Derivatives Pricing and Financial Modelling

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Tutorial 9

1. (*) The Vasicek model.

   Derive the form of the forward rate curve, $f(t,T)$ for the Vasicek model in terms of α, μ, σ and $r(t)$, given the pricing formula in the lecture notes.

2. (*) Suppose that X and Y are normal random variables with mean zero, variance 1 and correlation ρ under a measure Q.

   (a) For any real numbers ω and ν, write down:

   $$E_Q\left[e^{-\omega Y - \nu X}\right]$$

   (b) Let P be another measure equivalent to Q defined by the Radon-Nikodym derivative:

   $$\frac{dP}{dQ} = \frac{e^{-\omega Y}}{E_Q\left[e^{-\omega Y}\right]}$$

   Find an expression for $E_Q[\exp(-\omega Y - \nu X)]$ in terms of $E_Q[\exp(-\omega Y)]$ and $E_P[\exp(-\nu X)]$.

   (c) Hence show that under P, X has a normal distribution with mean $-\omega\rho$ and variance 1.

   (d) Use your answers to parts (b) and (c) to show that:

   $$E_Q\left[e^{-Y} I(X < x)\right] = e^{1/2}\Phi(x + \rho)$$

   where $I(X < x) = 1$ if $X < x$ and 0 otherwise and $\Phi(z)$ is the cumulative distribution function of the standard normal distribution.

   (e) Use the fact that $E_Q[e^{-Y} I(X < x)] = E_Q[I(X < x) E_Q[e^{-Y} \mid X]\ ]$ to verify directly (that is, without using the change of measure) the result in part (d).

3. (*) More on the Vasicek model....

Make use of the following facts to investigate the following problems relating to the Vasicek model:

A If $X = \int_0^T g(s)dW_s$ for some deterministic function $g(s)$ then X has a Normal distribution with mean 0 and variance $\int_0^T g(s)^2 ds$.

B If $\sigma(t,u)$ is such that

$$E\left[\int_0^T \left|\int_0^u \sigma(t,u)dW_t\right| du\right] < \infty$$

then

$$\int_0^t \int_0^u \sigma(s,u)dW_s\, du = \int_0^t \int_s^t \sigma(s,u)du\, dW_s$$

C If Y has a normal distribution with mean μ and variance σ^2 then $E[e^Y] = \exp[\mu + \frac{1}{2}\sigma^2]$.

Under the Vasicek model $dr(t) = \alpha(\mu - r(t))dt + \sigma d\tilde{W}_t$ where $\tilde{W}_t$ is a Brownian motion under the equivalent martingale measure Q.

(a) Show that $r(t) = \mu + (r(0) - \mu)e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} d\tilde{W}_s$.

(b) Show that

$$X(T) = \int_0^T r(t)dt = \mu T + (r(0) - \mu)\frac{1 - e^{-\alpha T}}{\alpha} + \sigma \int_0^T \frac{1 - e^{-\alpha(T-s)}}{\alpha} d\tilde{W}_s$$

(c) Find the joint distribution for $(r(T), X(T))$.

(d) Using (b) and (c) find an expression for $P(t,T) = E_Q[\exp\{-(X(T) - X(t))\} \mid \mathcal{F}_t]$, expressing this in the form $\exp[A(t,T) - B(t,T)r(t)]$.

(e) Using part (b), find the joint distribution for $(r(T), X(S))$ where $T < S$.

(f) Consider a European call option with exercise date T and strike price K with the zero-coupon bond maturing at time S (where $T < S$) as the underlying. The price at time 0 of this bond is:

$$C(0) = E_Q\left[e^{-X(T)} \max\{P(T,S) - K, 0\}\right]$$

Let P_1 and P_2 be two measures equivalent to Q defined by the Radon-Nikodym derivatives:

$$\frac{dP_1}{dQ} = \frac{e^{-X(S)}}{E_Q\left[e^{-X(S)}\right]}$$

$$\text{and} \quad \frac{dP_2}{dQ} = \frac{e^{-X(T)}}{E_Q\left[e^{-X(T)}\right]}$$

i. Show that:

$$C(0) = P(0,S)E_{P_1}[I(r(T) < r^*)] - KP(0,T)E_{P_2}[I(r(T) < r^*)]$$

where $r^* = (A(T,S) - \log K)/B(T,S)$ and $I(r(T) < r^*)$ equals 1 if $r(T) < r^*$ and zero otherwise.

ii. Use your answers to question 2 to show that

$$\begin{aligned} C(0) &= P(0,S)\Phi(d_1) - KP(0,T)\Phi(d_2) \\ \text{where } d_1 &= \frac{1}{\sigma_p}\log\frac{P(0,S)}{K.P(0,T)} + \frac{\sigma_p}{2} \\ d_2 &= d_1 - \sigma_p \\ \sigma_p &= \frac{\sigma}{\alpha}\left(1 - e^{-\alpha(S-T)}\right)\sqrt{\frac{1-e^{-2\alpha T}}{2\alpha}} \end{aligned}$$

4. (Cox-Ingersoll-Ross (CIR) etc.)

   (a) Under the CIR model we have

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}d\tilde{W}_t$$

   It is required that $\alpha\mu > \frac{1}{2}\sigma^2$ to keep the process strictly positive. By considering the process $X(t) = \log r(t)$ discuss why this requirement might be necessary (a rigorous proof is not required).

   (b) Under CIR we have prices of the form

$$\begin{aligned} P(t,T) &= \exp[A(t,T) - B(t,T)r(t)] \\ \text{where } B(t,T) &= \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\alpha)\left(e^{\gamma(T-t)}-1\right)+2\gamma} \\ A(t,T) &= \left[\frac{2\gamma e^{(\alpha+\gamma)(T-t)/2}}{(\gamma+\alpha)\left(e^{\gamma(T-t)}-1\right)+2\gamma}\right]^{2\alpha\mu/\sigma^2} \\ \gamma &= \sqrt{\alpha^2+2\sigma^2} \end{aligned}$$

   The Pearson-Sun model uses $dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)-\beta}d\tilde{W}_t$.
   Make use of the CIR pricing formula to derive prices for the PS model.

   (c) Under the Brennan-Schwartz model we have $dr(t) = \alpha(\mu - r(t))dt + \sigma r(t)d\tilde{W}_t$. Other than $\alpha > 0$, $\mu > 0$ and $\sigma > 0$ do we need any special conditions to ensure that $r(t)$ does not hit 0?

5. (*) (CIR)

   Consider the model:

$$dr(t) = \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}d\tilde{W}(t)$$

   where $\tilde{W}(t)$ is standard Brownian motion under the equivalent martingale measure Q and the parameter values are:
   $\alpha = 0.0125$, $\mu = 0.05$, $\sigma = 0.05$
   with $r(0) = 0.1$.

   Prices can be expressed in the form $P(0,T) = \exp[\bar{A}(T) - \bar{B}(T)r(0)]$. For the above parameters:

u	$\bar{A}(u)$	$\bar{B}(u)$
1	-0.0003111	0.993365
10	-0.0294161	9.048922
11	-0.0353140	9.819592

(a) Calculate the prices $P(0,1)$ and $P(0,11)$.

(b) Suppose $Z \sim N(0,1)$ and $Y = (Z + \sqrt{\lambda})^2$.
Show that $Pr(Y < y) = \Phi(\sqrt{y} - \sqrt{\lambda}) - \Phi(-\sqrt{y} - \sqrt{\lambda})$.

(c) A European call option has been written on $P(t,11)$ has an exercise date of $T = 1$ and a strike price of $K = \exp(-0.8)$.
Calculate the price of this call option at time 0.

6. (Black-Karasinski)

Suppose that $X(t)$ is an Ito process such that $dX(t) = \alpha(\mu - X(t))dt + \sigma dW_t$.

Let $r(t) = \exp X(t)$.

Use Ito's formula to derive the SDE for $r(t)$.

7. (*) (Futures prices)

Suppose $dr(t) = a(t).dt + b(t).dW_t$ for some suitable previsible functions $a(t)$ and $b(t)$ where W_t is a Brownian motion under the real-world measure P.

Consider a futures contract for delivery of the underlying $P(t,S)$ at time $T < S$. The futures price at time t for this contract is $F(t)$. The variation margin payable to the holder of the futures contract at the end of the time interval $[t, t+dt)$ is $dF(t) = F(t+dt) - F(t)$ for infinitesimally small dt. Immediately after the payment the price for the contract is always 0.

Prove that for all $t < T$: $F(t) = E_Q[P(T,S)|\mathcal{F}_t]$.

You should try to prove this using two different methods.

Method A: Follow similar steps to those in Section 6 of the lecture notes on equity derivatives and equity futures contracts (see OHP slide 6.9: note that step 4 should state that the portfolio contains $B_t\phi_t$ rather than ϕ_t forward contracts). You should start by assuming that $P(T,S) = E_Q[B(T)/B(S)|\mathcal{F}_T$ as usual. However, then assume that $P(t,S)$ is not tradeable between 0 and T. Then establish how you can replicate $P(T,S)$ using cash plus futures contracts.

Method B: (a) Let $\pi(t)$ be the accumulation in cash of the netcashflows $dF(t)$ to the holder of the long position: that is, $d\pi(t) = r(t)\pi(t)dt + dF(t)$. (b) Also note that $B(t)E_Q\left[\int_t^T B(s)^{-1}dF(s) \mid \mathcal{F}_t\right] = 0$ (state why!). (c) Define $\tilde{\pi}(t) = \pi(t)/B(t)$. Show that $d\tilde{\pi}(t) = B(t)^{-1}dF(t)$. (d) Show that $\tilde{\pi}(t)$ is a martingale under Q. (e) Hence show that $F(t)$ is a martingale under Q and complete the result.