A MULTIFACTOR GENERALISATION OF THE OLIVIER-SMITH MODEL FOR STOCHASTIC MORTALITY

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PLAN FOR TALK

- Motivating example
- Systematic and non-systematic mortality risk
- Mathematical concepts
  - forward survival probabilities
  - zero-coupon survivor bonds
- Short-rate versus Market models
- The Olivier-Smith model
- An open problem involving copulas

THE ANNUITY GUARANTEE PROBLEM – simplified

- Policyholder will retire at time T
- Lump sum k available for annuity purchase
- Market annuity rate: \$ $a(T)$ per \$ 1 of annuity
- Guaranteed purchase price of \$ g per \$ 1 of annuity
- Value of option at T is thus

$$V(T) = \frac{k}{g} \max\{a(T) - g \ , \ 0\}$$

THE ANNUITY GUARANTEE PROBLEM – simplified

- Value of option at T is thus

$$V(T) = \frac{k}{g} \max\{a(T) - g\ ,\ 0\}$$

- What is the price of the option at time $t < T$?
- Looking forwards from time t:
  - What is the distribution of $a(T)$?
  - What mortality table is in use at time T?

    Mortality market models tackle this problem.

STOCHASTIC MORTALITY

n lives, probability p of survival, N survivors

- Unsystematic mortality risk:

  $\Rightarrow N|p \sim$ Binomial(n, p)

  $\Rightarrow$ risk is diversifiable, $N/n \longrightarrow p$ as $n \to \infty$

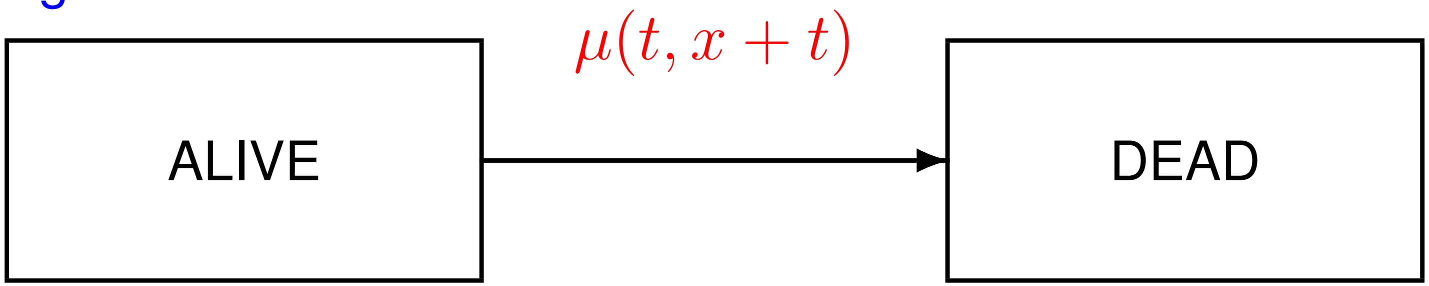
- Systematic mortality risk:

  $\Rightarrow p$ is uncertain

  $\Rightarrow$ risk associated with p is not diversifiable

MORTALITY MODEL

Initial age x



- $\mu(t, y) =$ transition intensity at t, age y at t
- $S(t, x) = \exp\left[- \int_0^t \mu(u, x+u) du \right]$

  $=$ survivor index

Filtration: $\mathcal{M}_t \Rightarrow$

history of $\mu(u, y)$ up to time t, for all ages y

$\mathcal{M}_t \not\Rightarrow$ individual histories

Single individual aged x at time 0:

$$I(t) = \begin{cases} 1 & \text{if alive at } t \\ 0 & \text{otherwise} \end{cases}$$

$$Pr(\, I(t) = 1 \,|\, I(0) = 1,\ \mathcal{M}_t\,) \quad = \quad S(t, x)$$

$$Pr(\, I(t) = 1 \,|\, I(0) = 1,\ \mathcal{M}_0\,) \quad = \quad E[S(t, x)|\mathcal{M}_0]$$

FORWARD SURVIVAL PROBABILITIES

Real-world probabilities, P

For $T_1 > T_0$, and any t

$$p_P(t, T_0, T_1, x) = Pr_P\Big(I(T_1) = 1 \,\Big|\, I(T_0) = 1,\ \mathcal{M}_t\Big)$$
$$= \frac{E_P[S(T_1)|\mathcal{M}_t]}{E_P[S(T_0)|\mathcal{M}_t]}$$

Pricing measure $Q \sim P$

$$p_Q(t, T_0, T_1, x) = Pr_Q\Big(I(T_1) = 1 \,\Big|\, I(T_0) = 1,\ \mathcal{M}_t\Big)$$

BRIEF DIGRESSSION

$P \longrightarrow Q$ covers:

- change in dynamics of $\mu(t, x+t)$
- change in individual histories given $\mu(t, x+t)$

Zero-coupon survivor bonds

$B(t, T, x) =$ price at t for $S(T, x)$ payable at T

for simplicity: assume interest rates are zero

The market is abritrage-free if there exists $Q \sim P$ under which the $B(t, T, x)$ are martingales, for all T, x

$$
\begin{aligned}
B(t,T,x) &= E_Q[S(T,x)|\mathcal{M}_t] \\
&= p_Q(t,0,T,x) \\
&= p_Q(1,0,1,x) \times \ldots \times p_Q(t,t-1,t,x) \\
&\quad \times p_Q(t,t,t+1,x) \times \ldots \times p_Q(t,T-1,T,x)
\end{aligned}
$$

At $t+1$:

$$
\begin{aligned}
\rightarrow \quad & p_Q(1,0,1,x) \times \ldots \times p_Q(t,t-1,t,x) \\
& \times p_Q(t+1,t,t+1,x) \\
& \times p_Q(t+1,t+1,t+2,x) \times \ldots \times p_Q(t+1,T-1,T,x)
\end{aligned}
$$

TWO TYPES OF MODEL – (Interest-rate terminology:)

- Short-rate models: (state variable $X(t)$)

  $\Rightarrow$ model for dynamics of $p_Q(t, t-1, t, x)$ for all x

  as a function of $X(t)$

  Forward survival probabilities are output

- Market (forward-rate) models:

  $\Rightarrow$ model for dynamics of $p_Q(t, T-1, T, x) \; \forall \; T, x$

  Forward survival probabilities are input

SHORT-RATE MODELS: state variable $X(t)$

- Good for pricing zero-coupon survivor bonds and longevity bonds   e.g. by simulation up to T

$$
\begin{aligned}
B(t,T,x) \;=\; & p(t,0,t,x) \times \\
& E_Q[p_Q(t+1,t,t+1,x) \times \ldots \\
& \ldots \times p_Q(T,T-1,T,x) \mid X(t)\,]
\end{aligned}
$$

- Very few biologically reasonable models have an analytical form for $B(t,T,x)$ as a function of $X(t)$

SHORT-RATE MODELS: Markov state variable $X(t)$

Pricing annuity guarantees is difficult: Recall

$$V(T) = \frac{k}{g}\max\{a(T) - g\ ,\ 0\}$$

- $a(T) \equiv a(T, X(T)) =$

  price at T for annuity of $1 per annum from time T.

$$a(T, X(T)) = \sum_{u=T}^{\infty} v^{u-T} p_Q(T, T, u, x;\ X(T))]$$

- No analytical form for

$$p_Q(T,T,u;X(T)) \equiv B(T,u,x)/B(T,T,u)$$

$\Rightarrow$ evaluating $V(T)$ is computationally expensive

- Hence pricing annuity guarantees is difficult using short-rate models

Plus point:

- Statistically: very flexible

MARKET MODELS

- NOT good for pricing zero-coupon survivor bonds and longevity bonds prices are *input* at time 0
- Prices are *output*

$\Rightarrow$ for all $u \geq T$, $B(T, u, x)$ is automatically available at T

$\Rightarrow$ Calculating $a(T)$ is easy

$\Rightarrow$ Pricing annuity guarantees (more) straightforward

THE OLIVIER & SMITH MODEL

$$p_Q(t+1, T-1, T, x) = p_Q(t, T-1, T, x)^{b(t,T-1,T,x)G(t+1)}$$

- Discrete time
- $G(t+1) \sim \text{Gamma}(\alpha, \alpha)$ under Q
- $b(t+1, T-1, T, x) =$ bias correction factors $\Rightarrow$

$$E_Q\left[p_Q(t+1, t, T, x) | \mathcal{M}_t\right] = p_Q(t, t, T, x)$$

WHY GAMMA?

- $0 < p_Q(t, T-1, T, x) < 1$

$$\Rightarrow \quad 0 < p_Q(t+1, T-1, T, x) < 1$$

- Gamma + martingale property of $p_Q(u, t, T, x)$ for $u = t, t+1$ implies

$$b(t, T, T+1, x) \quad = \quad - \frac{\alpha p_Q(t, t, T, x)^{-1/\alpha} \left(p_Q(t, T, T+1, x)^{-1/\alpha} - 1 \right)}{\log p_Q(t, T, T+1, x)}$$

(Note $b \approx 1$ if α is large and p close to 1.)

$\Rightarrow$ exact simulation in discrete time possible

STATISTICALLY: IS IT A GOOD MODEL?

- Same $G(t+1)$ applies to all $p_Q(t+1, T-1, T, x)$

  $\Rightarrow$

  – Single-factor model

  – No flexibility over the volatility term structure (except through the choice of α)

- Model $\Rightarrow$ testable hypothesis

STATISTICALLY IS IT A GOOD MODEL?

- Problem: there is *no market*

  $\Longrightarrow$ no forward survival probabilities

- Compromise:

  Concentrate on observed 1-year survival probabilities

STATISTICALLY IS IT A GOOD MODEL?

- Assumption: for age x at time t

$$p_Q(t, t, t+1, x-t) = \theta(x) p_Q(t, t-1, t, x-(t-1))$$

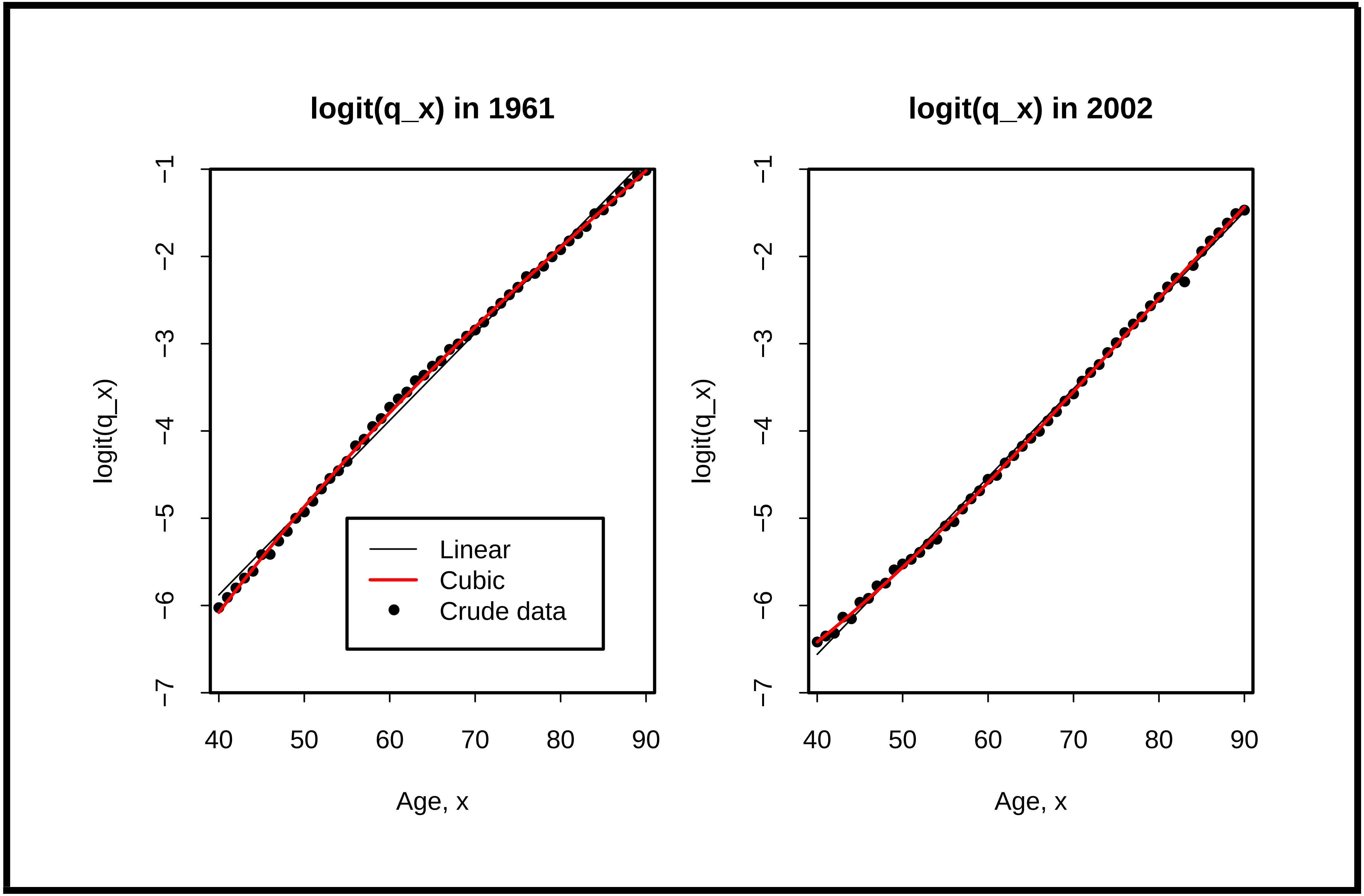
- $\theta(x) =$ age x predicted improvement

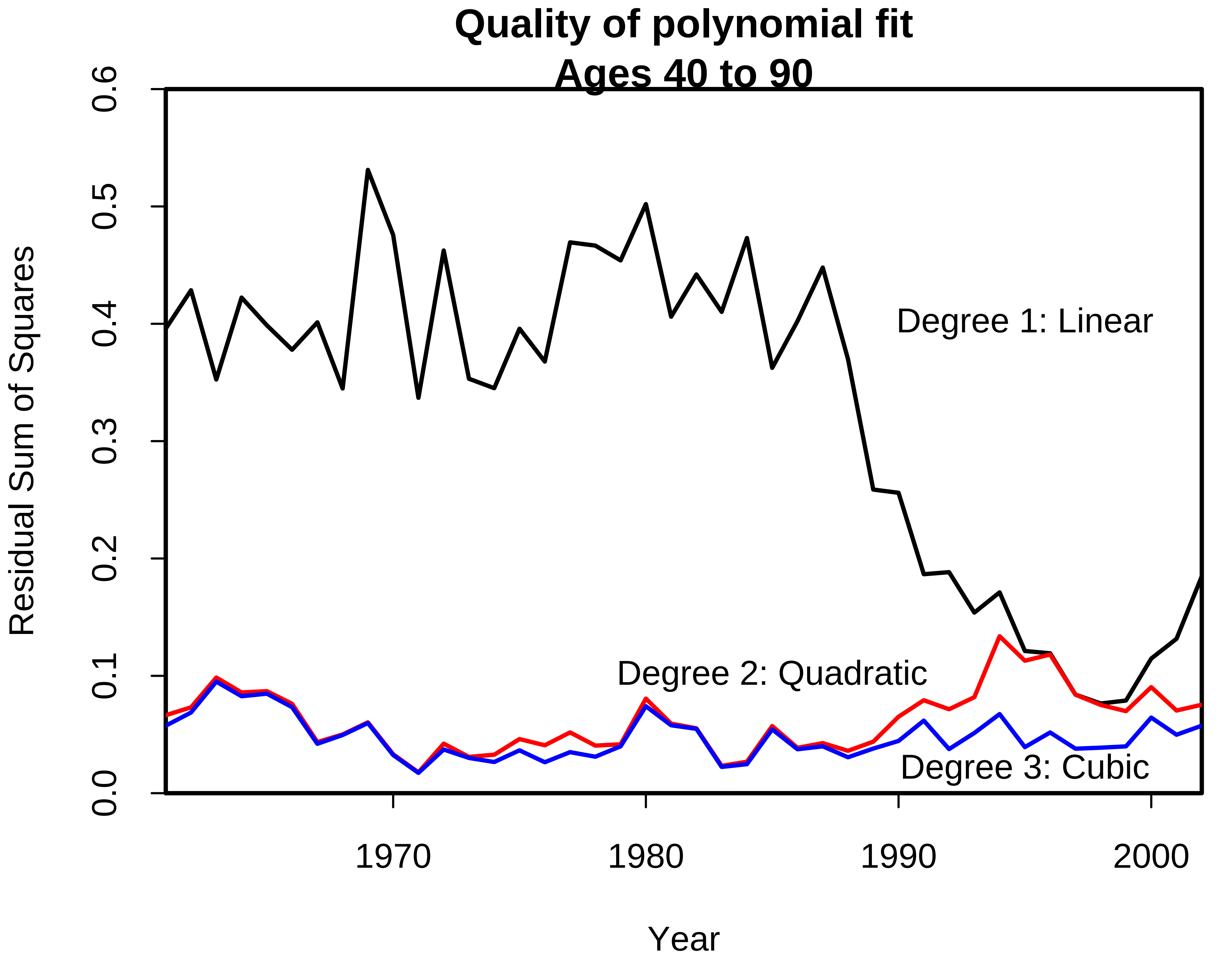
- Second approximation:

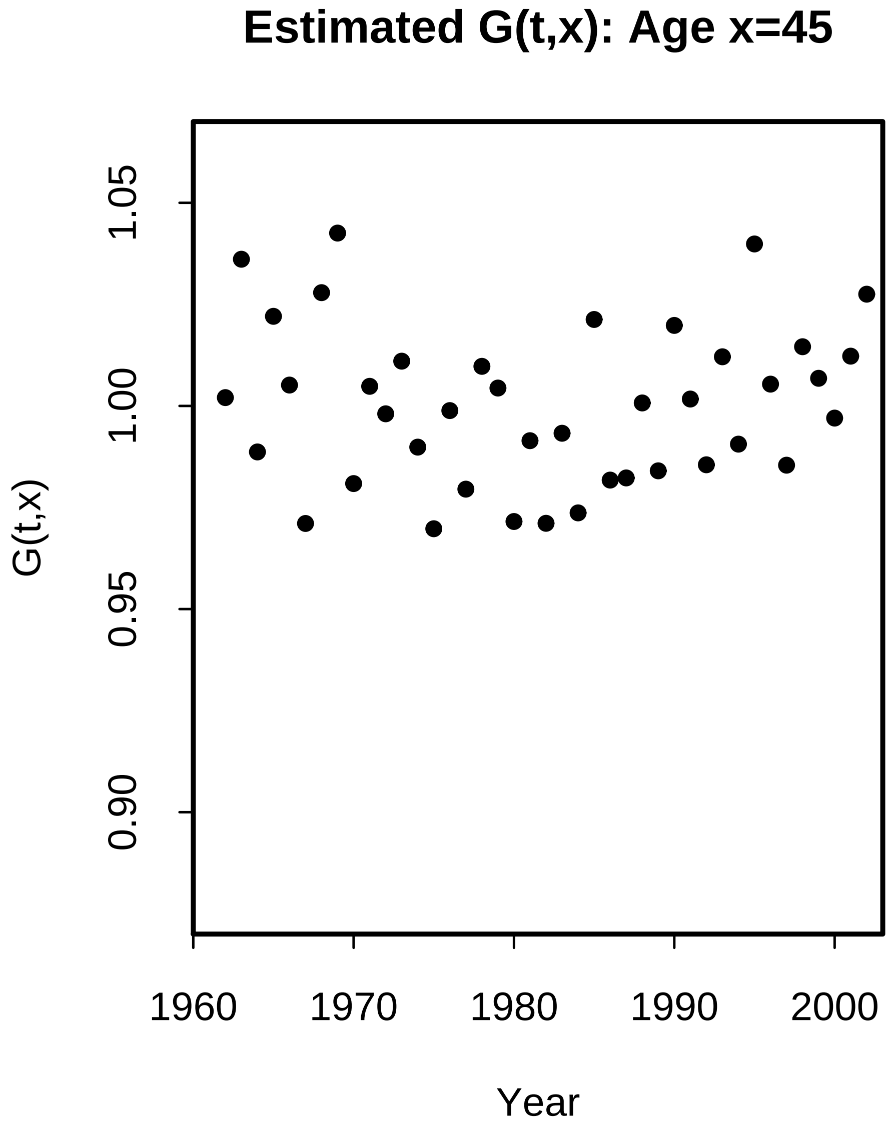
$$p_Q(t+1, t, t+1, x-t) = p_Q(t, t, t+1, x-t)^{1 \times G(t+1, x)}$$

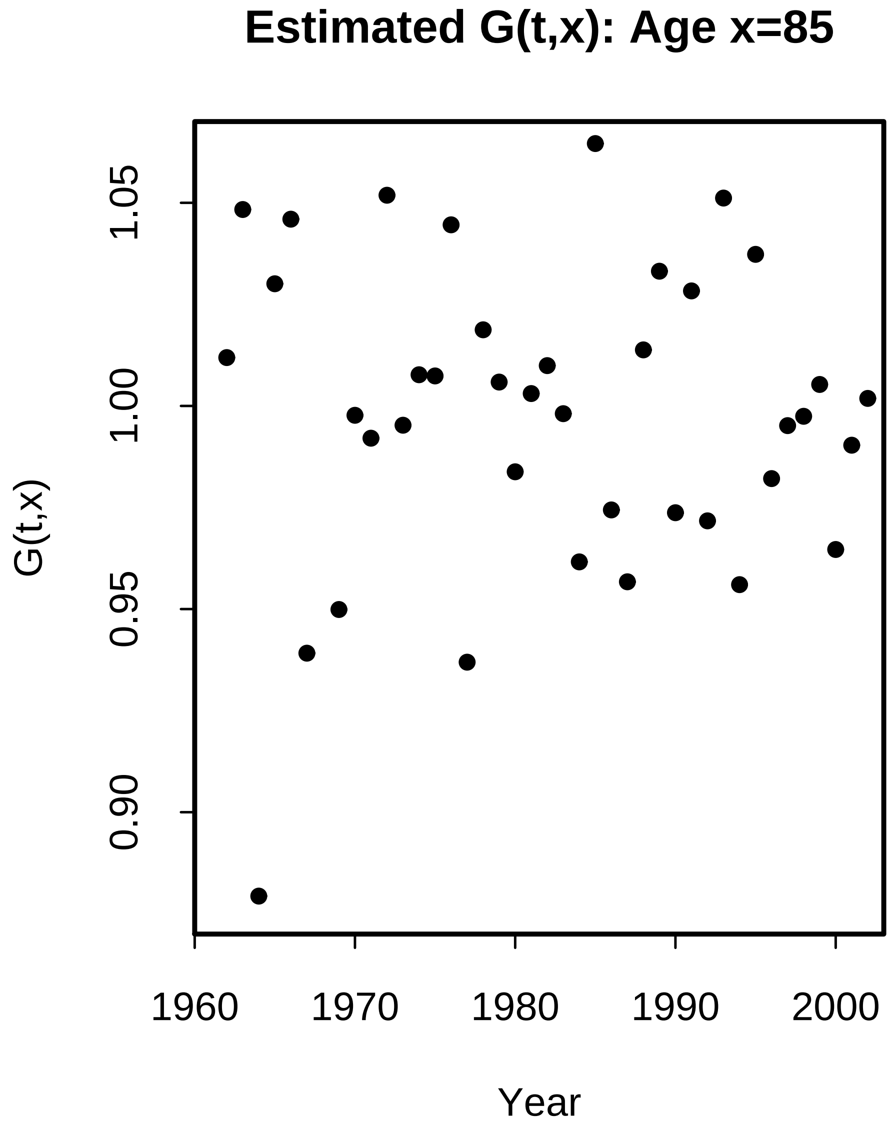
- Data: England and Wales mortality, males, 1961-2002
- Individual calendar years smoothed first
- $G(t, x)$ calculated for each year t and age x
- Results:
  - First factor explains $\sim$ 80% of variability
  - For a single x:

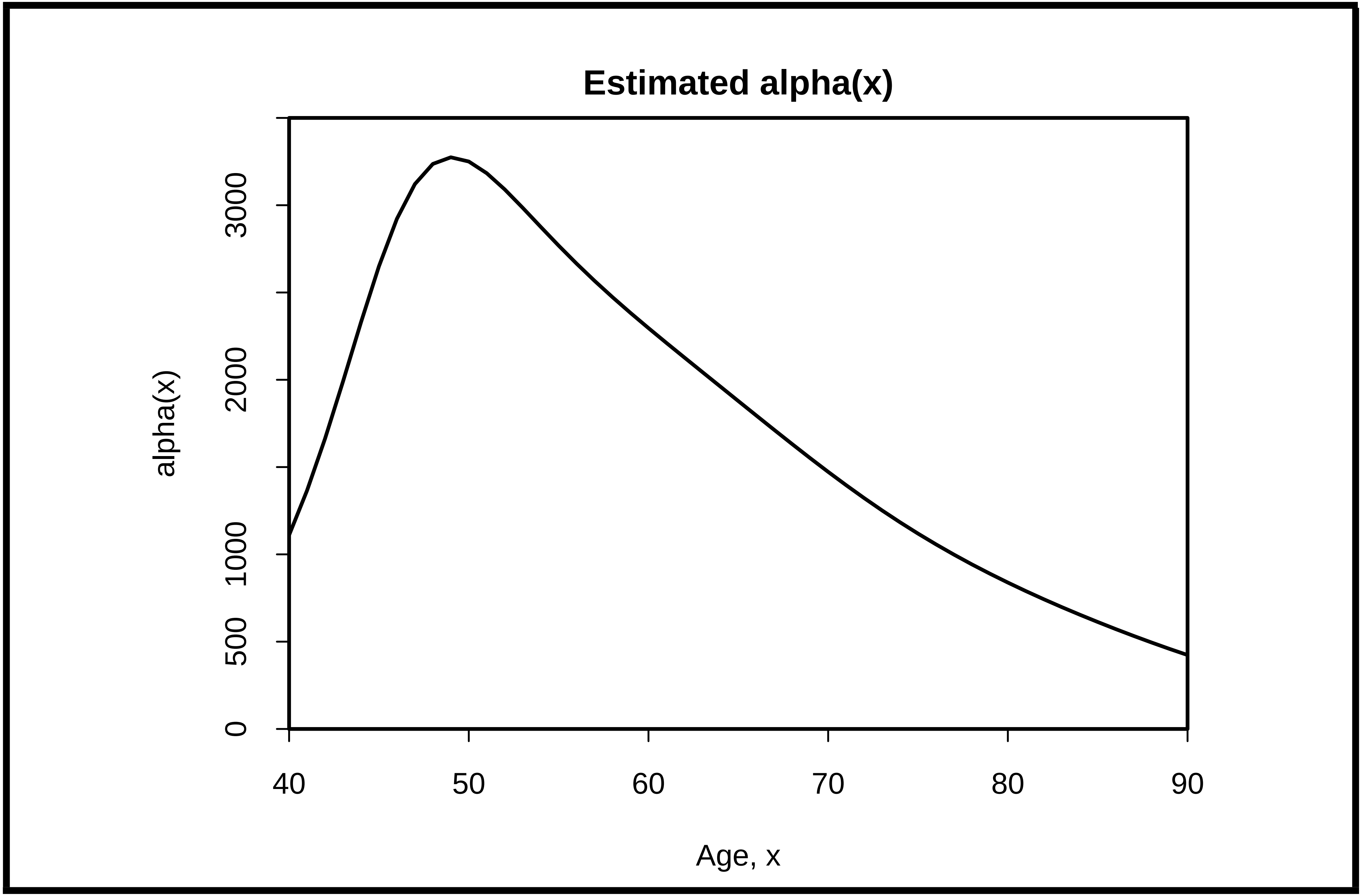
    Estimate $\alpha(x) = 1/Var[G(t, x)]$
  - $\alpha(x)$ is clearly dependent on x


logit(q_x) in 1961
logit(q_x) in 2002
logit(q_x)
Age, x
−1
−2
−3
−4
−5
−6
−7
40
50
60
70
80
90
Linear
Cubic
Crude data

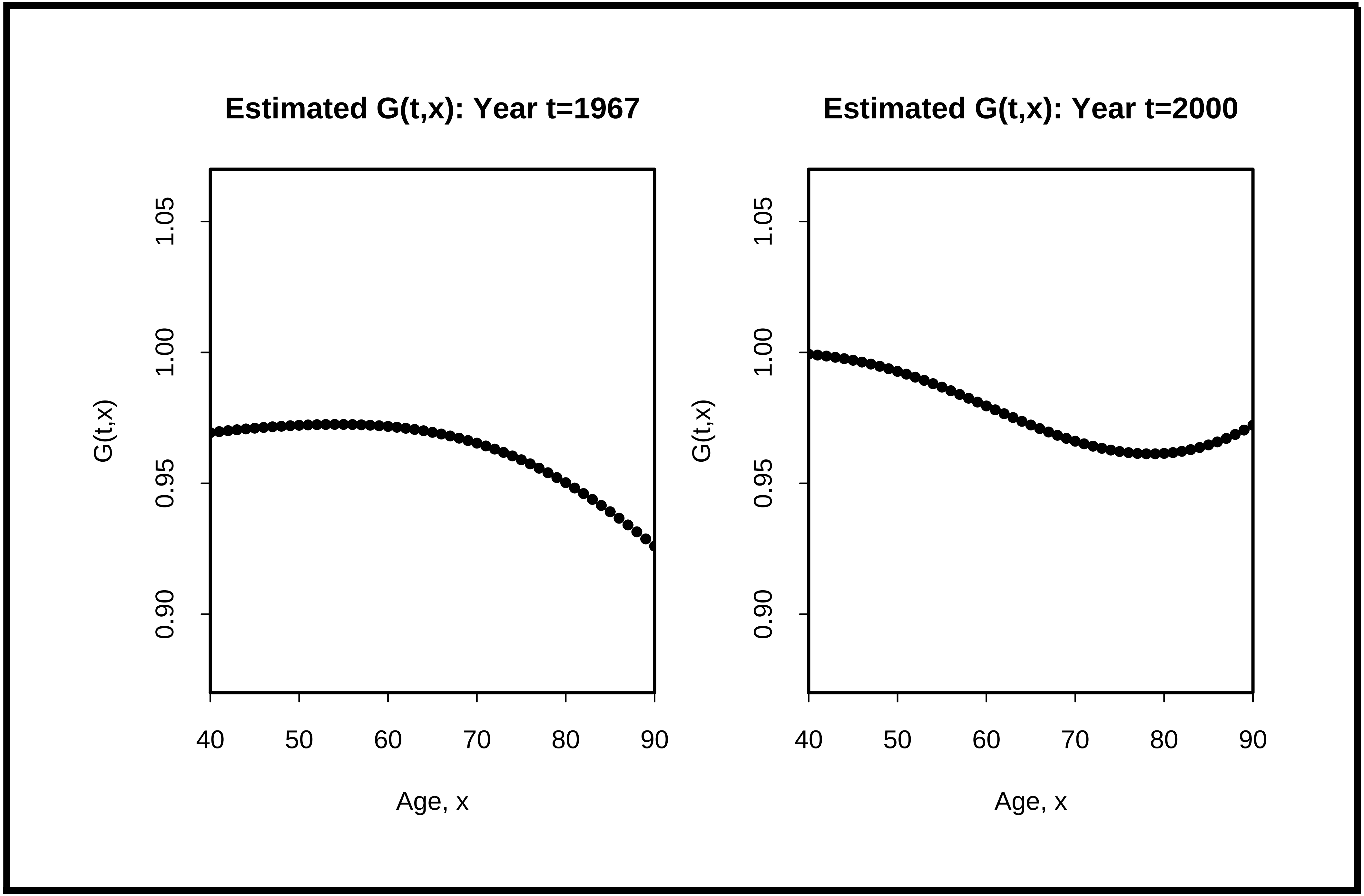

Quality of polynomial fit
Ages 40 to 90
Residual Sum of Squares
0.0
0.1
0.2
0.3
0.4
0.5
0.6
Degree 1: Linear
Degree 2: Quadratic
Degree 3: Cubic
1970
1980
1990
2000
Year


Estimated G(t,x): Age x=45
G(t,x)
0.90
0.95
1.00
1.05
1960
1970
1980
1990
2000
Year

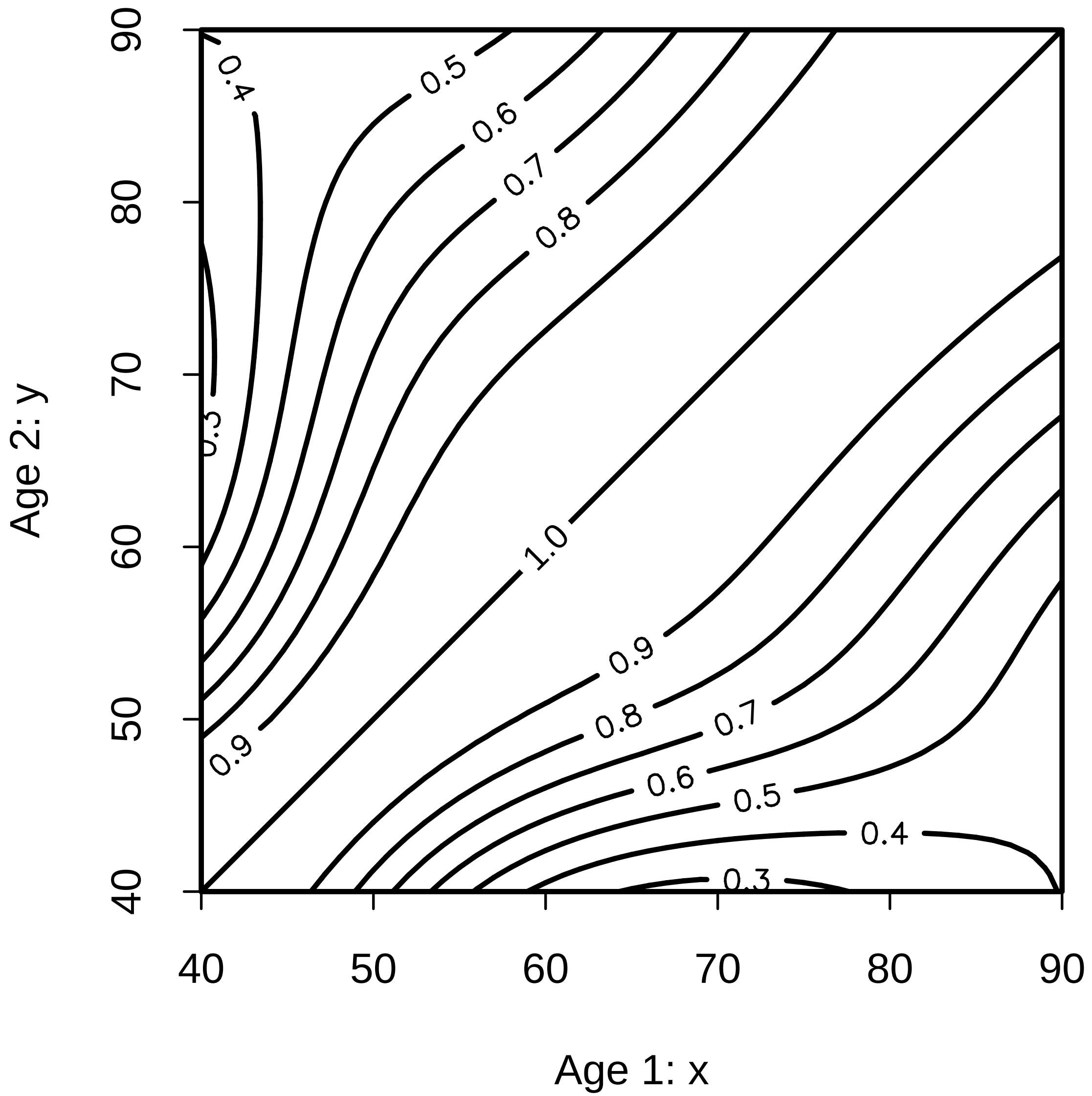

Estimated G(t,x): Age x=85
G(t,x)
0.90
0.95
1.00
1.05
1960
1970
1980
1990
2000
Year



Estimated alpha(x)
alpha(x)
0
500
1000
2000
3000
40
50
60
70
80
90
Age, x


Estimated G(t,x): Year t=1967
Estimated G(t,x): Year t=2000
G(t,x)
0.90
0.95
1.00
1.05
40
50
60
70
80
90
Age, x

Contour plot: correlation between mortality improvements at different ages: G(t,x) and G(t,y)



A GENERALISED OLIVIER-SMITH MODEL

Solution: use copulas

Stage 1: one-year-ahead spot survival probabilities

$$p_Q(t+1, t, t+1, x) = p_Q(t, t, t+1, x)^{b(t,t,t+1,x)G(t+1,x)}$$

- $G(t+1, x) \sim$ Gamma$\big(\alpha(x), \alpha(x)\big)$ under Q
- $cor\big(G(t+1, x_1),\ G(t+1, x_2)\big) \ = \ \rho(x_1, x_2)$
- $\{G(t+1, x) : x_l \leq x \leq x_u\}$ generated e.g. using the multivariate Gaussian copula

Stage 2A: all spot survival probabilities

$$p_Q(t+1, T-1, T, x) = p_Q(t, T-1, T, x)^{b(t,T-1,T,x)G(t+1,x)}$$

- $G(t+1, x) \sim \text{Gamma}\big(\alpha(x), \alpha(x)\big)$ under Q
- Same $G(t+1, x)$ for each T
- $cor\big(G(t+1, x_1),\ G(t+1, x_2)\big) \ = \ \rho(x_1, x_2)$
- $\{G(t+1, x) : x_l \le x \le x_u\}$ generated e.g. using the multivariate Gaussian copula

Stage 2B: all forward survival probabilities

$$p_Q(t+1, t, T, x) = p_Q(t, t, T, x)^{g(t,T,x)G(t+1,T,x)}$$

- $G(t+1, T, x) \sim$ Gamma$\big(\alpha(T, x), \alpha(T, x)\big)$under$Q$
- Different $G(t+1, T, x)$ for each (T, x)
- Specified correlation structure
- $\{G(t+1, T, x) : x_l \leq x \leq x_u;\ T > t\}$ generated e.g. using the multivariate Gaussian copula

ONGOING ISSUES

- Problem: all $0 < p_Q(t+1, t, T, x) < 1$

  BUT with small probability

  Gaussian copula $\Rightarrow p_Q(t+1, t, T, x)$ not decreasing with T

Some thoughts on how to resolve this: Let

$p_Q(t,t,T,x) = \exp(-M_1)$;

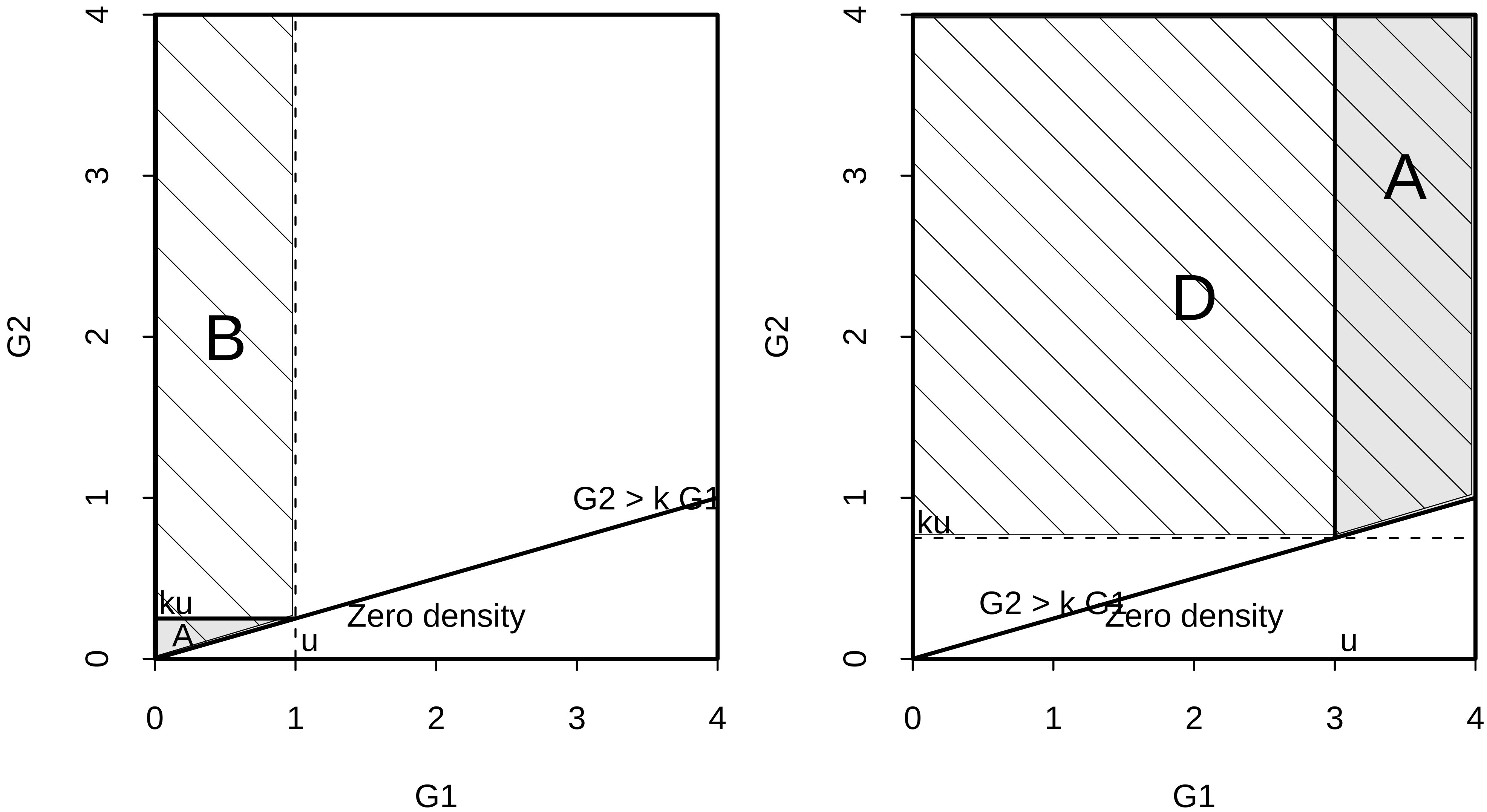
$p_Q(t,t,T+1,x) = \exp(-M_2)$

$$p_Q(t,t,T,x)^{g(t,T,x)G(t+1,T,x)} \equiv e^{-M_1 g_1 G_1}$$

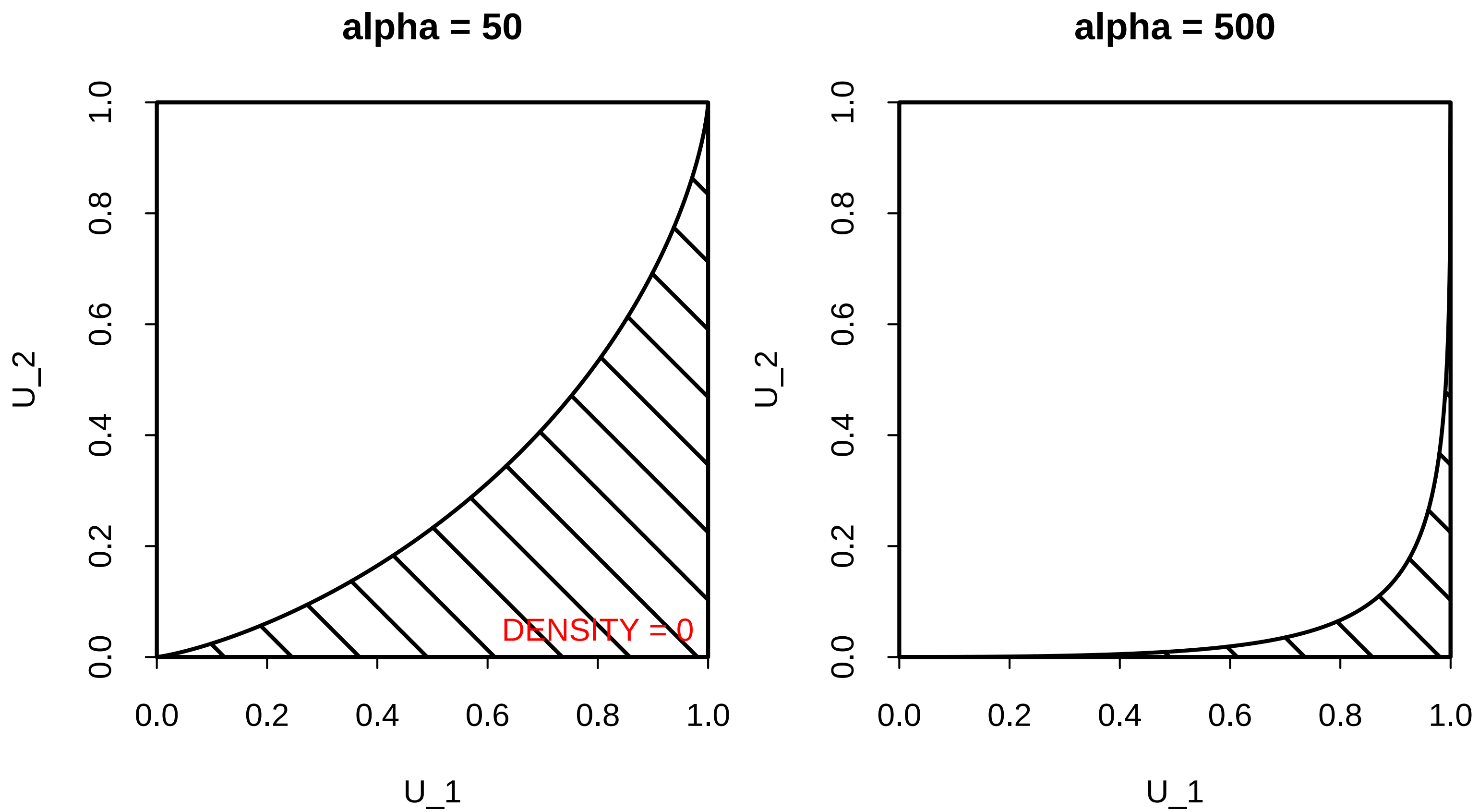
$$p_Q(t,t,T+1,x)^{g(t,T+1,x)G(t+1,T+1,x)} \equiv e^{-M_2 g_2 G_2}$$

- M_1, M_2, g_1, g_2 known at t
- $M_2 > M_1$
- Require $M_1 g_1 G_1 < M_2 g_2 G_2$

Lemma: $1 \leq \alpha_2/\alpha_1 \leq g_2 M_2/g_1 M_1$



$M_1 g_1 G_1 < M_2 g_2 G_2 \Rightarrow$ constraints on copula:



e.g. $M_1 g_1 / M_2 g_2 = 0.9$, $\alpha_1 = \alpha_2$

A step in the right direction...

- U_1 and $U_2 \sim$ i.i.d. $U[0,1]$
- Define: $\bar{U}_2 = \max\{f(U_1), U_2\}$
- Let $\bar{F}(u_2) =$ c.d.f. of $\bar{U}_2$.
- Define $V_1 = U_1$ and $V_2 = \bar{F}(\bar{U}_2)$.

  $\Rightarrow V_1$ and V_2 are dependent $U[0,1]$
- Given V_1: the minimum value taken by V_2 is $V_1 f(V_1)$.
- Define $G_1 = F_1^{-1}(V_1)$ and $G_2 = F_2^{-1}(V_2)$.

ISSUES STILL TO BE RESOLVED

- How to control correlation between V_1 and V_2?
- Algorithm results in probability mass on the

  $V_2 = V_1 f(V_1)$ boundary.

  $V_2 = V_1 f(V_1) \Rightarrow M_1 g_1 G_1 = M_2 g_2 G_2$.

  $\Rightarrow$ mortality rate between T and $T + 1$ will be zero.

CONCLUSIONS

Provided we can find a suitable copula ...

($\Rightarrow$ simulation of $U(T, x)$ for all (T, x) easy)

generalised Olivier-Smith model could prove a useful tool

for modelling stochastic mortality.