# *Optimal Dynamic Asset Allocation for Defined Contribution Pension Plans*

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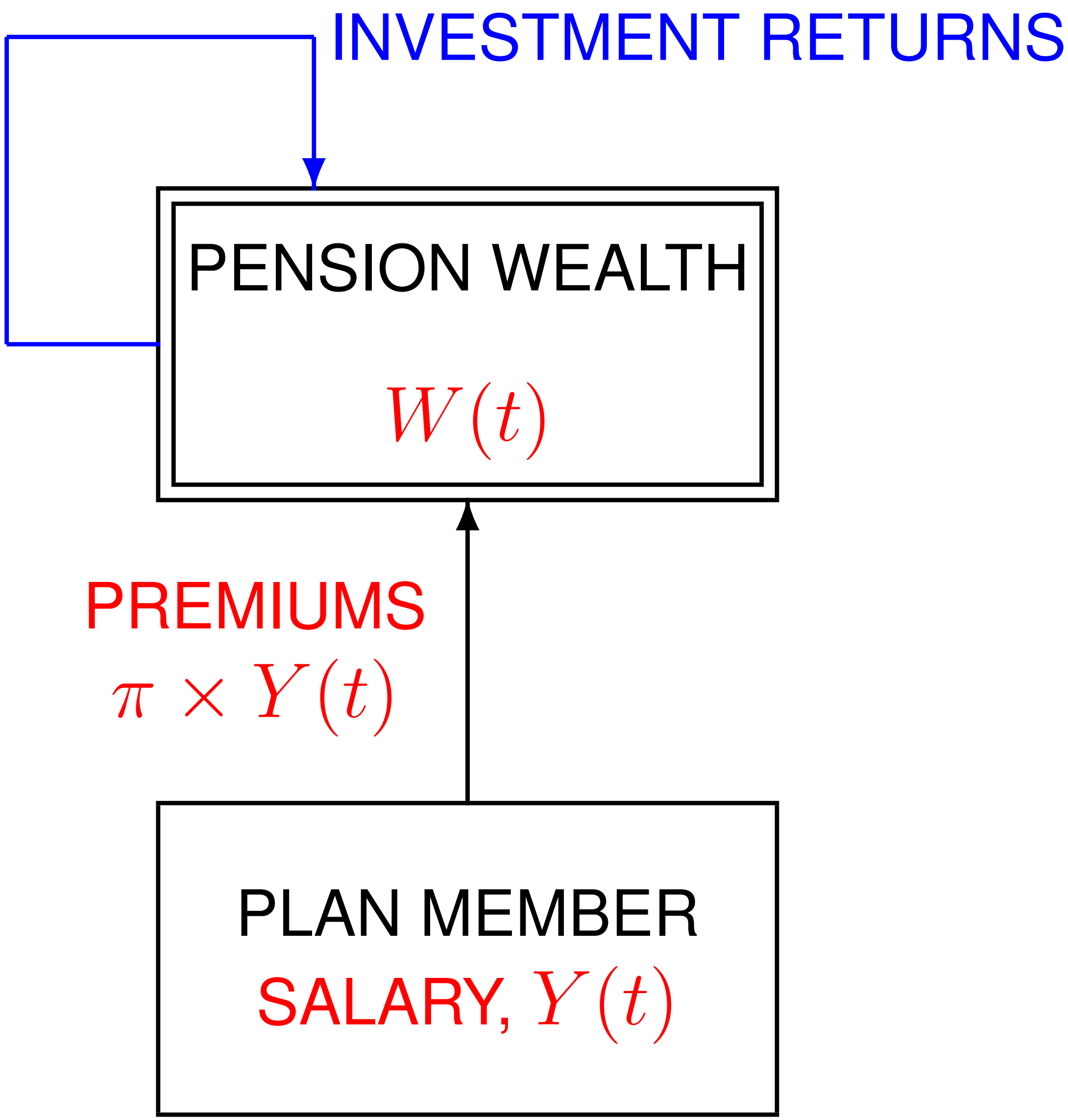
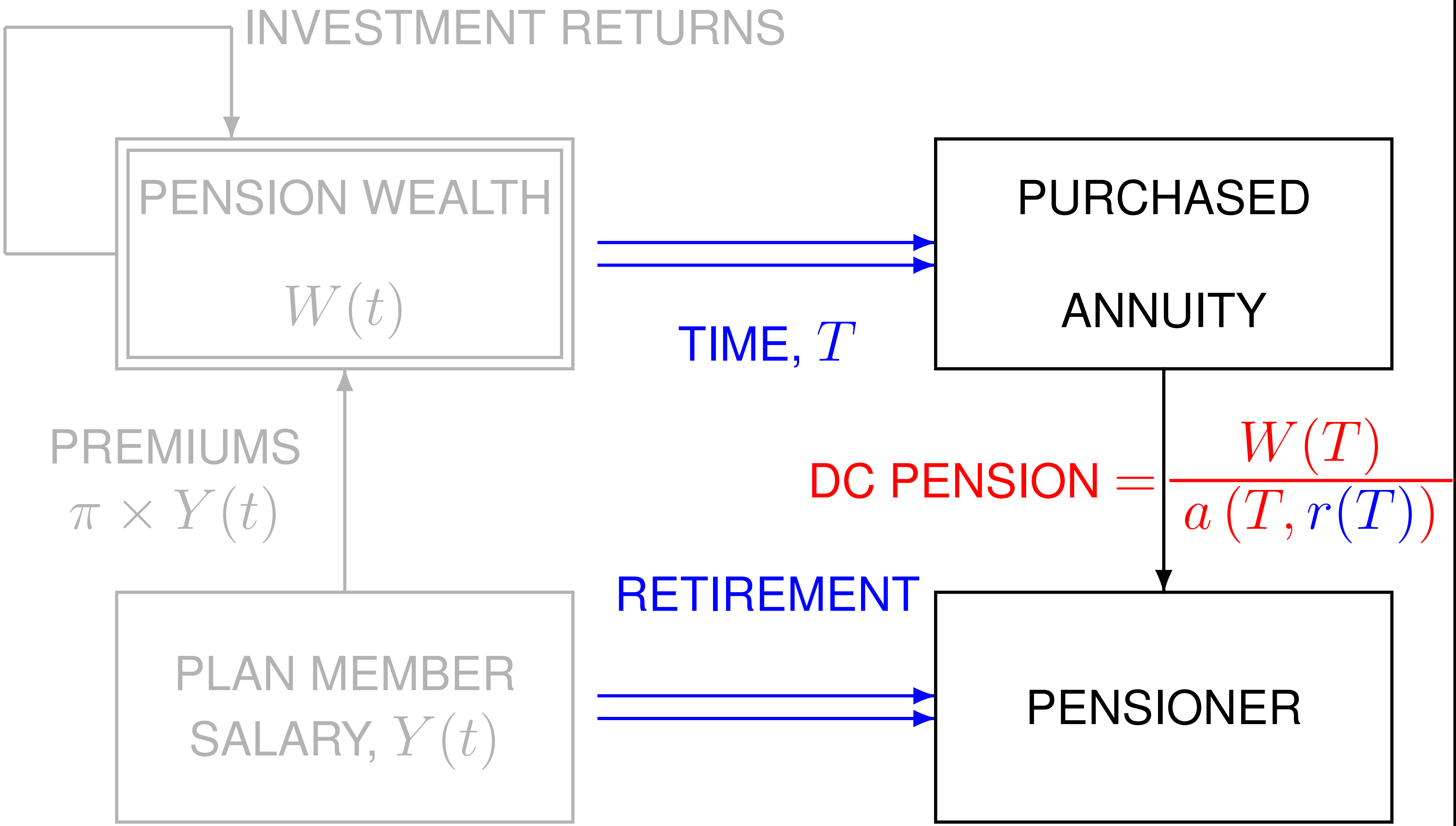
## Outline for talk

- A basic Defined Contribution (DC) pension plan
- Model formulation
- Optimal investment strategy up to retirement
- Development of numerical results
  - Comparison of optimal strategy with commercial strategies

Using a toy model: how much room for improvement?




A simple occupational DC plan




A simple occupational DC plan

## How do we measure the success of a DC plan?

Informally:

$$\text{Replacement Ratio} = \frac{\text{DC pension}}{\text{final salary}}$$

"Model" Occupational DC plan

- Contributions = fixed % of salary
- Various asset classes
- Compare:
  - "commercial" investment strategies
  - optimal dynamic asset strategy

## The model

State variables:

$$Y(t) = \text{Salary or } \textit{Labour Income}$$

$$W(t) = \text{Accumulated pension wealth}$$

$$r(t) = \text{Risk-free interest rate (one-factor model)}$$

## The model: Assets

$N+1$ sources of risk: $Z_0(t)$, $Z_1(t)$, $\ldots$, $Z_N(t)$

Cash account, $R_0(t)$:

$$dR_0(t) \; = \; r(t)R_0(t)dt$$

$$dr(t) \; = \; \mu_r(r(t))dt + \sum_{j=1}^{N} \sigma_{rj}(r(t))dZ_j(t)$$

# The model: Assets

Risky assets, $R_1(t), \ldots, R_N(t)$:

$$dR_i(t) = R_i(t)\left[\left(r(t) + \sum_{j=1}^{N} \sigma_{ij}\xi_j\right) dt + \sum_{j=1}^{N} \sigma_{ij} dZ_j(t)\right]$$

$$C \quad = \quad \left(\ \sigma_{ij}\ \right) \quad = \quad \text{volatility matrix } (N \times N)$$

(non-singular)

$$\xi \quad = \quad \left(\ \xi_j\ \right) \quad = \quad \text{market prices of risk } (N \times 1)$$

## The model: Salary and contributions

$$dY(t) = Y(t)\Big[(r(t) + \mu_Y(t))\,dt + \sum_{j=1}^{N} \sigma_{Yj} dZ_j(t) + \sigma_{Y0} dZ_0(t)\Big]$$

$\mu_Y(t)$ deterministic

Plan member contributes continuously into DC pension plan at the rate $\pi Y(t)$ for constant $\pi$.

The model: Pension wealth, $W(t)$:

$$p(t) = \big(p_1(t), \ldots, p_N(t)\big)$$
$$= \text{proportion of wealth in risky assets}$$

$$dW(t) = W(t)\Big[\,(r(t) + p(t)'C\xi)\,dt + p(t)'C\,dZ(t)\Big]$$

$$+\ \pi Y(t)dt$$

The model: The pension:

Retirement at a fixed date $T$.

Annuity: At $T$ the cost of \$1 for life is

$$a(r(T)) = \sum_{u=0}^{\infty} p(65,u) P(T, T+u;\ r(T))$$

$$p(65,u) = \text{survival probability from } 65 \text{ to } 65+u$$

$$P(T,\tau;\ r) = \text{price at } T \text{ for \$1 at } \tau$$

$$\text{given } r(T) = r$$

## Replacement ratio:

$$RR(T) = \frac{Pension(T)}{Y(T)} = \frac{W(T)/a(r(T))}{Y(T)}$$

## Teminal utility

$$u(w, y, r) = \frac{1}{\gamma} RR(T)^{\gamma} = \frac{1}{\gamma}\left(\frac{w/a(r)}{y}\right)^{\gamma}$$

($\Rightarrow$ type of habit formation:

plan member is accustomed to consuming at the rate of $(1 - \pi)Y(T)$.)

## Reduction of state space:

Sufficient to model $r(t)$ and $X(t) = W(t)/Y(t)$

$$
\begin{aligned}
dX(t) &= \pi dt \\
&+X(t)\Big[\Big(-\mu_Y(t) + p(t)'C(\xi - \sigma_Y) + \sigma_{Y0}^2 + \sigma_Y'\sigma_Y\Big)dt \\
&\qquad -\sigma_{Y0}dZ_0(t) + \Big(p(t)'C - \sigma_Y'\Big)dZ(t)\Big]
\end{aligned}
$$

$$
\begin{aligned}
\sigma_Y &= (\sigma_{Y1}, \ldots, \sigma_{Yn})' \\
\xi &= (\xi_1, \ldots, \xi_n)'
\end{aligned}
$$

Optimisation: Given strategy $p(t)$

Expected terminal utility is $J(t, x, r; p) =$

$$E\left[\gamma^{-1}\left(\frac{X_p(T)}{a(r(T))}\right)^{\gamma} \middle| \; X(t) = x, \; r(t) = r\right]$$

$X_p(t) =$ path of $X(t)$ given strategy $p$.

Objective:

Maximise expected terminal utility

over Markov strategies $p = \{p(t) : 0 \leq t \leq T\}$

$$V(t,x,r) = \sup_p J(t,x,r;p)$$

HJB equation $\Rightarrow$ nonlinear PDE

$$\begin{aligned}
&V_t \\
&+\mu_r(r)V_r \\
&+\big(\pi - \tilde{\mu}_Y(t)x + \sigma_Y'(\xi-\sigma_Y)x\big)V_x \\
&+\frac{1}{2}\sigma_r(r)'\sigma_r(r)V_{rr} \\
&+\frac{1}{2}\sigma_{Y0}^2 x^2 V_{xx} \\
&-\frac{1}{2}(\xi-\sigma_Y)'(\xi-\sigma_Y)\frac{V_x^2}{V_{xx}} \\
&-(\xi-\sigma_Y)'\sigma_r(r)\frac{V_xV_{xr}}{V_{xx}} \\
&-\frac{1}{2}\sigma_r(r)'\sigma_r(r)\frac{V_{xr}^2}{V_{xx}} \quad = \quad 0.
\end{aligned}$$

## Problem factors

Combination of

- Premiums $\pi Y(t) > 0$
- non-hedgeable salary risk

$\Rightarrow$ no analytical solution

AIM: to develop a full numerical solution

Build up and learn from:

- Constant interest case
- Hedgeable salary case

## Constant-interest case: $r(t) = r$

One risky asset

$$
\begin{aligned}
dR_0(t) &= rR_0(t)dt \\
dR_1(t) &= R_1(t)\left[\left(r + \xi_1\sigma_1\right)dt + \sigma_1 dZ_1(t)\right]
\end{aligned}
$$

$$
\begin{aligned}
dY(t) &= Y(t)\Big[\left(r + \mu_Y\right)dt + \sigma_{Y0} dZ_0(t) \\
&\qquad + \sigma_{Y1} dZ_1(t)\Big]
\end{aligned}
$$

Wealth, $W(t)$:

$$dW(t) = W(t)\left[(r + p(t)\xi_1\sigma_1)dt + p(t)\sigma_1 dZ_1(t)\right]$$

$$+ \ \pi Y(t)dt$$

$p(t) =$ proportion of wealth in equities

$X(t) = W(t)/Y(t) =$ sufficient state variable

## Terminal utility

$$U(W(T), Y(T)) = \frac{1}{\gamma}\left(\frac{W(T)}{Y(T)}\right)^{\gamma}$$

Equivalent to

$$U(X(T)) = \frac{1}{\gamma}(X(T))^{\gamma}$$

$$V(t, x) = \sup_{p(s,X(s))} \left\{ E\left[U\Big(X_p(T)\Big) \,\Big|\, X(t) = x\right] \right\}$$

Problem case: $\pi > 0$ and $\sigma_{Y0} > 0$.

Partial solution: For $0 < t < T$ and $x > 0$:

$$p^*(t,x) = p^*(t,x;V) = \frac{1}{\sigma_1}\left(\sigma_{Y1} - \frac{V_x}{xV_{xx}}(\xi_1 - \sigma_{Y1})\right)$$

Partial solution: For $0 < t < T$ and $x > 0$:

$$p^*(t,x) = p^*(t,x;V) = \frac{1}{\sigma_1}\left(\sigma_{Y1} - \frac{V_x}{xV_{xx}}(\xi_1 - \sigma_{Y1})\right)$$

$\Rightarrow$HJB-PDE:

$$V_t + \left(\pi + \delta x\right)V_x + \frac{\sigma_{Y0}^2 x^2}{2}V_{xx} - \frac{(\xi_1 - \sigma_{Y1})^2}{2}\frac{V_x^2}{V_{xx}} = 0$$

$$\delta = -\mu_Y + \sigma_{Y0}^2 + \sigma_{Y1}\xi_1$$

Boundary condition: $V(T,x) = \gamma^{-1}x^\gamma$.

## Numerical solution: Explicit Finite Difference Method

Problem (e.g. $\gamma < 0$) as $x \to 0$:

$$V(t,x) \;\to\; \begin{cases} -\infty, & \text{if } t = T \\ l(t), & -\infty < l(t) < 0,\ t < T \end{cases}$$

$$\frac{\partial V}{\partial t}(t,0) \;\to\; -\infty \quad \text{as } t \to T$$

$\Rightarrow$ numerical solution: unstable near $x = 0$ ??

## Solution (??):

- Transform $V(t,x)$ to $U(t,x) = \big(\gamma V(t,x)\big)^{1/\gamma}$

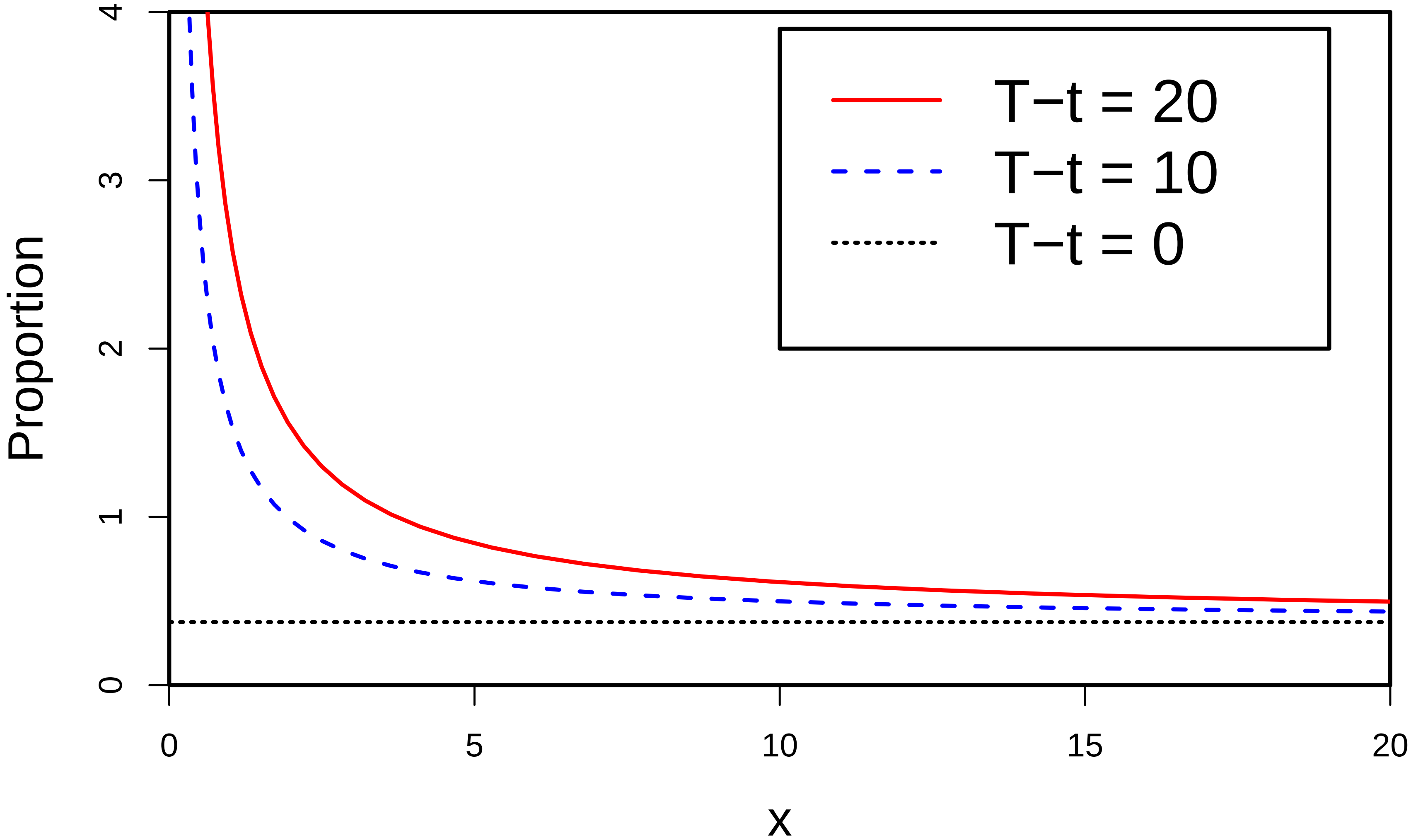
  $\Rightarrow U(T,x) = x$
- Transform $x$ to $y = \log x$
- Stability in FD scheme $\Rightarrow$

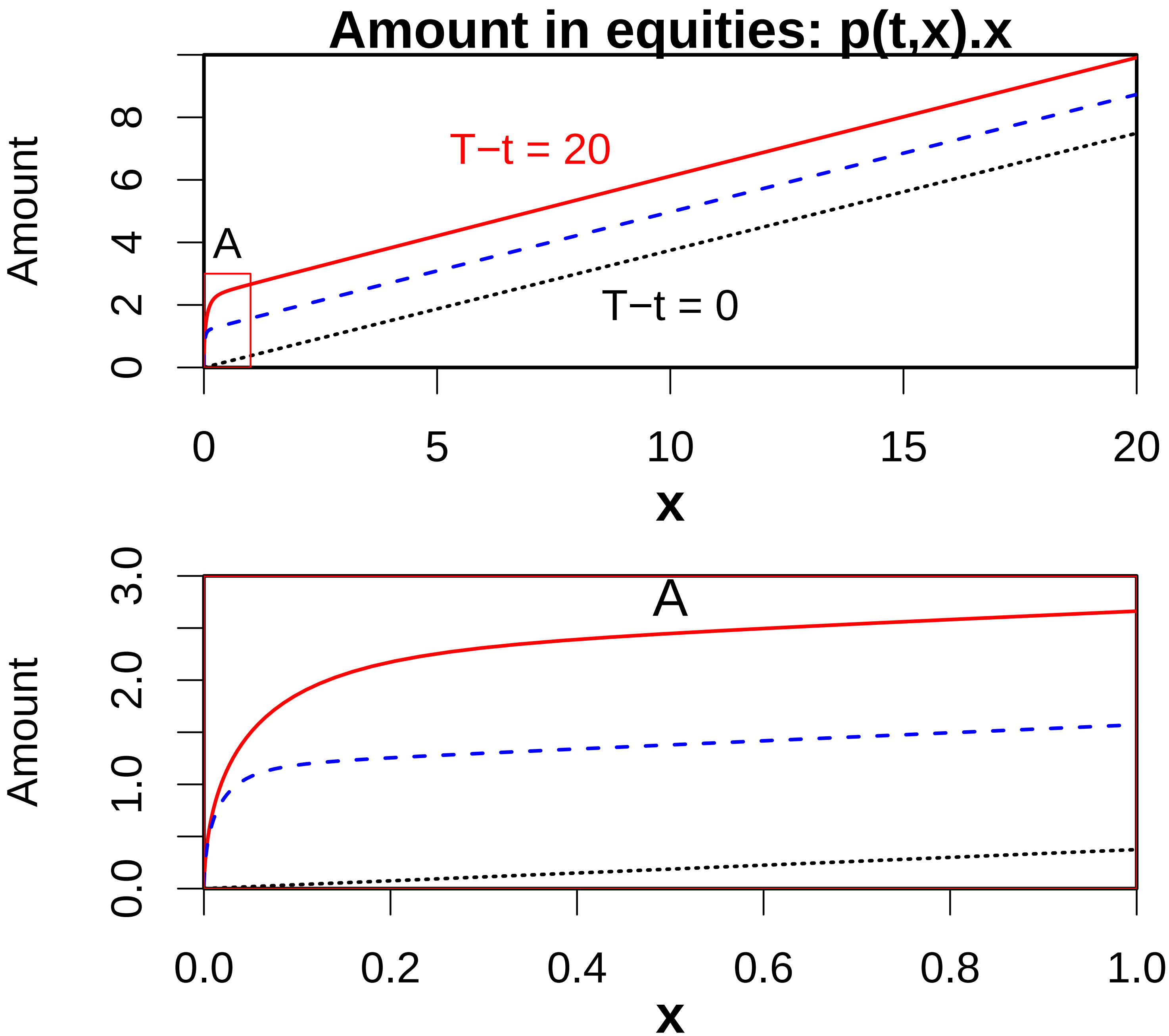
$$\sigma_Y^{p^*\,2} \frac{\Delta t}{\Delta y^2} < 1$$

over the range of $x$.

$\Rightarrow$ keep to $-\infty < y_0 < y < y_1 < \infty$

# Proportion in Equities: p(t,x)




Amount in equities: p(t,x).x
Amount
0 2 4 6 8
A
T−t = 20
T−t = 0
0 5 10 15 20
x
Amount
0.0 1.0 2.0 3.0
A
0.0 0.2 0.4 0.6 0.8 1.0
x

Numerical results $\Rightarrow$ for $t < T$

$$p^*(t,x)\sqrt{x} \to \phi \text{ as } x \to 0$$

Value of $\phi$ is critical!

- $\phi = \infty \quad \Rightarrow \quad X(t)$ might hit 0 or become $-$ve

** $0 < \phi < \infty \quad \Rightarrow \quad X(t)$ might or might not hit zero

** $\phi = 0 \quad \Rightarrow \quad X(t)$ never hits 0

Numerical solutions suggest (**): $\phi < \infty$.

## Upper bound

Introduce an extra asset, $R_2(t)$, to complete the market.

$$dR_2(t) \;=\; R_2(t)\Big[\,(r + \xi_0\sigma_{Y0})\,dt + \sigma_{Y0}dZ_0(t)\Big].$$

$\xi_0 =$ arbitrary market price of risk: to be specified

More choice $\Rightarrow$ increased $E[u(W(T), Y(T))]$

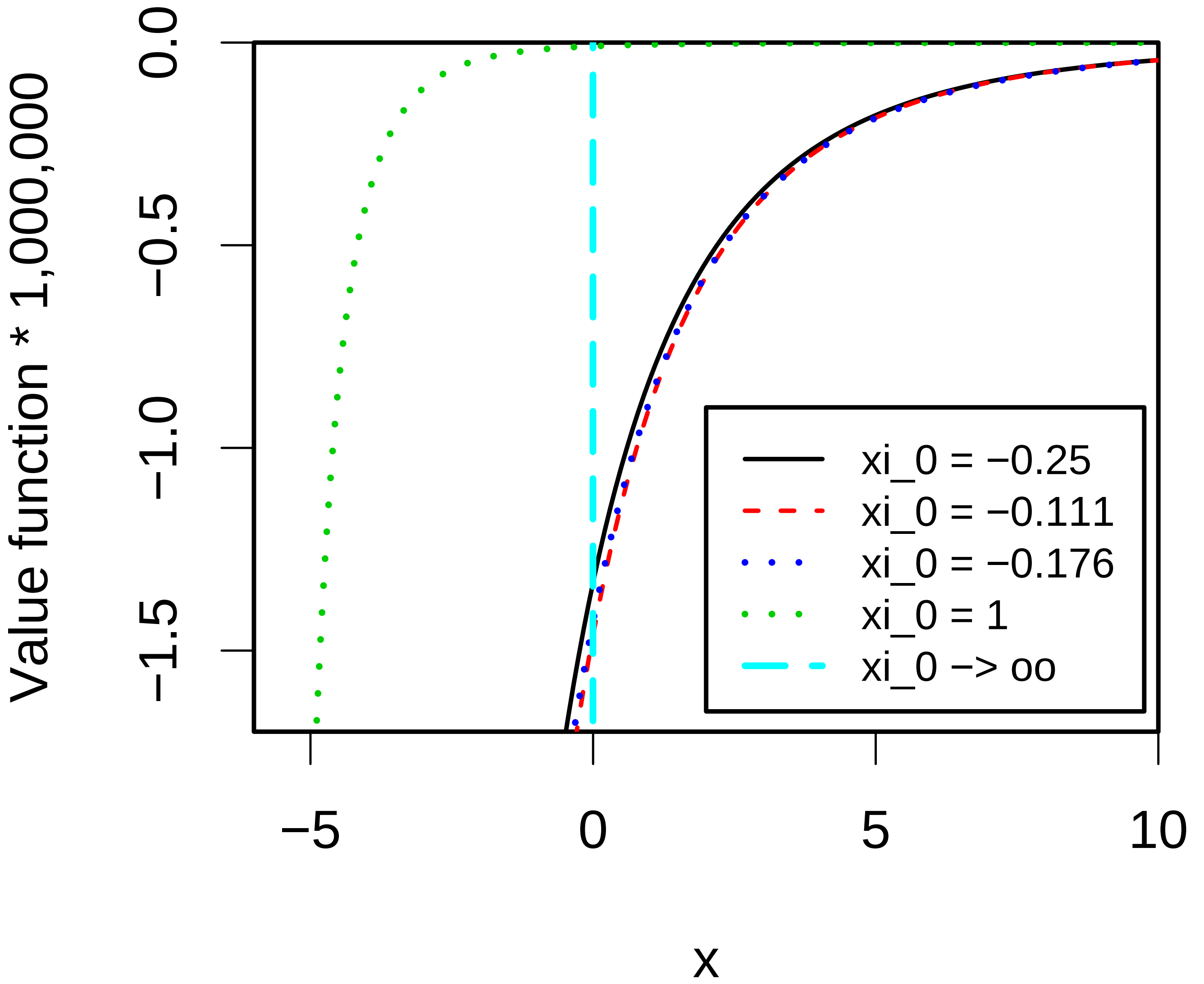
## Upper bound

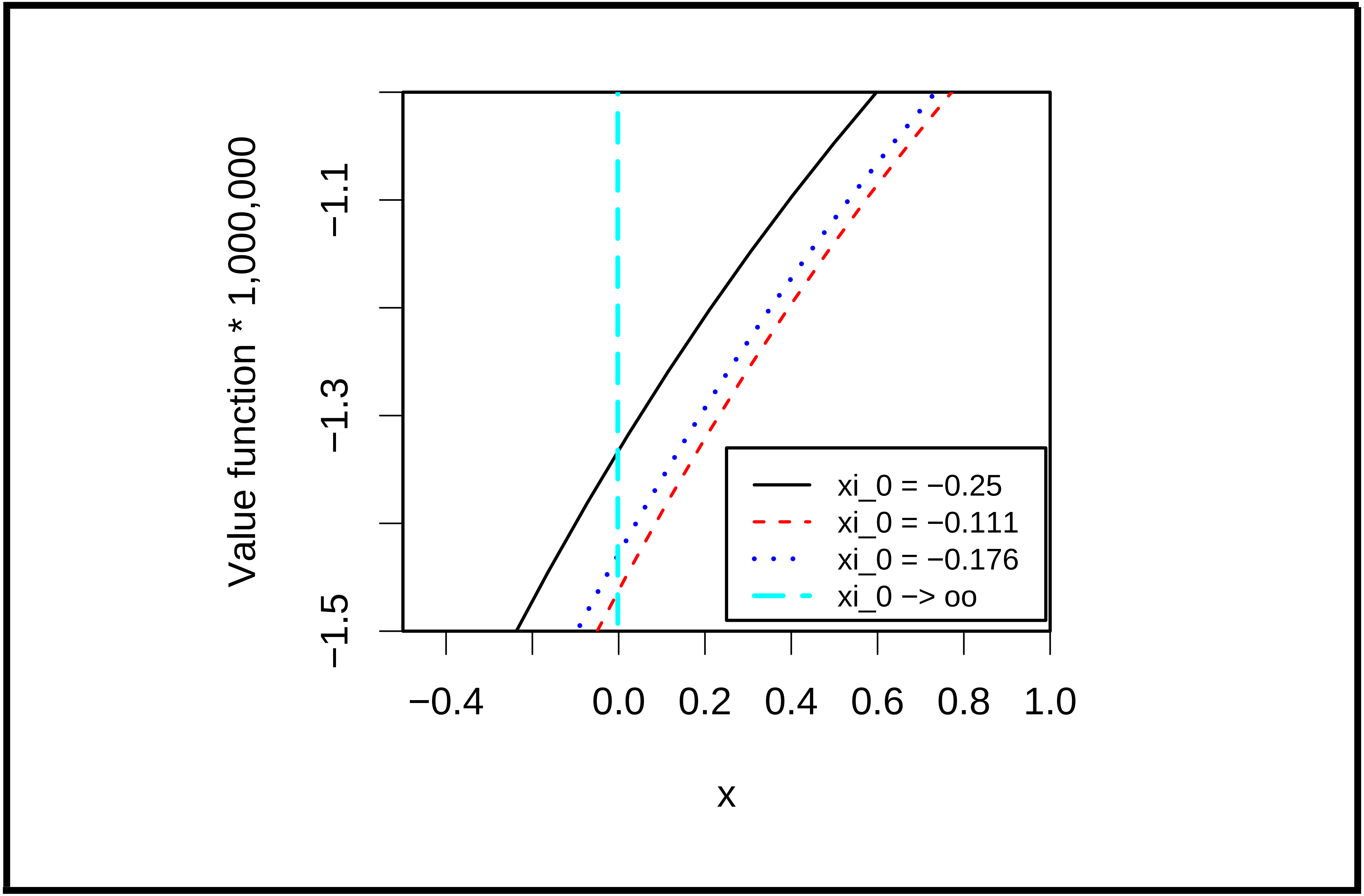
Complete market

$\Rightarrow$ analytical upper bound, $V^u(t, x; \xi_0)$, for each $\xi_0$.

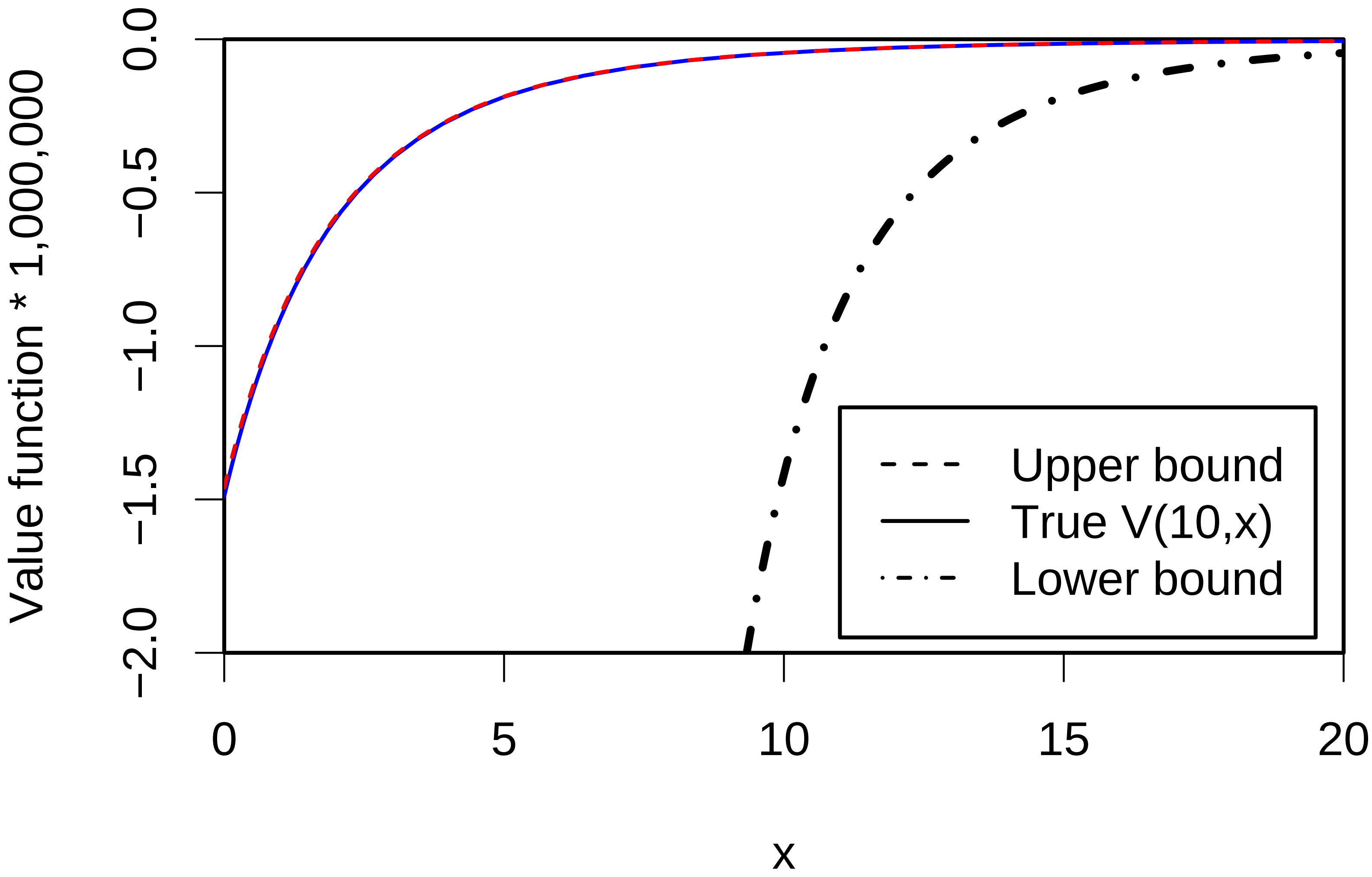
Then $V(t, x) \le V^u(t, x) = \inf_{\xi_0 \in R} V^u(t, x; \xi_0)$

# Construction of the upper bound: $\gamma = -5,\ T - t = 10$





Value function * 1,000,000
−1.1
−1.3
−1.5
−0.4
0.0
0.2
0.4
0.6
0.8
1.0
x
xi_0 = −0.25
xi_0 = −0.111
xi_0 = −0.176
xi_0 −> oo




True $V(t,x)$ versus upper and lower bounds.

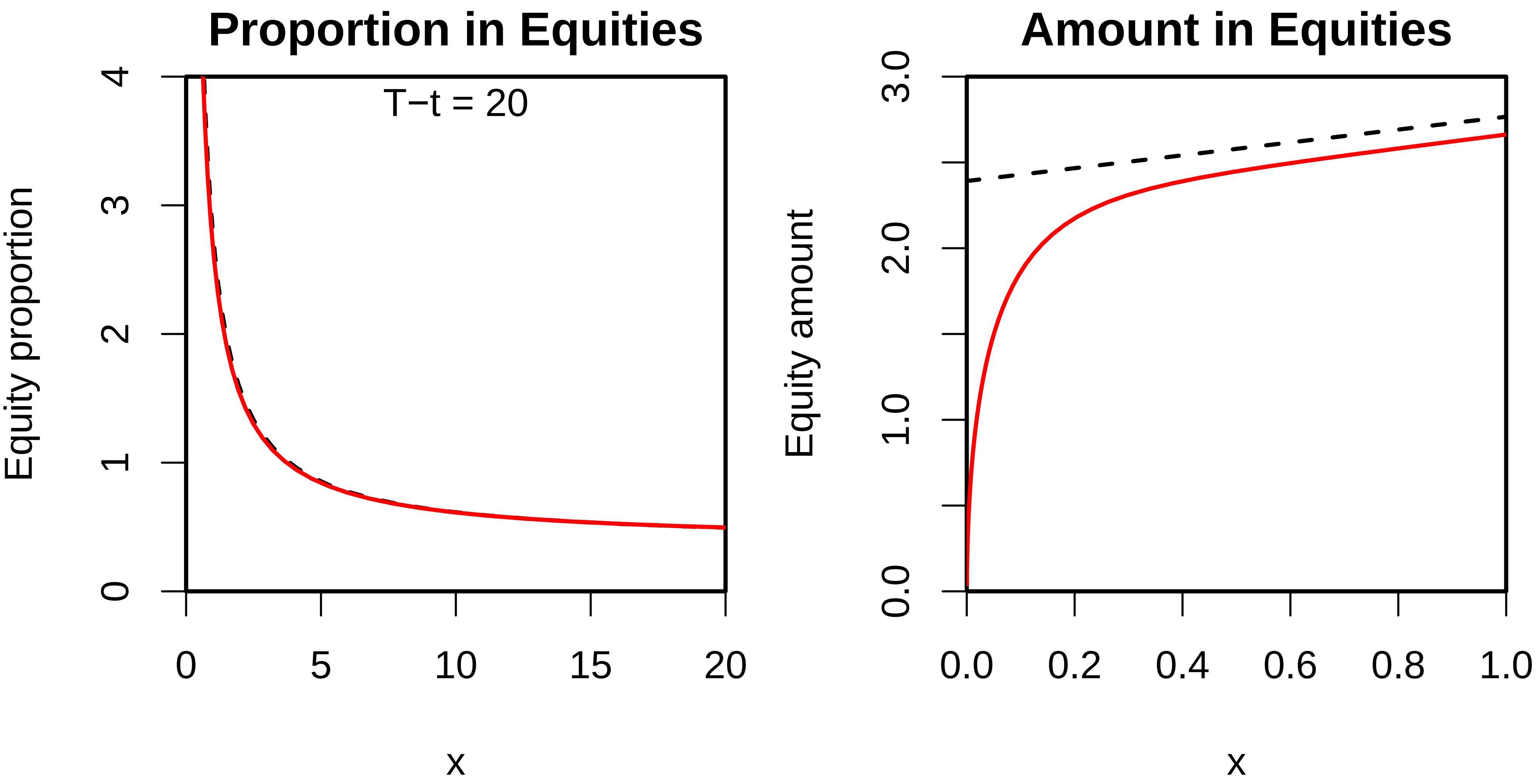
All three $= O(x^\gamma)$ as $x \to \infty$.

## Upper bound: Caution

- Upper bound is close to true $V(t,x)$
- BUT e.g.

$$p^u(t,x) = \frac{1}{\sigma_1}\left(\sigma_{Y1} - \frac{V_x^u}{xV_{xx}^u}(\xi_1 - \sigma_{Y1})\right)$$

is unsuitable as a good (?) but suboptimal strategy.


Proportion in Equities
T−t = 20
Equity proportion
0
1
2
3
4
0
5
10
15
20
x
Amount in Equities
Equity amount
0.0
1.0
2.0
3.0
0.0
0.2
0.4
0.6
0.8
1.0
x

## Cost of suboptimality

Start from $X(t) = 0$

Benchmark: $\pi = 10\% +$ optimal $p^*(t, x)$

Suboptimal strategy: $p(t, x) =$ constant

What contribution rate, $\pi$, do we need to pay to get the same expected utility?

What contribution rate, $\pi$, do we need to pay to get the same expected utility?

| | Contribution rate | | |
|---|---|---|---|
| $T-t$ | $p^*(t,x)$ | $p=0.375$ | $p=0.16667$ |
| 10 | 10% | 10.06% | 10.35% |
| 20 | 10% | 10.12% | 10.71% |

$p = 0.375$ (Accounts for salary/asset correlation)

$p = 0.16667$ (No accounting for salary/asset correlation)

## Hedgeable salary risk

- stochastic interest (Vasicek)
- many assets
- $\sigma_{Y0} = 0$

$$U\Big(W(T), Y(T), r(T)\Big) = \frac{1}{\gamma}\left(\frac{W(T)/a(r(T))}{Y(T)}\right)^{\gamma}$$

## Key features of solution

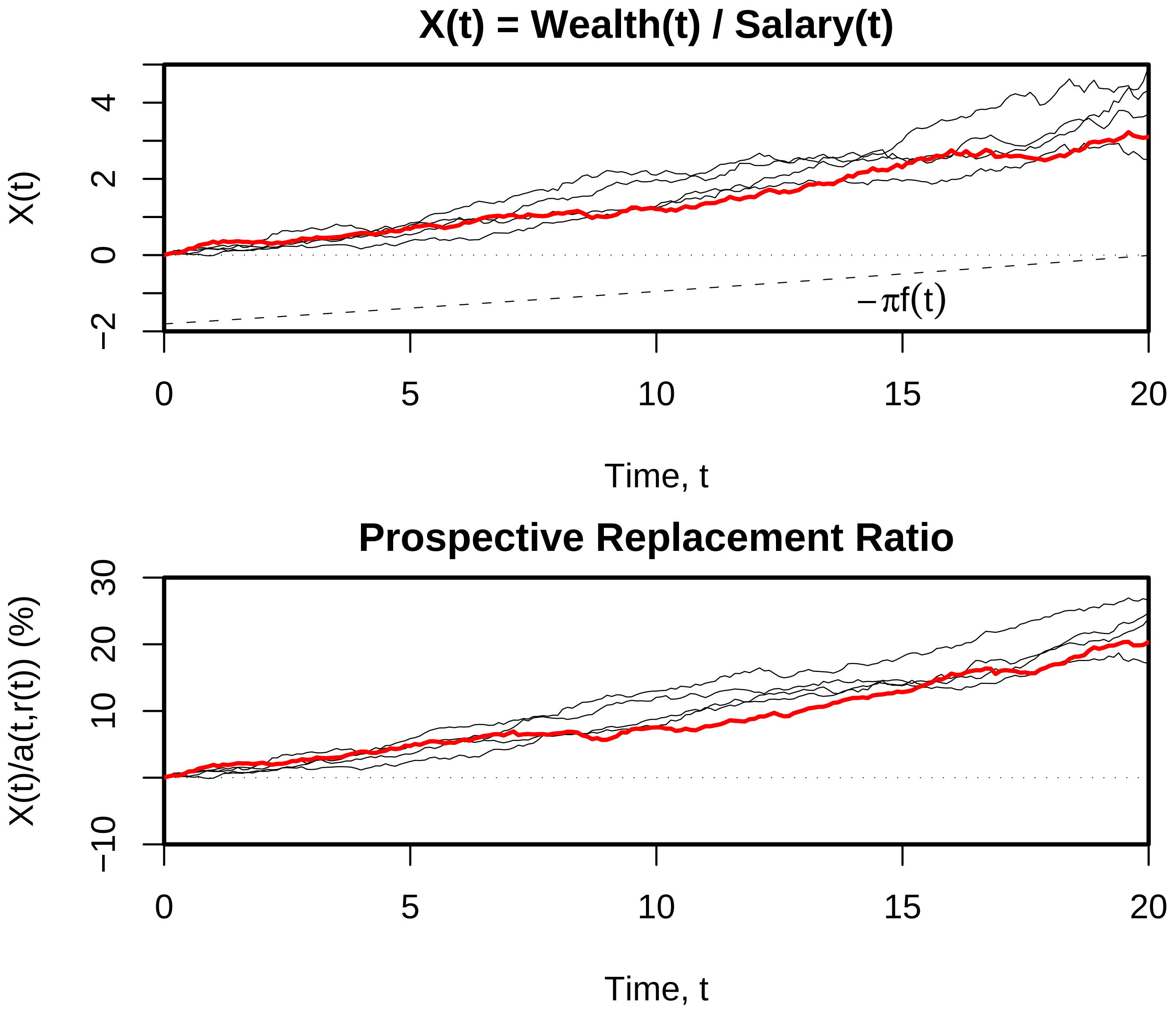
- Hedgeable salary $\Rightarrow$ can capitalise future premiums
- augmented fund $=$ wealth $+$ p.v. future premiums
- 3-fund theorem $\Rightarrow$
  - risky mutual fund (fixed % of augmented fund)
  - low-risk cash fund (% of augmented fund $\searrow$ with $t$)
  - low-risk bonds fund (% of augmented fund $\nearrow$ with $t$)
- "Low risk" $\Rightarrow$ relative to salary ($+$ annuity) numeraire

## Numerical example: $r(t) \sim$ Vasicek

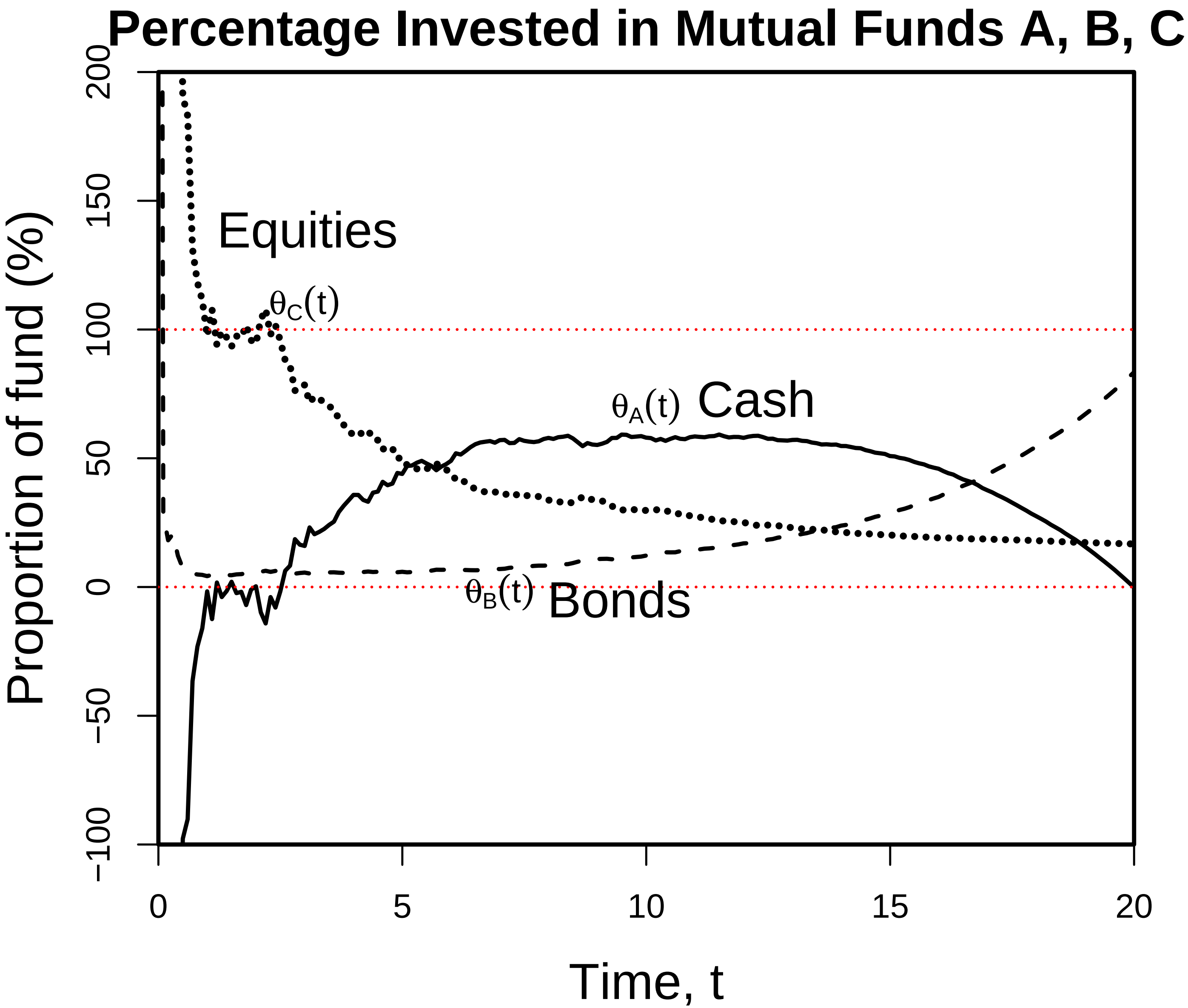
### Example 1:

- Relative risk aversion: $RRA = 6$ ($\gamma = -5$)
- Duration of contract: $T = 20$ years
- Contribution rate: 10% of salary

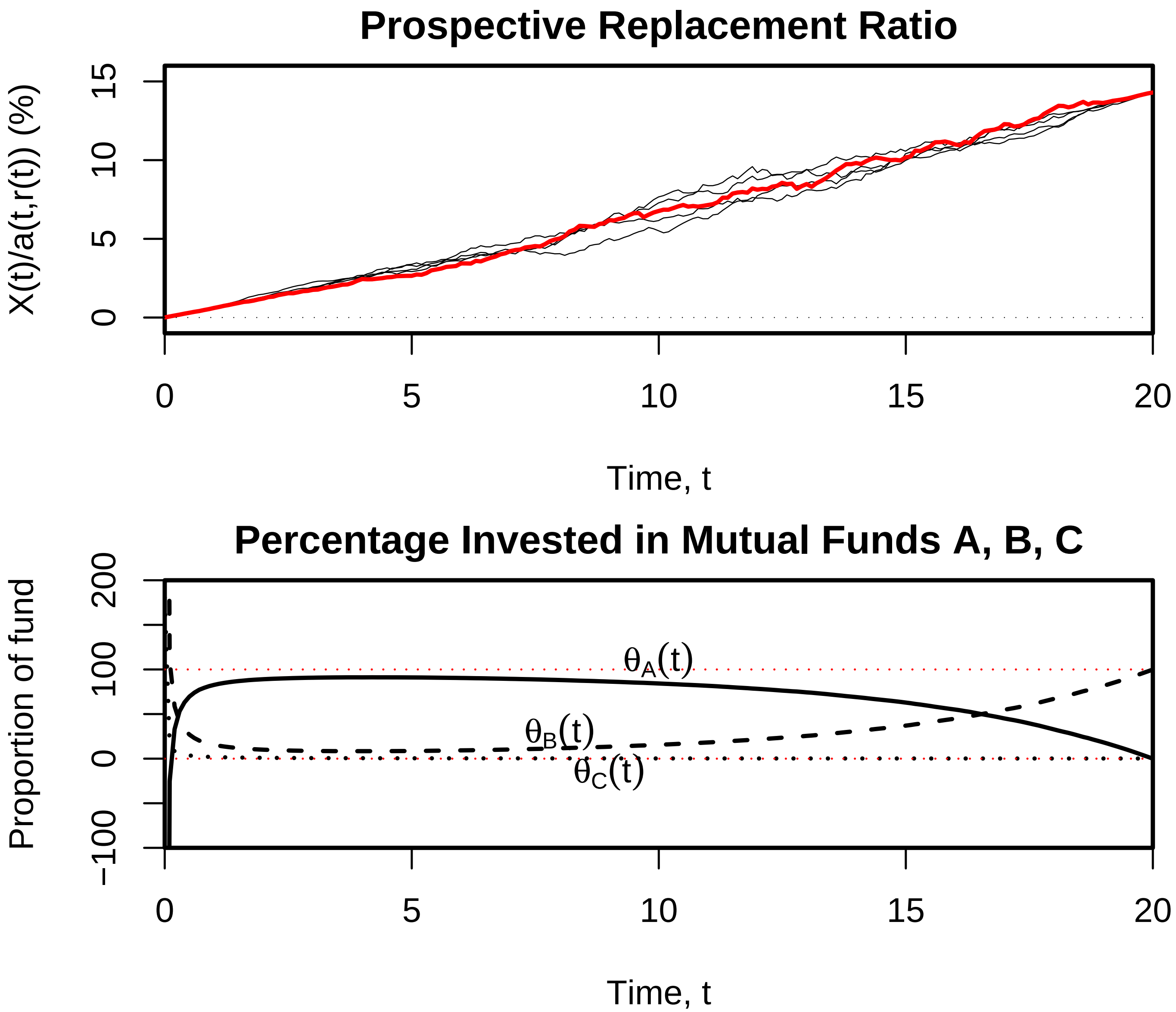
# Example 1: $RRA = 6, T = 20$



# Example 1: $RRA = 6, T = 20$



# Example 2: Very high $RRA$, $T = 20$



## The cost of suboptimality

Optimal strategy versus:

- Salary-hedged static strategy (S)
- Merton-static strategy (M)
- Deterministic lifestyle strategies:
  - initially 100% in equities
  - gradual switch over last 10 years into 100% bonds (B-10) or 100% cash (C-10)

| (c) | $RRA = 6, \quad T = 20$ | | | | |
|---|---|---|---|---|---|
| Strategy: | Optimal stochastic | Static | | Deterministic lifestyle | |
| | | S | M | B-10 | C-10 |
| $V(0,0)$ | -100 | -134.58 | -205.42 | -141.00 | -191.47 |
| Cost | 10.00% | 10.61% | 11.55% | 10.71% | 11.39% |

Cost:

- Benchmark: 10% cont. rate with optimal strategy
- Other strategies: % contribution rate to match optimal utility

| (c) | $RRA = 6, \quad T = 20$ | | | | |
|---|---|---|---|---|---|
| Strategy: | Optimal | Static | | Deterministic lifestyle | |
| | stochastic | S | M | B-10 | C-10 |
| Cost | 10.00% | 10.61% | 11.55% | 10.71% | 11.39% |

| (d) | $RRA = 6, \quad T = 40$ | | | | |
|---|---|---|---|---|---|
| Strategy: | Optimal | Static | | Deterministic lifestyle | |
| | stochastic | S | M | B-10 | C-10 |
| Cost | 10.00% | 11.52% | 12.58% | 12.86% | 13.67% |

| (a) | $RRA = 1, \quad T = 20$ | | | | |
|---|---|---|---|---|---|
| Strategy: | Optimal | Static | | Deterministic lifestyle | |
| | stochastic | S | M | B-10 | C-10 |
| Cost | 10.00% | 13.79% | 13.78% | 20.18% | 21.39% |

| (c) | $RRA = 6, \quad T = 20$ | | | | |
|---|---|---|---|---|---|
| Strategy: | Optimal | Static | | Deterministic lifestyle | |
| | stochastic | S | M | B-10 | C-10 |
| Cost | 10.00% | 10.61% | 11.55% | 10.71% | 11.39% |

| (e) | $RRA = 12, \quad T = 20$ | | | | |
|---|---|---|---|---|---|
| Strategy: | Optimal | Static | | Deterministic lifestyle | |
| | stochastic | S | M | B-10 | C-10 |
| Cost | 10.00% | 10.61% | 12.08% | 11.70% | 12.65% |

| (b) | $RRA = 1, \quad T = 40$ | | | | |
|---|---|---|---|---|---|
| Strategy: | Optimal | Static | | Deterministic lifestyle | |
| | stochastic | S | M | B-10 | C-10 |
| Cost | 10.00% | 17.37% | 17.36% | 32.21% | 34.33% |

| (d) | $RRA = 6, \quad T = 40$ | | | | |
|---|---|---|---|---|---|
| Strategy: | Optimal | Static | | Deterministic lifestyle | |
| | stochastic | S | M | B-10 | C-10 |
| Cost | 10.00% | 11.52% | 12.58% | 12.86% | 13.67% |

| (f) | $RRA = 12, \quad T = 40$ | | | | |
|---|---|---|---|---|---|
| Strategy: | Optimal | Static | | Deterministic lifestyle | |
| | stochastic | S | M | B-10 | C-10 |
| Cost | 10.00% | 12.38% | 13.17% | 16.57% | 17.82% |

## Summary

- Numerical methods important to assess the cost of suboptimality.
- Commercial strategies can be costly
- Stochastic interest $\Rightarrow$ important dynamic element in asset strategy
- Next step: to combine results for
  - constant interest, incomplete market
  - stochastic interest, complete market