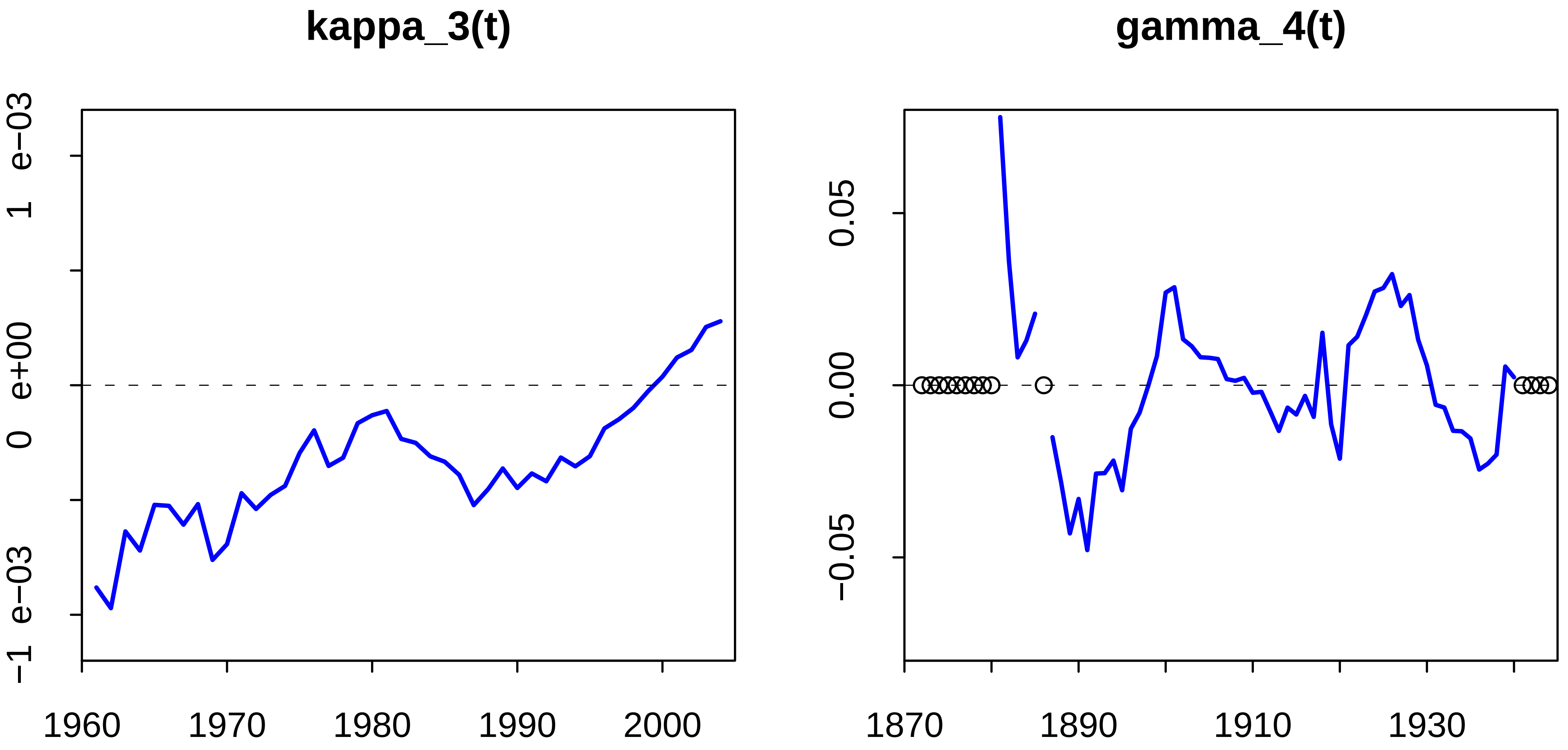
(e.g. see Renshaw & Haberman (2006))

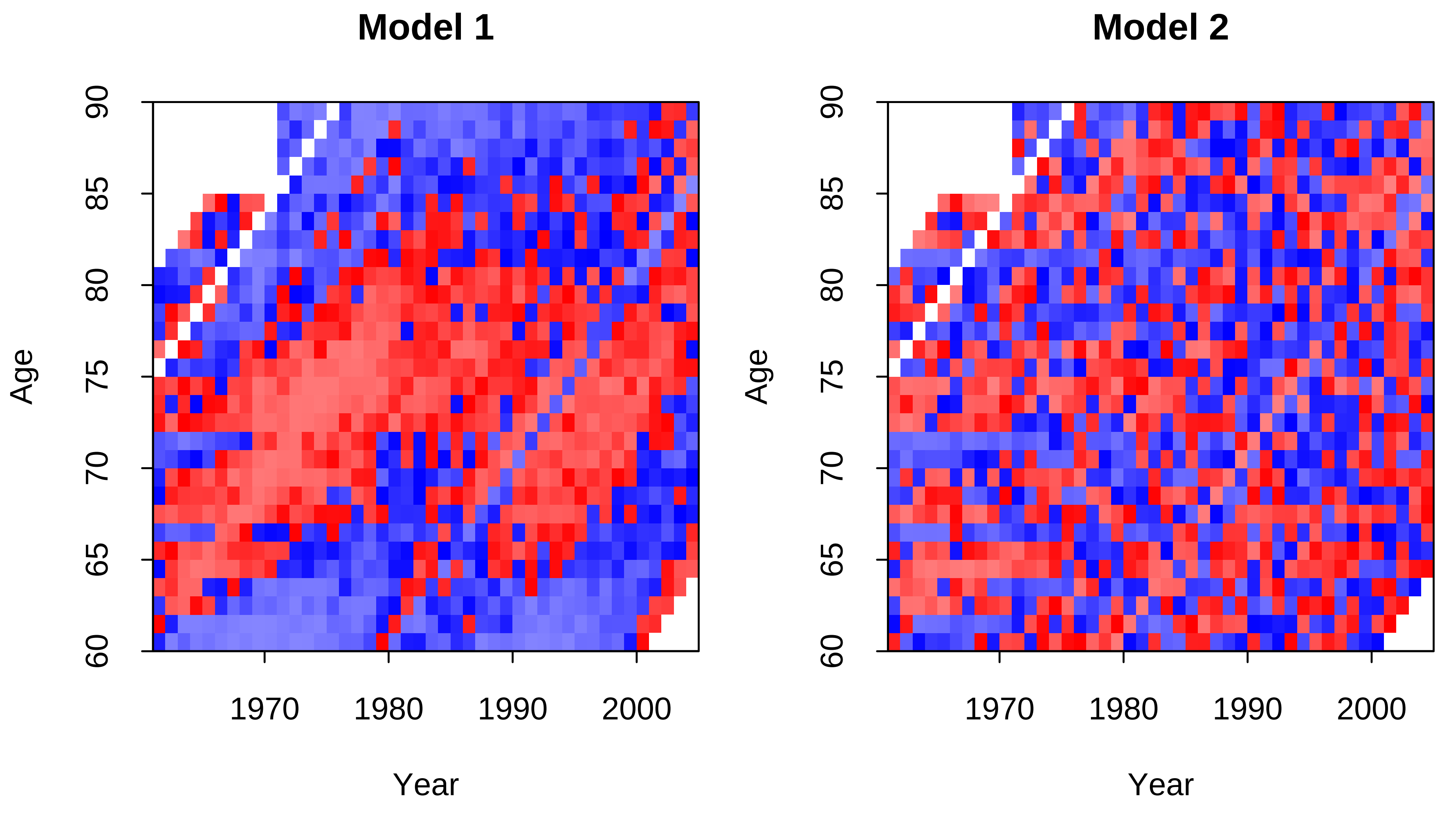
Model 1 versus Model 2



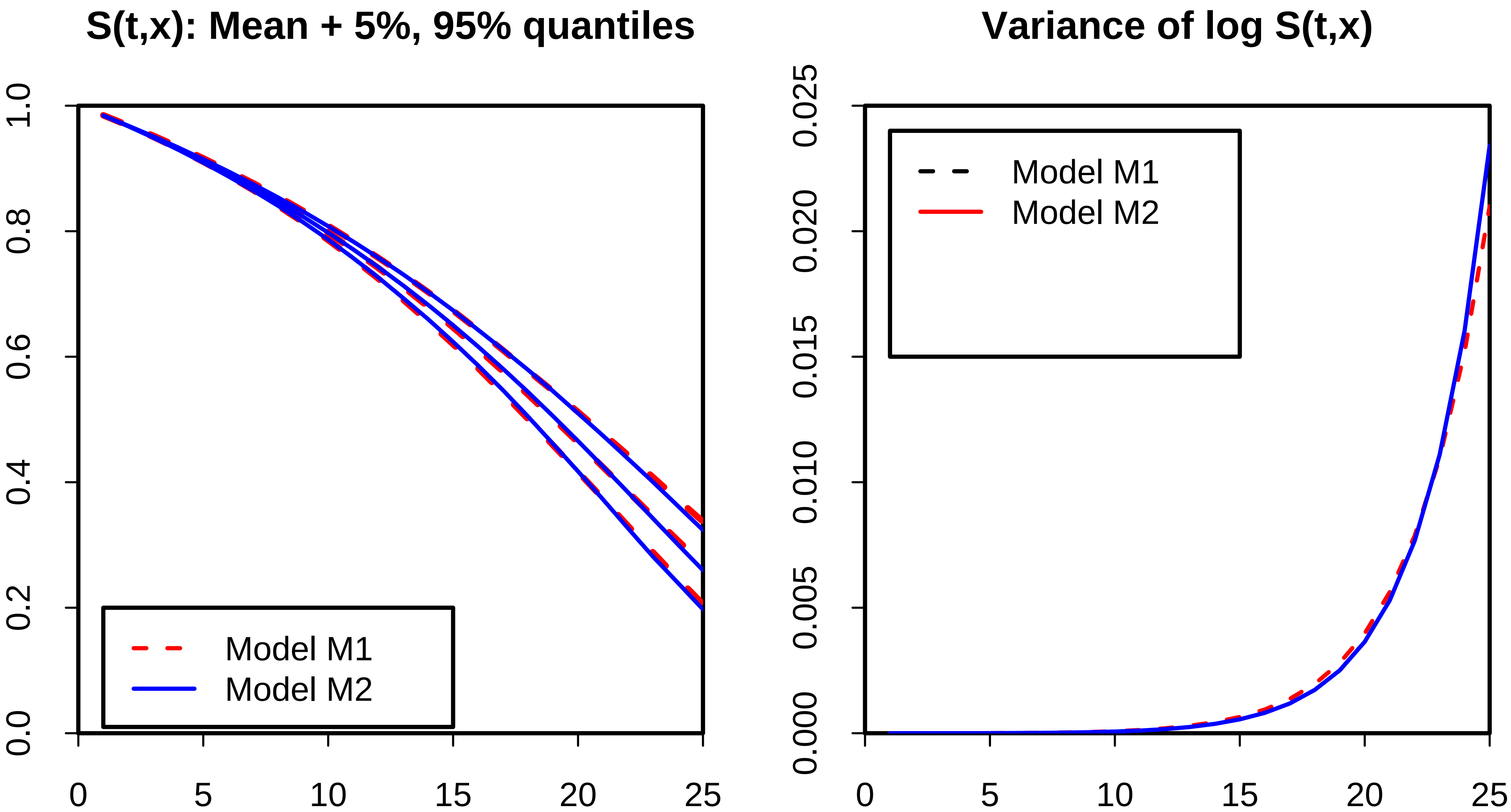
Model 2: extra factors



Standardised residuals



Survivor index projections



4% Annuity Values

<table>
<tr><th rowspan="2"></th><th rowspan="2">Model 1</th><th colspan="2">Model 2</th></tr>
<tr><th>$\gamma_{1944}^{(4)} = -0.0398$</th><th>$\gamma_{1944}^{(4)} = 0.0402$</th></tr>
<tr><td>$x = 60$</td><td>13.472</td><td>13.557</td><td>13.350</td></tr>
<tr><td>$x = 65$</td><td>11.449</td><td colspan="2">11.451</td></tr>
<tr><td>$x = 70$</td><td>9.325</td><td colspan="2">9.354</td></tr>
<tr><td>$x = 75$</td><td>7.220</td><td colspan="2">7.240</td></tr>
</table>

Conclusions 1

- Stochastic models important for
  - risk measurement and management
  - valuing life policies with option characteristics
- Two models out of many possibilities
- Significant longevity risk in the medium/long term

Conclusions 2

- Parameter risk is important
- Model risk might be important
- The significance of longevity risk varies from one problem to the next:
  - In absolute terms
  - As a percentage of the total risk

How do you price a longevity bond?

- Hedgers are prepared to pay a premium
- Two approaches:
  - Take *real-world* expected values

    use a risk-adjusted discount rate
  - Take *risk-adjusted* expected values

    use the risk-free discount rate

Risk-neutral pricing (risk-adjusted expected values)

$$\begin{pmatrix} \kappa_{t+1}^{(1)} \\ \kappa_{t+1}^{(2)} \end{pmatrix} - \begin{pmatrix} \kappa_{t}^{(1)} \\ \kappa_{t}^{(2)} \end{pmatrix}$$

$$= \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \tilde{Z}_1(t+1) + \lambda_1 \\ \tilde{Z}_2(t+1) + \lambda_2 \end{pmatrix}$$

where $\tilde{Z}_1(t+1)$ and $\tilde{Z}_2(t+1)$ are i.i.d. $\sim N(0,1)$

<u>under a risk-neutral pricing measure $Q(\lambda)$</u>

λ_1 and λ_2 are market prices of risk

How does the market price of risk work?

- Two independent sources of risk $Z_1(t)$, $Z_2(t)$
- Tradeable security has corresp. volatilities σ_1, σ_2
- Market price of risk is

  the additional expected return over the risk free rate

  per unit of risk
- Hence

$$\text{Risk premium} \quad = \left(\sigma_1\lambda_1 + \sigma_2\lambda_2\right)$$

Comments

- The market is highly incomplete
- The switch from P to Q is a modelling assumption
- (Simple) Key assumption:

  market prices of risk λ_1 and λ_2 are constant.
- As a market develops this assumption becomes a testable hypothesis

$\leq$ One data point: the EIB-BNP longevity bond

- Offer price (ultimately unsuccessful) $\Rightarrow$

  average risk premium of 20 basis points

  (paid by the buyer of the bond to the seller)

  if held to maturity

- What values of λ_1, λ_2 are consistent with the 20b.p.'s risk premium?

- One price, two parameters $\Rightarrow$ many solutions

Answer: 20 b.p. spread equates to

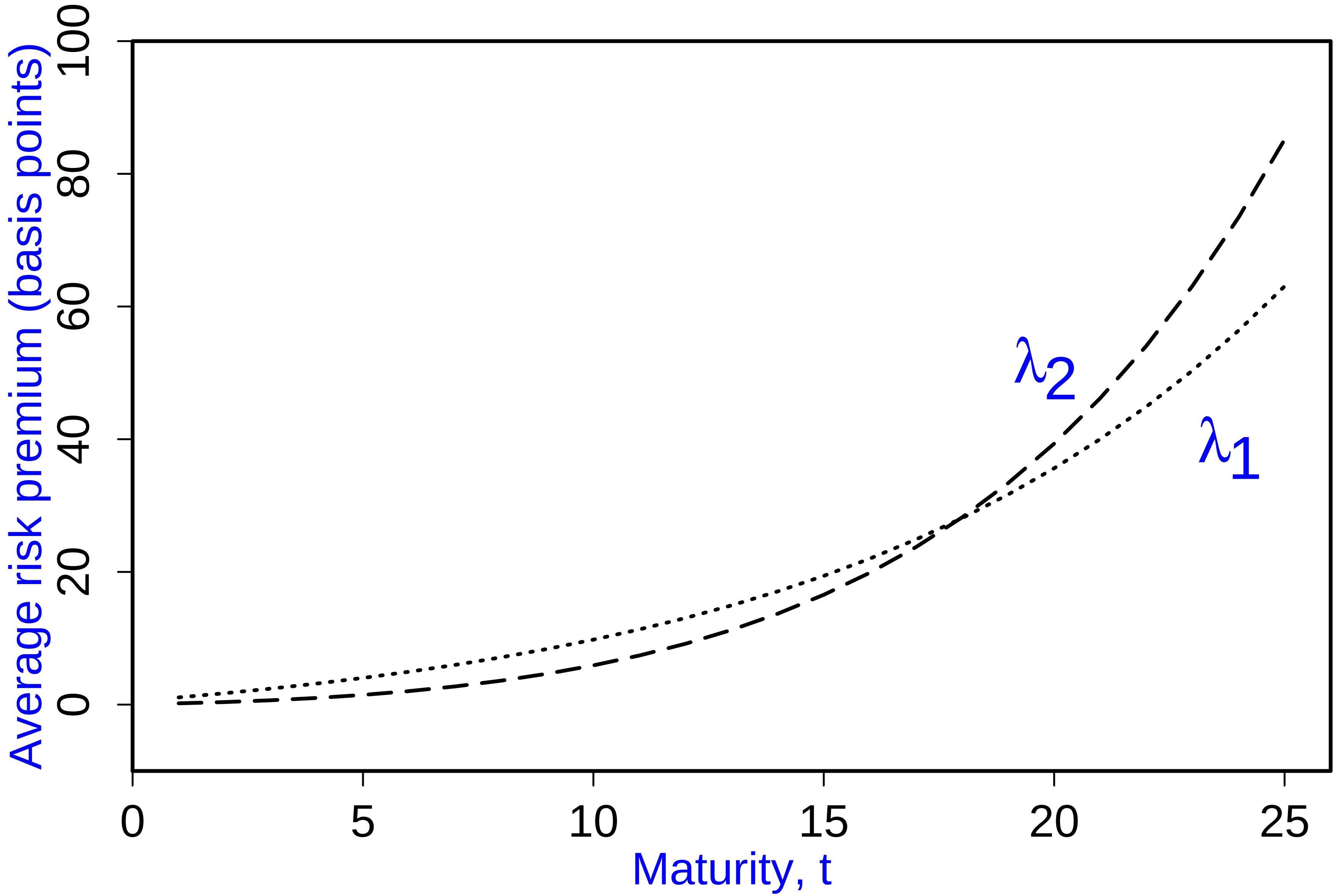
$$\begin{array}{ccc} \lambda_1 = & 0.375, & \lambda_2 = & 0 \\ & \downarrow & & \downarrow \\ \lambda_1 = & 0, & \lambda_2 = & 0.315 \end{array}$$

Do these values represent a *good deal*?

Why do we need to know λ_1, λ_2?

$\Rightarrow$ info. on how to price new issues in the future.

Zero-coupon Longevity bonds: avg. risk premium p.a.



Longevity Bond Risk Premiums: $\lambda = (0.375, 0)$

Dependency on term and initial age:

		Initial age of cohort, x		
		60	65	70
Bond	20	8.9	14.7	23.1
Maturity	25	12.7	20.0	28.7
T	30	16.9	24.3	31.5