STOCHASTIC MORTALITY

Andrew Cairns

Heriot-Watt University, Scotland

and

The Maxwell Institute, Edinburgh

Joint work with David Blake & Kevin Dowd

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Plan

- Introduction $+$ data
- Approaches to modelling mortality improvements
- A two-factor model for stochastic mortality
- Applications
  - The survivor index
  - Annuity reserves
  - How to price longevity bonds if you must
  - Annuity guarantees
  - Survivor caps and caplets
- Conclusions

The facts about mortality:

- Life expectancy is increasing.
- Future development of life expectancy is uncertain.

"Longevity risk"

Longevity Risk $=$ the risk that future mortality rates are lower than anticipated

Focus here: Mortality rates above age 60

STOCHASTIC MORTALITY

n lives, probability p of survival, N survivors

- Unsystematic mortality risk:

  $\Rightarrow N|p \ \sim$ Binomial(n, p)

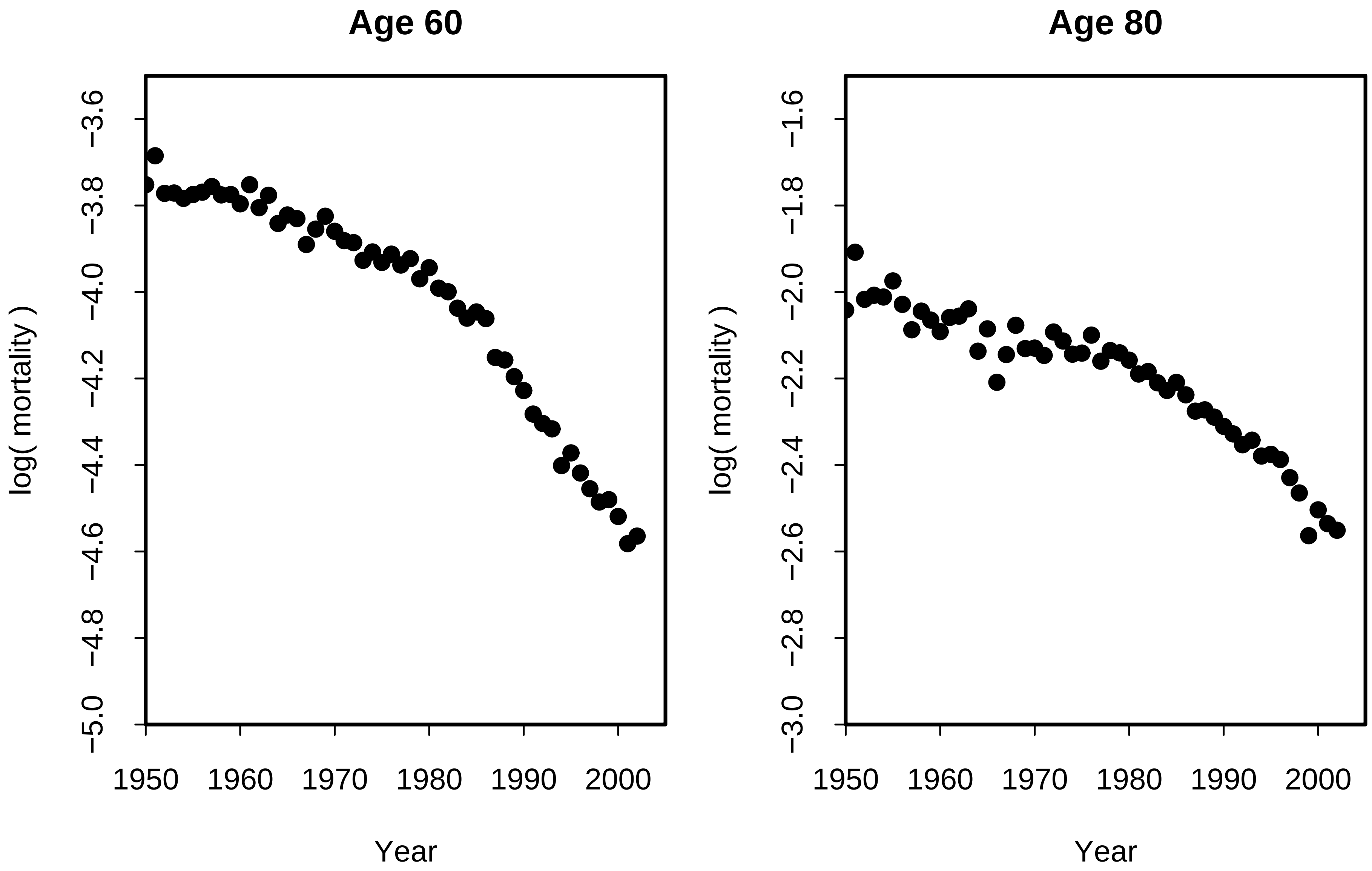
  $\Rightarrow$ risk is diversifiable, $\quad N/n \longrightarrow p \quad$ as $n \to \infty$

- Systematic mortality risk:

  $\Rightarrow p$ is uncertain

  $\Rightarrow$ risk associated with p is not diversifiable

England and Wales log mortality rates 1950-2002



Stochastic Models

Different approaches to modelling

- Lee-Carter
- P-splines
- Parametric, time-series models
- Market models
- Age-Period-Cohort extensions

Stochastic Models

Limited historical data $\Rightarrow$

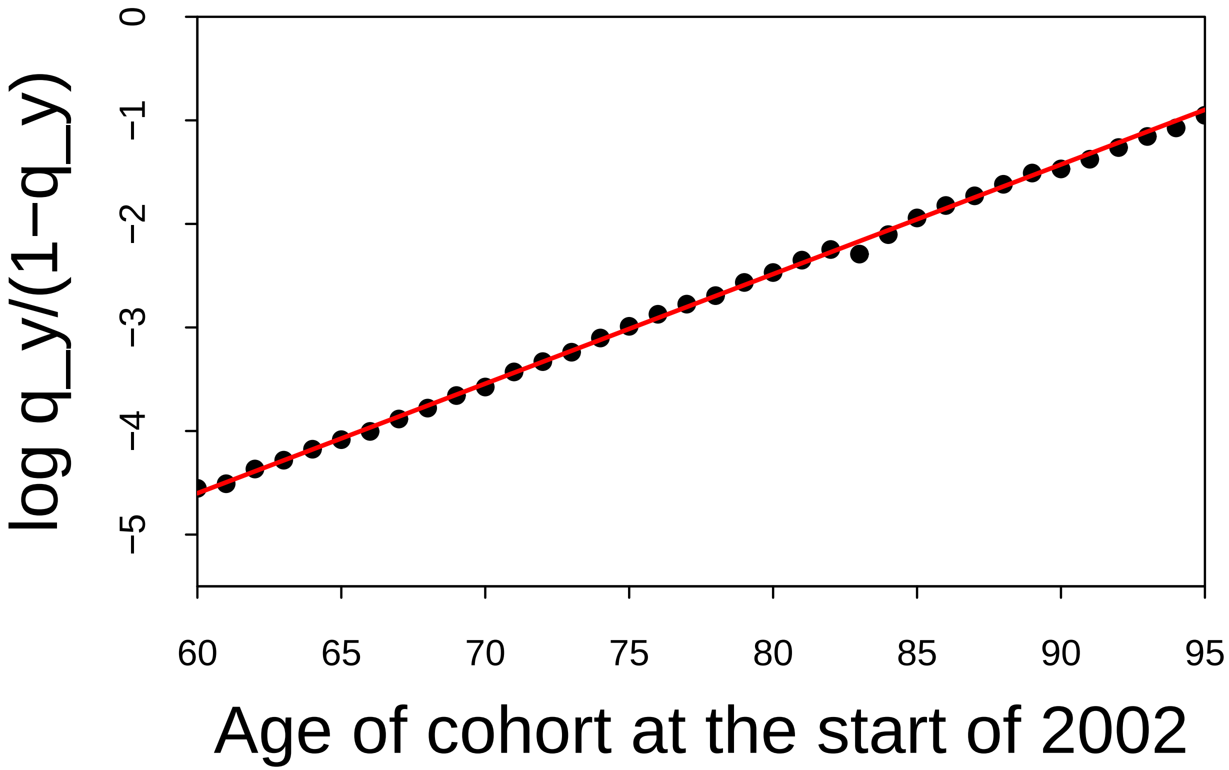
- No single model is 'the right one'

  limited data $\Rightarrow$ Model risk

- Even with the right model

  limited data $\Rightarrow$ Parameter risk

Case study: England and Wales males, age 60-95



$q_y =$ mortality rate at age y in 2002

Data suggests $\log q_y/(1 - q_y)$ is linear

A TWO-FACTOR PARAMETRIC TIME-SERIES MODEL

Cohort: Age x at time $t = 0$

Mortality rates for the year t to $t + 1$:

$$q(t, x) = \frac{e^{A_1(t)+A_2(t)(x+t)}}{1 + e^{A_1(t)+A_2(t)(x+t)}}$$

$(x + t)$= age at time t

We model $A(t) = (A_1(t), A_2(t))'$ as a random-walk with drift

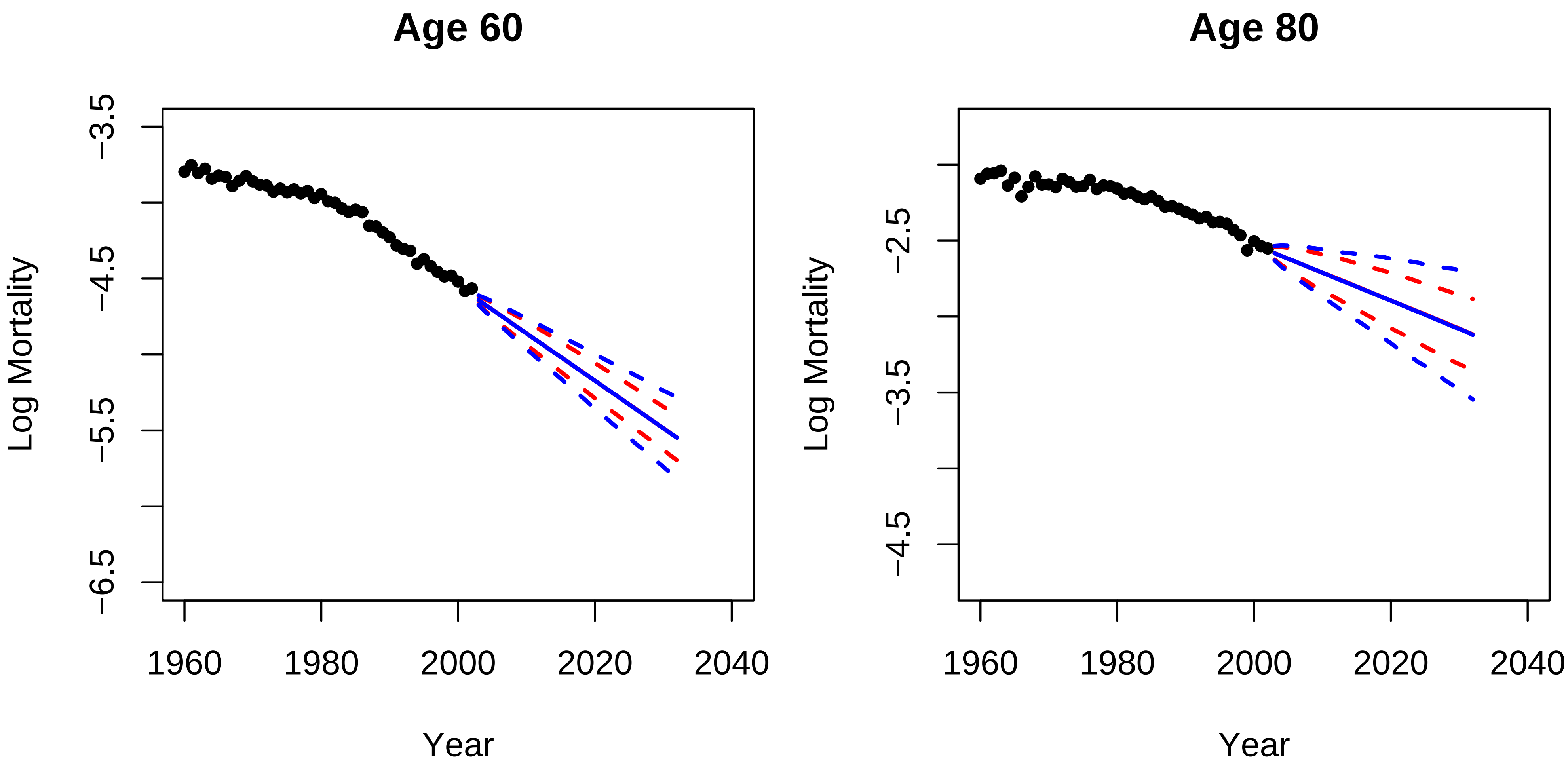
$A(t) = (A_1(t), A_2(t))'$

Model: Random walk with drift

$$A(t+1) - A(t) = \mu + CZ(t+1)$$

- $V = CC' =$ variance-covariance matrix
- Estimate μ and V
- Quantify parameter uncertainty in μ and V

Simulated mortality rates: 90% confidence intervals with and without parameter uncertainty



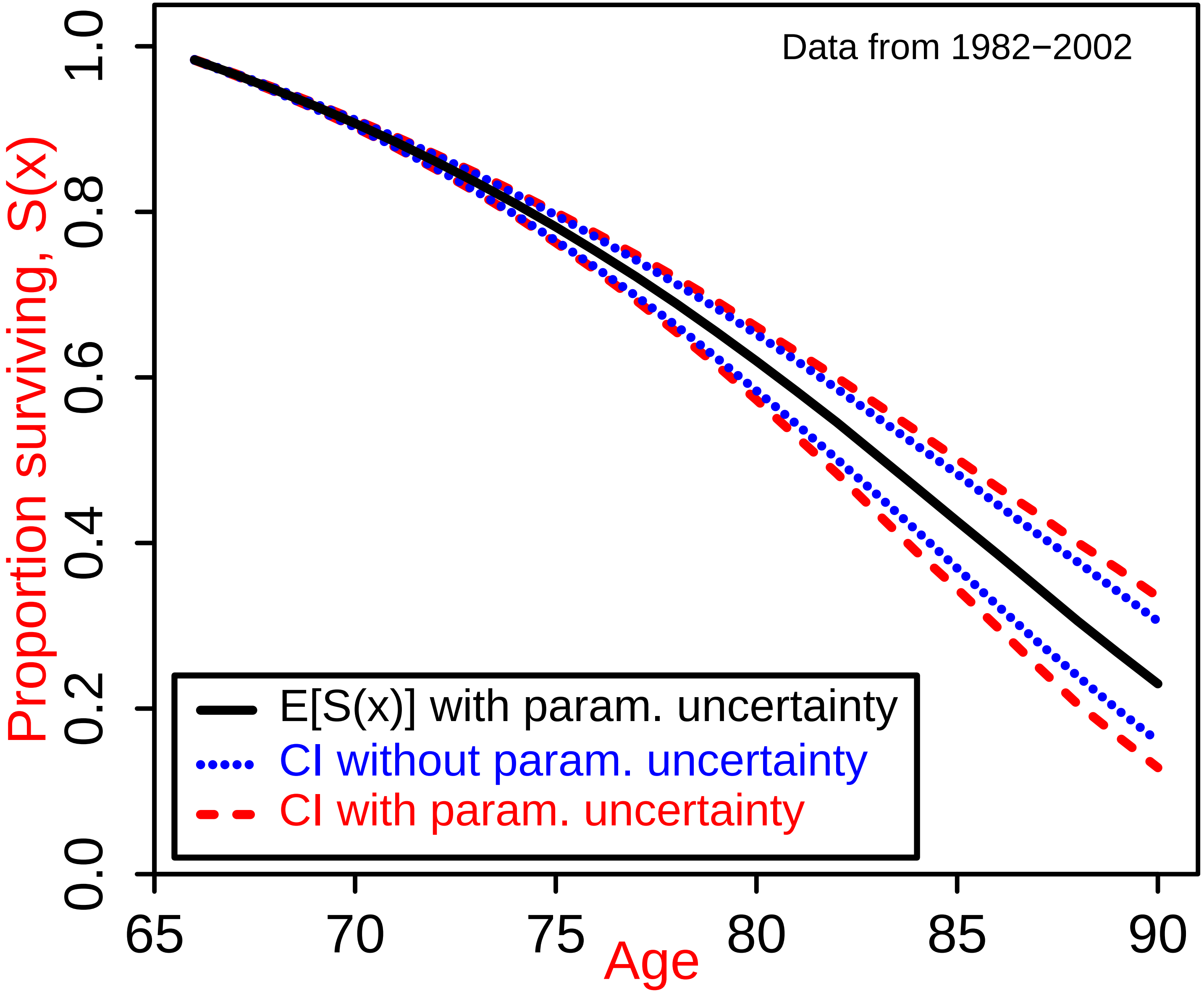
Application 1: cohort survivorship

- Cohort: Age x at time $t = 0$
- $S(t, x) =$ survivor index at t

  proportion surviving from time 0 to time t

$$S(t, x) \ = \ (1 - q(0, x)) \times \ldots \times (1 - q(t-1, x))$$

90% Confidence Interval (CI) for Cohort Survivorship



Cohort Survivorship: General Conclusions

- Less than 10 years:
  - Systematic risk not significant
- Over 10 years
  - Systematic risk becomes more and more significant over time
- Over 20 years
  - Model and parameter risk begin to dominate

Application 2: Reserving for annuities

- Large group of identical annuitants aged 65

  $\Rightarrow$ non-systematic risk negligible

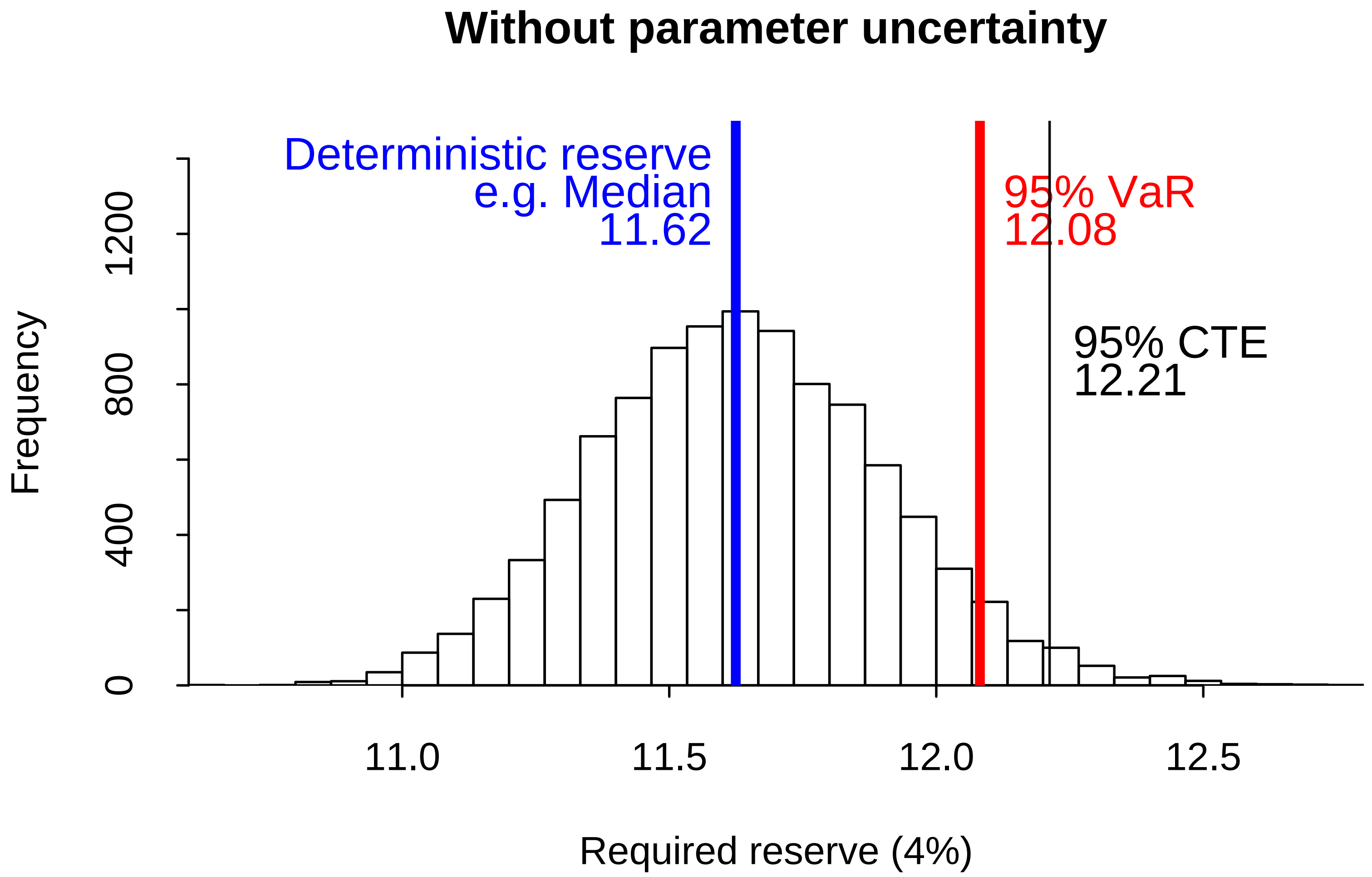
- Level annuity payable annually in arrears

- Interest rate 4% p.a.

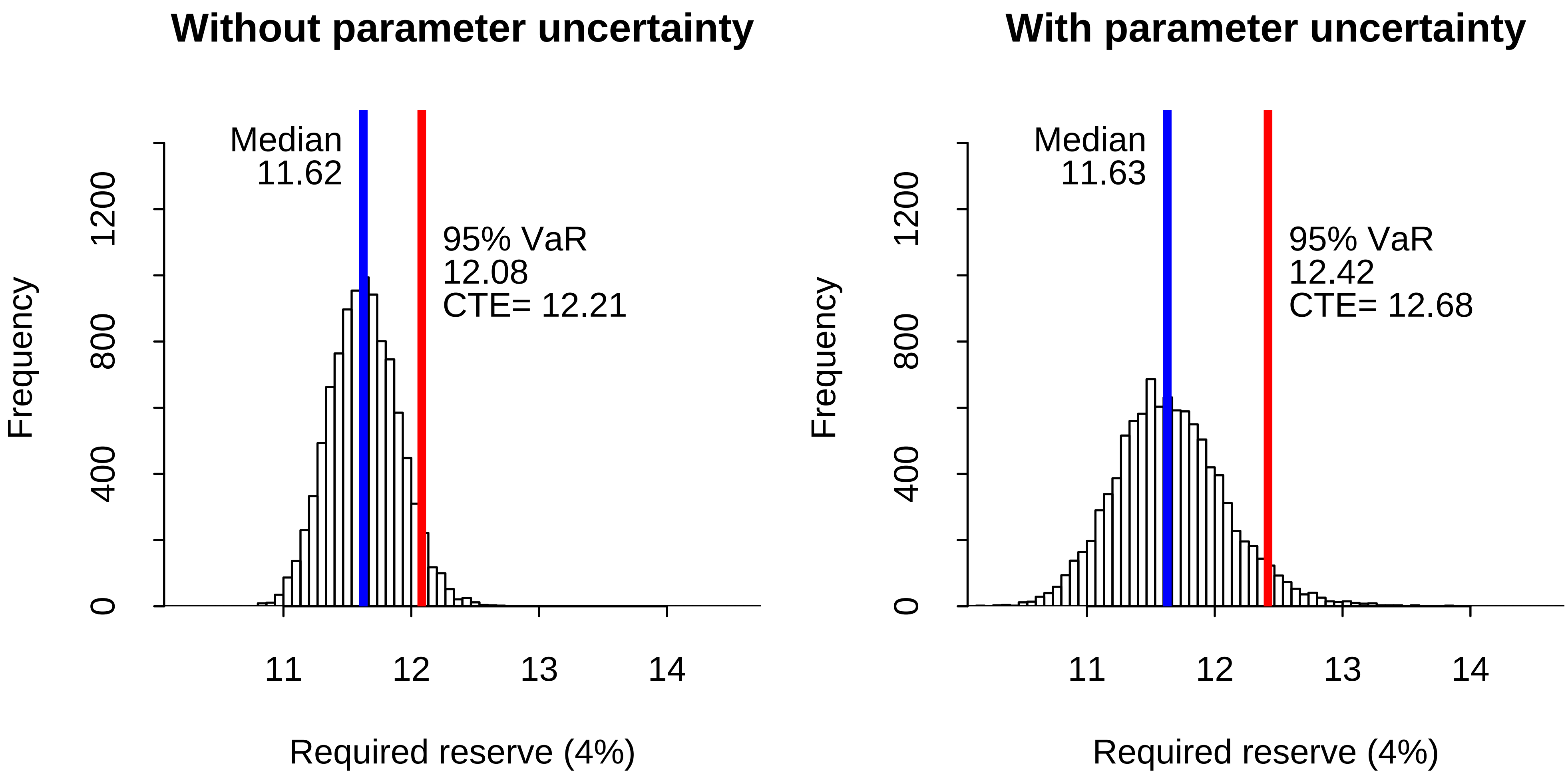
- For each £1 of annuity

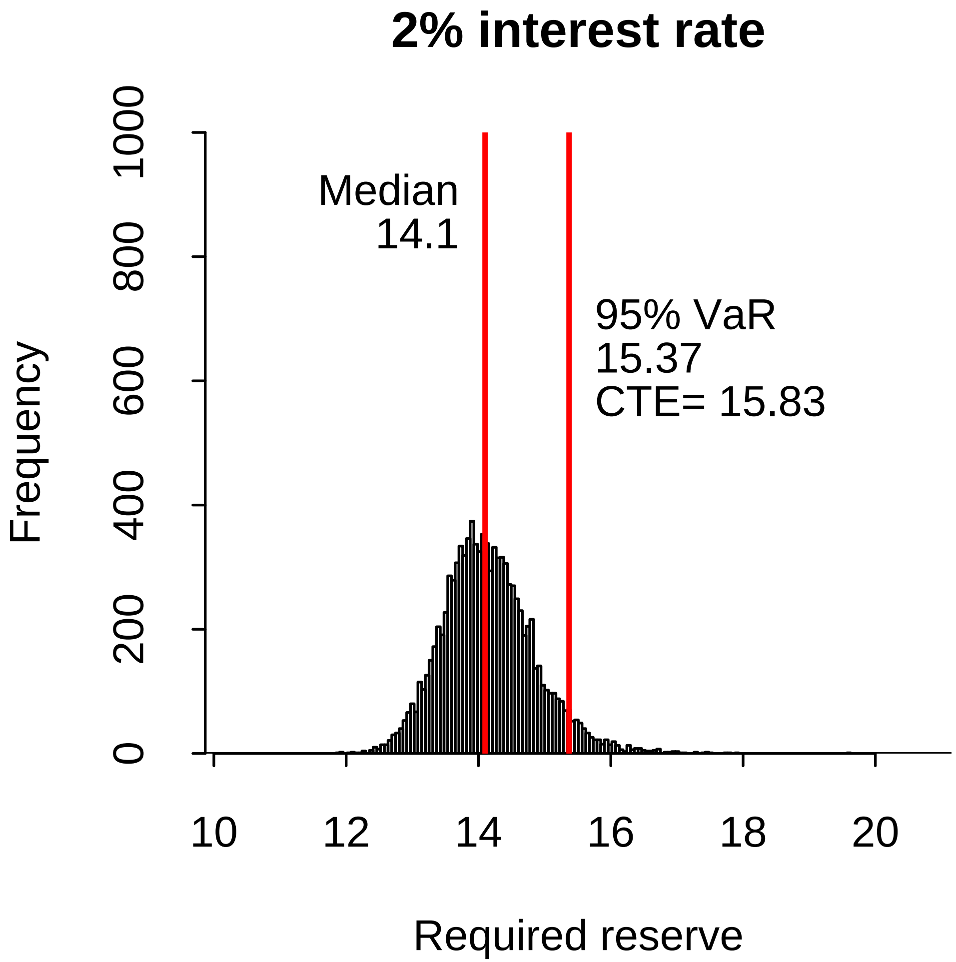
  How much of a reserve to we need at time 0?

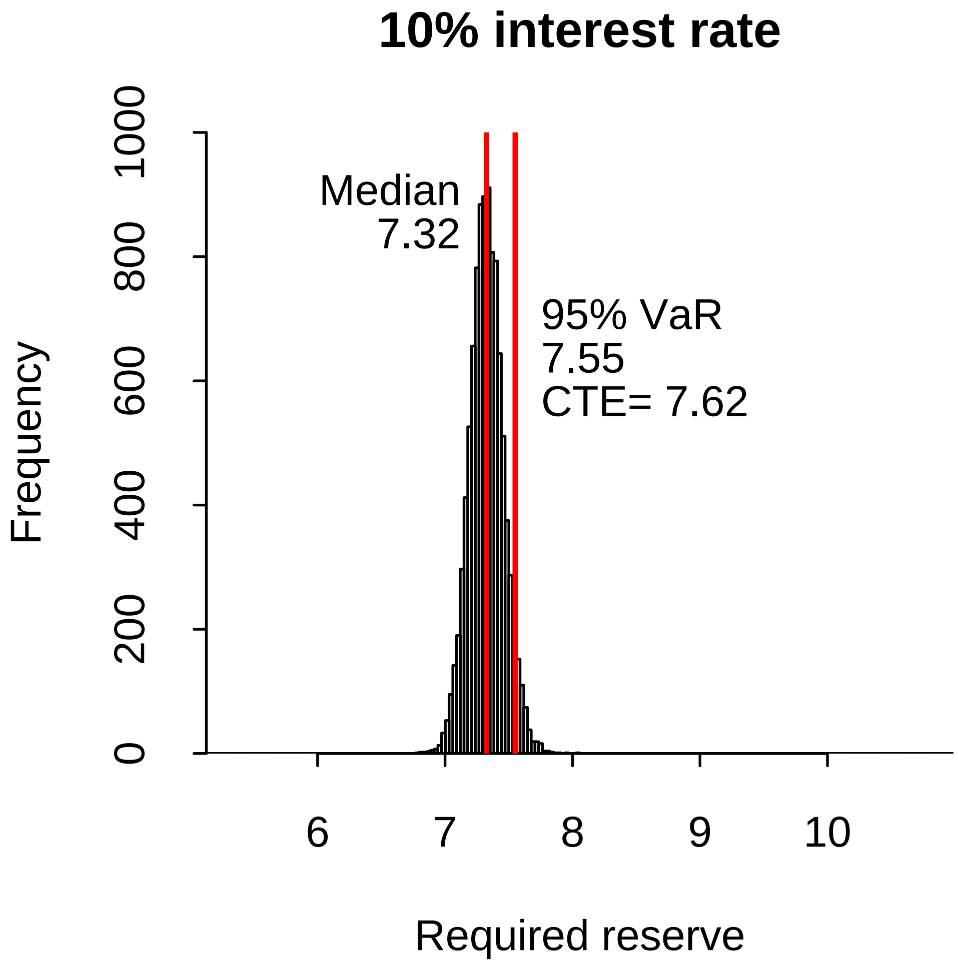
10000 simulations



Required reserve $= \sum_{t=1}^{\infty} S(t, 65) v^t$


Without parameter uncertainty
Median
11.62
95% VaR
12.08
CTE= 12.21
Frequency
0
400
800
1200
11
12
13
14
Required reserve (4%)
With parameter uncertainty
Median
11.63
95% VaR
12.42
CTE= 12.68
Frequency
0
400
800
1200
11
12
13
14
Required reserve (4%)

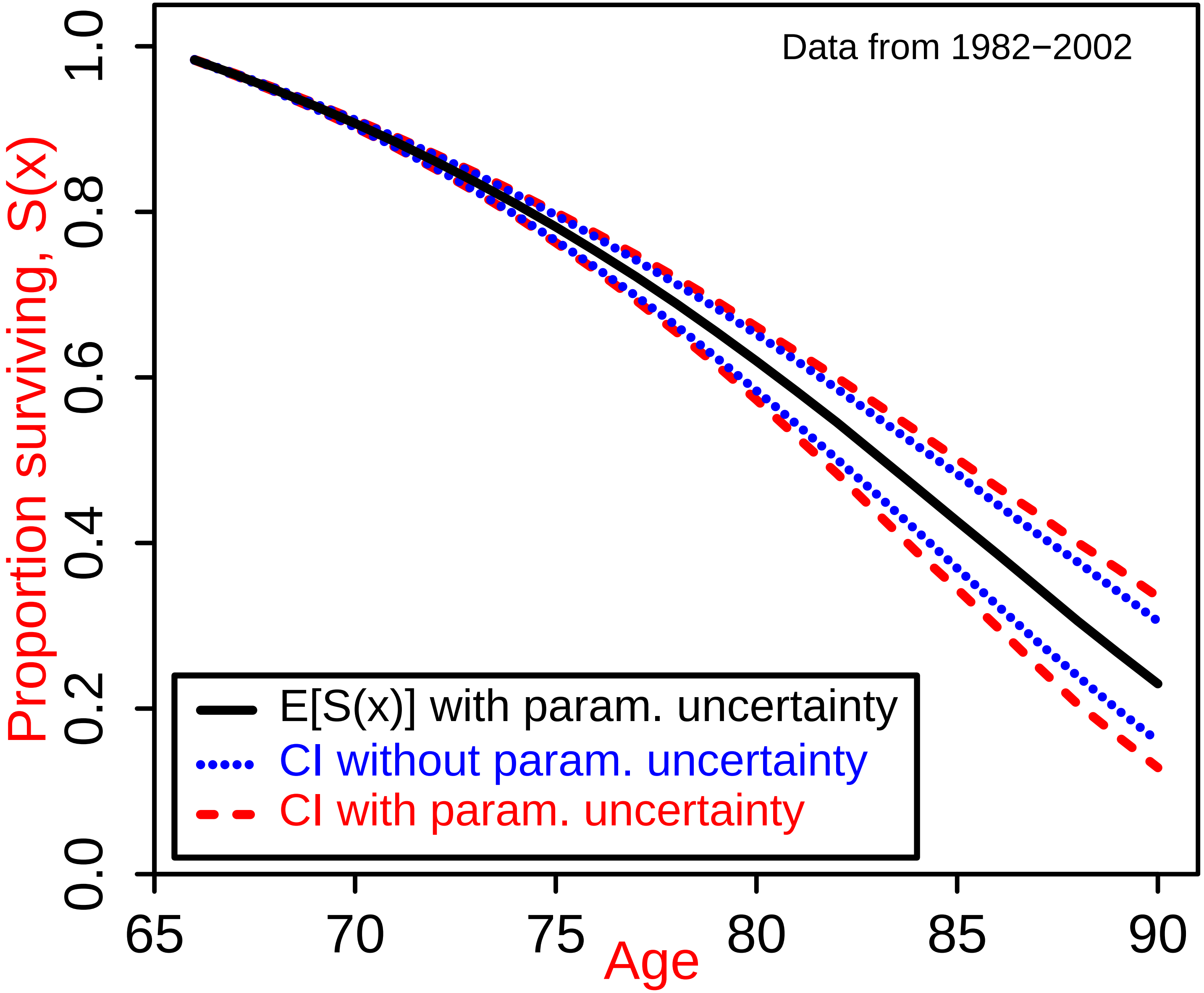

2% interest rate
Median
14.1
95% VaR
15.37
CTE= 15.83
Frequency
0
200
400
600
800
1000
10
12
14
16
18
20
Required reserve


10% interest rate
Median
7.32
95% VaR
7.55
CTE= 7.62
Frequency
0
200
400
600
800
1000
6
7
8
9
10
Required reserve

Application 3: Longevity bonds

- Cohort: Age x at time $t = 0$
- $S(t, x) =$ survivor index at t
- Longevity bond pays $S(t, x)$ at times $t = 1, \ldots, T$

Recap: 90% CI for Cohort Survivorship



How do you price a longevity bond

- Hedgers are prepared to pay a premium
- Two approaches:
  - Take *real-world* expected values

    use a risk-adjusted discount rate
  - Take *risk-adjusted* expected values

    use the risk-free discount rate

Risk-neutral pricing (risk-adjusted expected values)

$$\begin{pmatrix} A_1(t+1) \\ A_2(t+1) \end{pmatrix} = \begin{pmatrix} A_1(t) \\ A_2(t) \end{pmatrix} + \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} + \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \tilde{Z}_1(t+1) + \lambda_1 \\ \tilde{Z}_2(t+1) + \lambda_2 \end{pmatrix}$$

where $\tilde{Z}_1(t+1)$ and $\tilde{Z}_2(t+1)$ are i.i.d. $\sim N(0,1)$

<u>under a risk-neutral pricing measure $Q(\lambda)$</u>

λ_1 and λ_2 are market prices of risk

How does the market price of risk work?

- Two independent sources of risk $Z_1(t)$, $Z_2(t)$
- Tradeable security has corresp. volatilities σ_1, σ_2
- Market price of risk is

  the additional expected return over the risk free rate

  per unit of risk
- Hence

$$\text{Risk premium} \quad = \Big(\sigma_1\lambda_1 + \sigma_2\lambda_2\Big)$$

Comments

- The market is highly incomplete
- The switch from P to Q is a modelling assumption
- (Simple) Key assumption:

  market prices of risk λ_1 and λ_2 are constant.
- As a market develops this assumption becomes a testable hypothesis

One data point: the EIB-BNP longevity bond

- Offer price (ultimately unsuccessful) $\Rightarrow$

  average risk premium of 20 basis points

  (paid by the buyer of the bond to the seller)

  if held to maturity

- What values of λ_1, λ_2 are consistent with the 20b.p.'s risk premium?

- One price, two parameters $\Rightarrow$ many solutions

Answer: 20 b.p. spread equates to

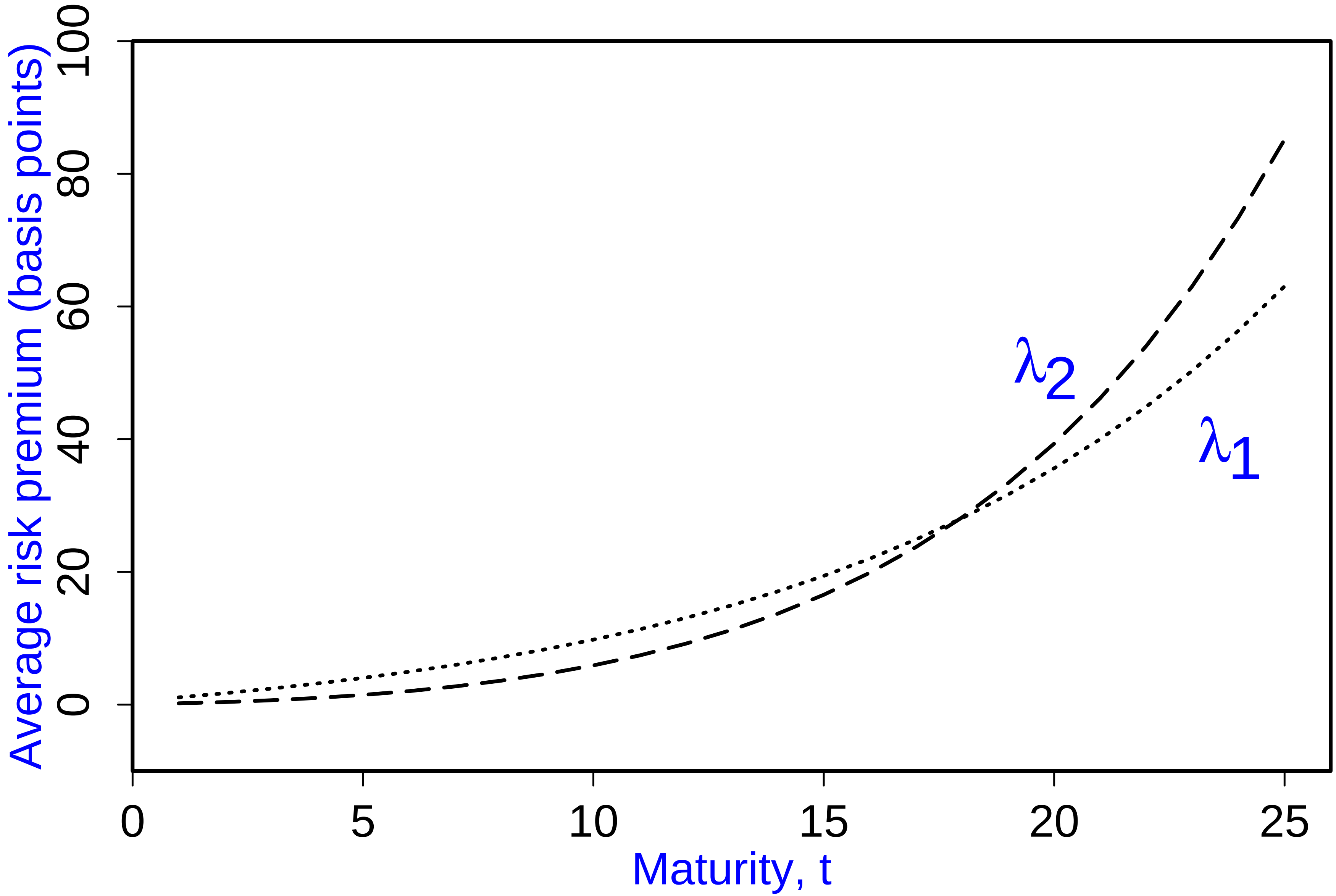
$$\begin{array}{llll} \lambda_1 = & 0.375, & \lambda_2 = & 0 \\ & \downarrow & & \downarrow \\ \lambda_1 = & 0, & \lambda_2 = & 0.315 \end{array}$$

Do these values represent a *good deal*?

Why do we need to know λ_1, λ_2?

$\Rightarrow$ info. on how to price new issues in the future.

Zero-coupon Longevity bonds: avg. risk premium p.a.



Longevity Bond Risk Premiums: $\lambda = (0.375, 0)$

Dependency on term and initial age:

		Initial age of cohort, x		
		60	65	70
Bond	20	8.9	14.7	23.1
Maturity	25	12.7	20.0	28.7
T	30	16.9	24.3	31.5

Application 4: Guaranteed annuity options

- Contract pays lump sum of £1,000 in $T = 30$ years
- Lump sum to be used to purchase an RPI-linked annuity
- Guarantee:

$$\text{Pension} \quad = \max\left\{\frac{1000}{a_{65}(T)}, \frac{1000}{g}\right\}.$$

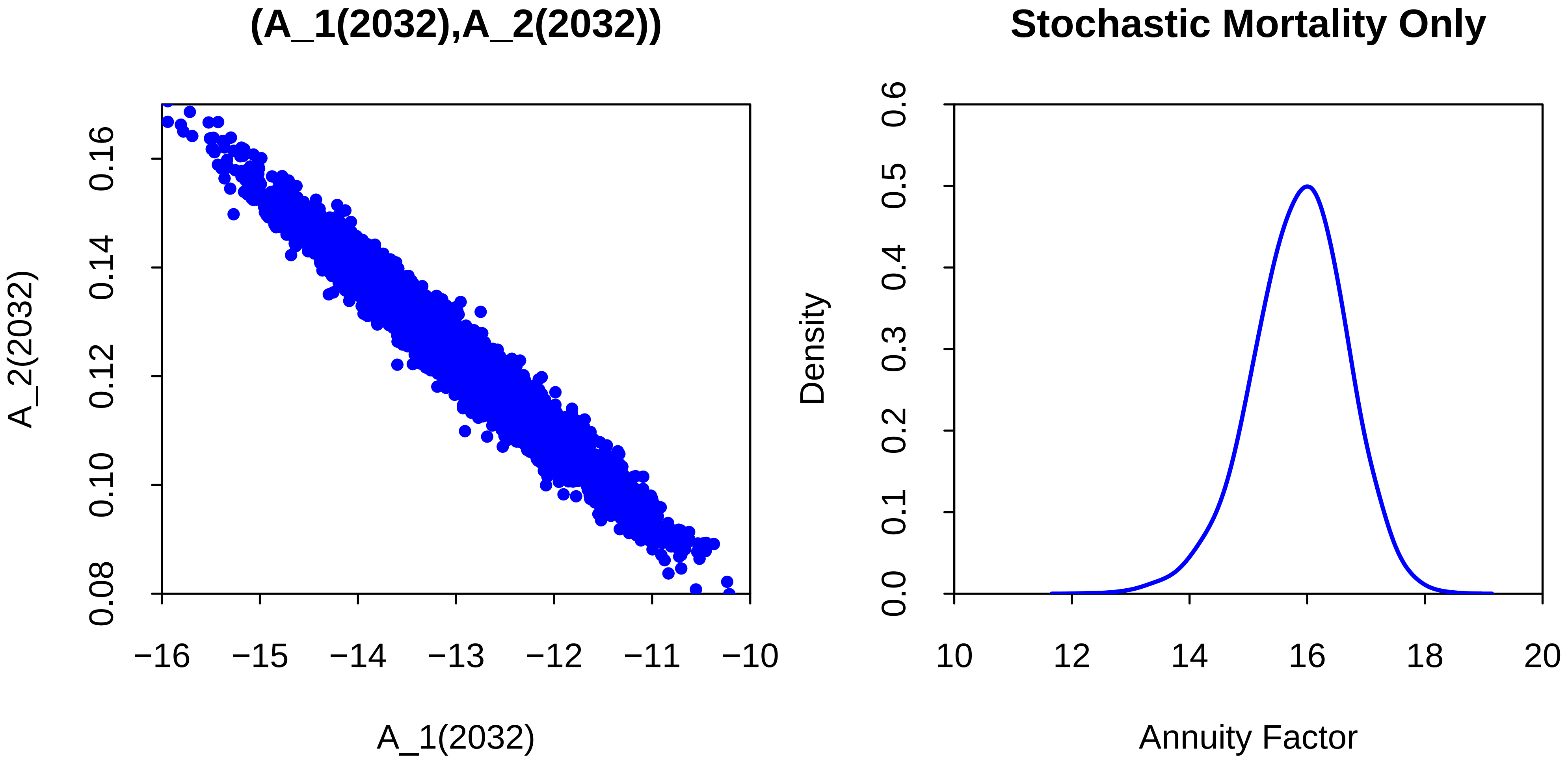
- Value at T is

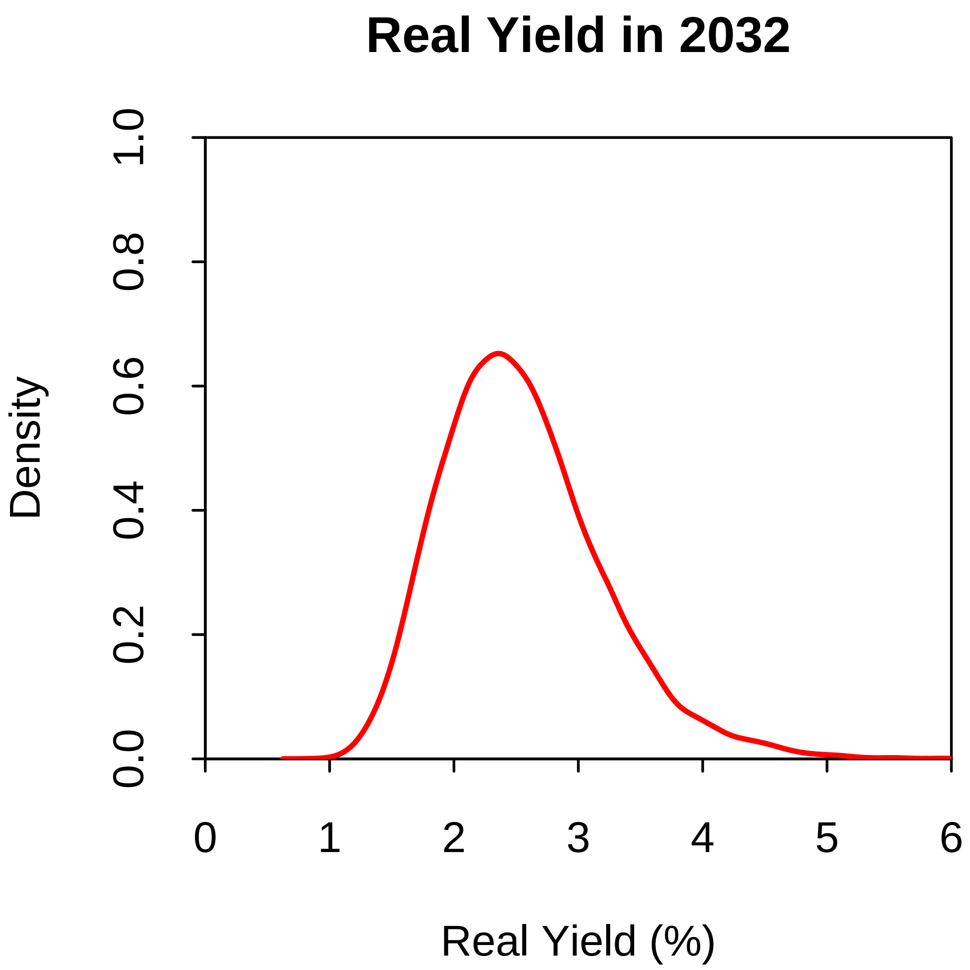
$$\max\left\{\frac{1000}{a_{65}(T)},\ \frac{1000}{g}\right\} \times a_{65}(T)$$

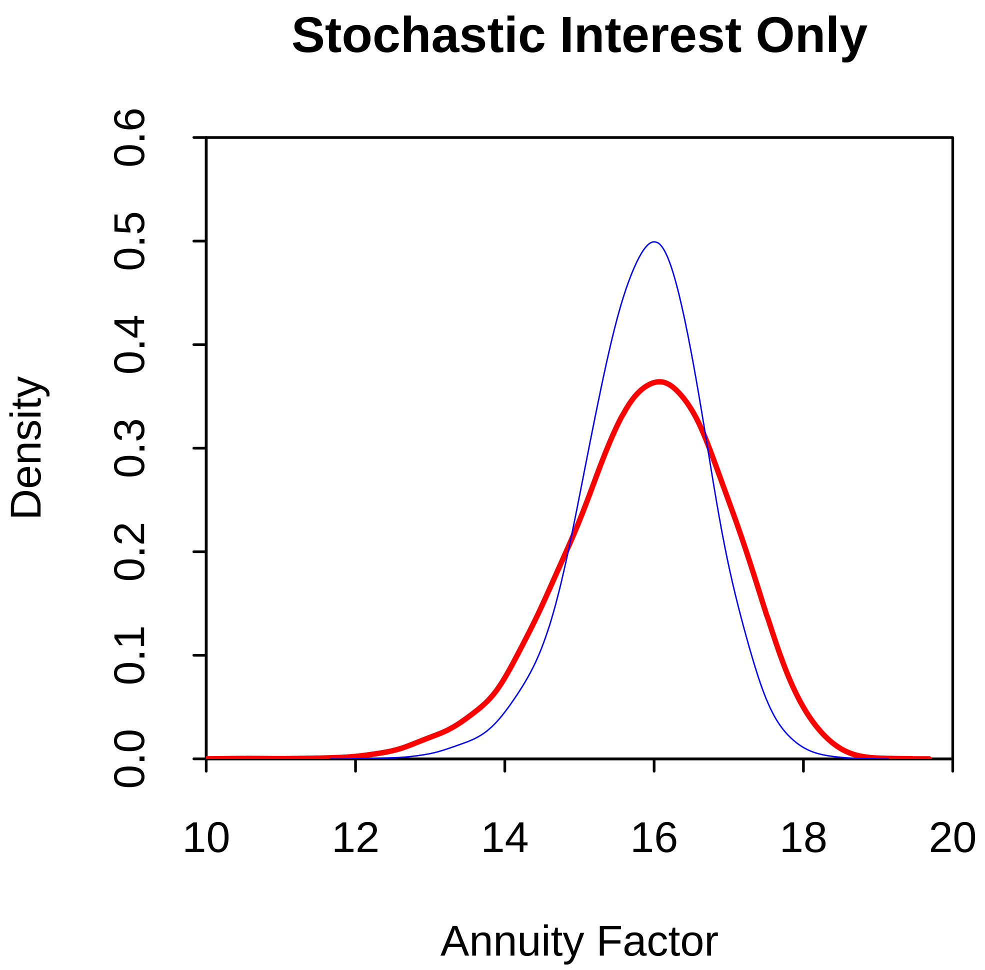
- Option value is

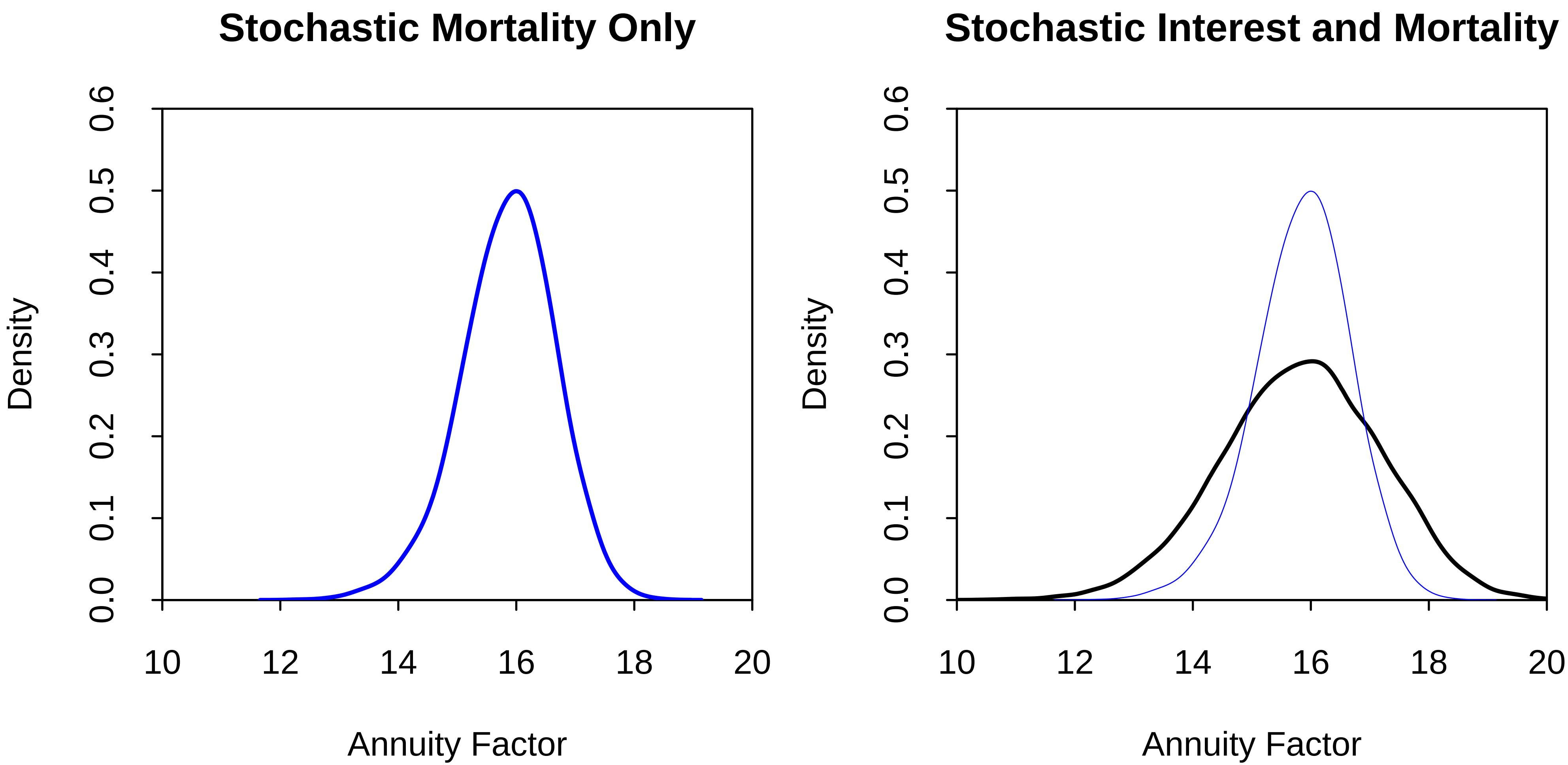
$$\frac{1000}{g}\max\{a_{65}(T) - g, 0\}$$

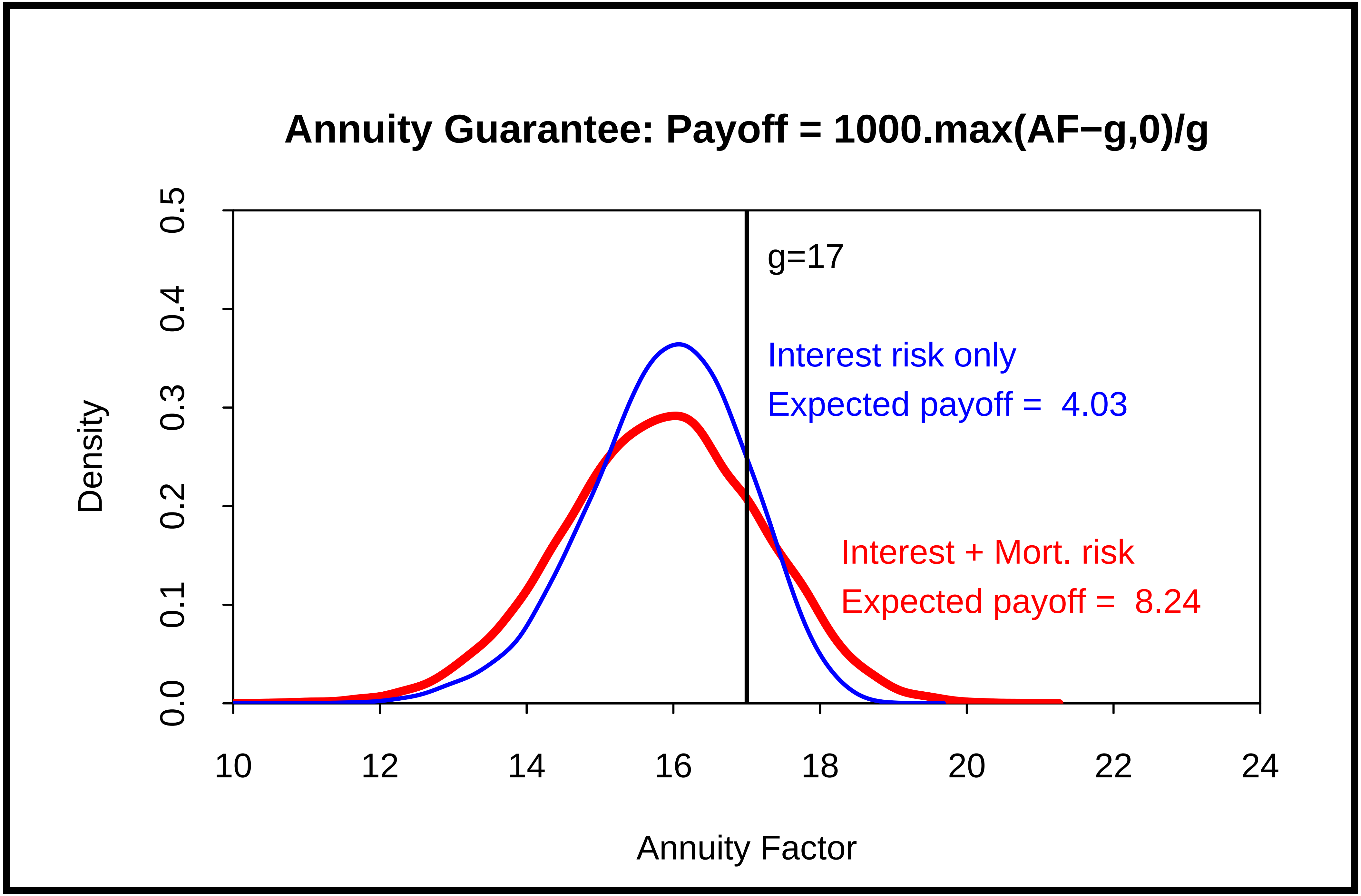
- $a_{65}(T)$ depends on real rates of interest, and on the mortality table in use at T


(A_1(2032),A_2(2032))
A_2(2032)
0.08
0.10
0.12
0.14
0.16
−16
−15
−14
−13
−12
−11
−10
A_1(2032)
Stochastic Mortality Only
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
10
12
14
16
18
20
Annuity Factor


Real Yield in 2032
Density
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5
6
Real Yield (%)

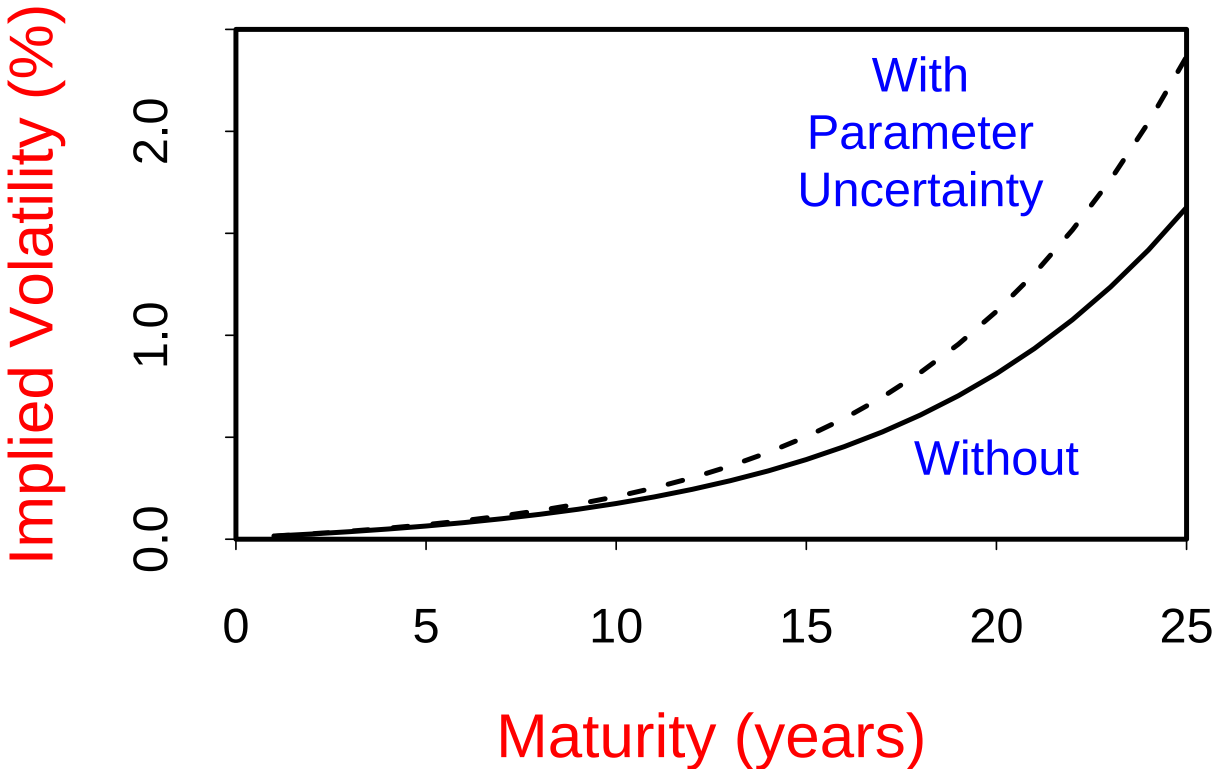

Stochastic Interest Only
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
10
12
14
16
18
20
Annuity Factor


Stochastic Mortality Only
Stochastic Interest and Mortality
Density
0.0
0.1
0.2
0.3
0.4
0.5
0.6
10
12
14
16
18
20
Annuity Factor


Annuity Guarantee: Payoff = 1000.max(AF−g,0)/g
Density
0.0
0.1
0.2
0.3
0.4
0.5
10
12
14
16
18
20
22
24
Annuity Factor
g=17
Interest risk only
Expected payoff = 4.03
Interest + Mort. risk
Expected payoff = 8.24

Application 5: At-the-Money Call Options on $S(t)$

Payoff: $\max\{S(T) - K,\ 0\}$ where $K = E_{Q(\lambda)}[S(T)]$



25-year survivor cap: parameter uncertainty adds 33%

Conclusions

- Stochastic models important for
  - risk measurement $\longrightarrow$ assessment of risk premium
  - pricing contracts with option characteristics
- One model out of many possibilities
- Significant longevity risk in the medium/long term
- Model and parameter risk is important

Conclusions

- The significance of longevity risk varies from one problem to the next:
  - In absolute terms
  - As a percentage of the total risk

Selected References

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Andrew Cairns

Heriot-Watt University

E-mail: A.Cairns@ma.hw.ac.uk

Internet: www.ma.hw.ac.uk/∼andrewc