# Derivatives Pricing and Financial Modelling


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# Tutorial 8

1. In a certain bond market the prices at time 0 of the zero-coupon bonds maturing at times are

| $T$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $P(0,T)$ | 0.96 | 0.92 | 0.87 | 0.82 | 0.77 | 0.73 |

Furthermore, using the notation of Chapter 3 of the lecture notes we have $u(2) = 1.01$ and $d(2) = 0.96$ and we have a recombining tree. Throughout the binomial tree the risk-neutral probability of an up-step in prices is constant (that is, it does not depend upon time or the history of the process).

A put option which expires at time 2 has been issued on the zero-coupon bond which matures at time 3. The exercise price of the option is 0.96.

(a) Determine the risk-neutral probability of an up-step at any point in the tree.

(b) Determine the values of $u(T)$ and $d(T)$ for $T = 3, 4, 5, 6$.

(c) For each of the outcomes at time 2 find the payoff on the option and the accumulation of cash up to time 2.

(d) Find the price of the put option at time 0 using risk-neutral expectations.

(e) Find the price of the put option by constructing a replicating strategy which uses (i) the zero-coupon bond maturing at time 3 and (ii) the zero-coupon bond maturing at time 6. Comment on the differences in the two replicating strategies.

2. (*) A recombining binomial equilibrium model for the one year interest rate $r_n$ (which applies between times $n$ and $n+1$) (continuously compounding) is as follows:

- $r_0 = 0.06$
- Given $r_n$, $r_{n+1} = r_n + 0.005(1 - 2I_{n+1})$ where $I_n$ equals 0 with probability $\hat{p} = 0.4$ (in the risk-neutral world) and equals 1 otherwise. Thus $\sum_{i=1}^{n} I_i$ is the number of down-steps in the one-year rate.

(a) Find the prices at time 0 of the zero-coupon bonds which mature at times 1, 2, 3 and 4.

(b) Find the coupon rate $\rho_4$ (payable annually) for a coupon bond which matures at time 4 and which stands at par at time 0.

(c) An *interest-rate swap* is a contract under which party A pays at time $n + 1$ to party B a fixed rate of interest $R^*$ in return for a payment from B to A of the variable one-year LIBOR rate of interest $R_n = \exp(r_n) - 1$, with $n = 0, 1, \ldots, N - 1$.

Show that this contract when $N = 4$ has zero value to each party when $R^* = \rho_4$.

What is the value of the contract to A if $R^* = 0.06$?

(d) Another contract called a *swaption* gives A the right but not the obligation to enter into a swap agreement at time 1 under which pays at time $n+1$ to party B a fixed rate of interest $R^*$ in return for a payment from B to A of the variable one-year LIBOR rate of interest $R_n = \exp(r_n) - 1$, with $n = 1, \ldots, N-1$. Here $N = 4$ and $R^* = 0.06$.

Under what circumstances will A exercise the option at time 1?

What is the value of this contract to A at time 0?

Are there any circumstances under which this contract could have a negative value to A?

(e) Investor A holds a convertible zero-coupon bond which, if unconverted, will pay 1 at time 3. However, the bond also gives A the option to convert the 1 unit of convertible bond at time 2 into 1.06 units of the (conventional) zero-coupon bond which matures at time 4.

i. Using the same model for $r_t$ above calculate the three possible values that the bond can have at time 2.

ii. Hence calculate the value of the convertible bond at time 0.

iii. Find the replicating strategies for the value at time 2 using cash (that is, $P(t, t+1)$) and $P(t, 3)$ first and then using cash and $P(t, 4)$.

3. (*) Consider the following random-walk model for $r(t)$:

$$r(t+1) = r(t) + \delta(2I_{t+1} - 1)$$

where $I_1, I_2, \ldots$ are independent and identically distributed under $Q$ with $Pr_Q(I_t = 1) = q$ and $Pr_Q(I_t = 0) = 1 - q$.

(a) Use the results in Section 3.3 in the book and lecture notes to determine the forward-rate curve at time 0 given $r(0)$, $\delta$ and $q$.

Hence derive the theoretical zero-coupon prices at time 0 for this model.

(b) Verify your formula for zero-coupon bond prices at time 0 by direct calculation of prices using the formula:

$$P(0, T) = E_Q\left[\exp\left(-\sum_{s=0}^{T-1} r(s)\right) \mid \mathcal{F}_0\right]$$

(Hint: prove the result by induction.)

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4. Consider the recombining binomial model in Section 3.3 of the lecture notes.

   We have previously proved that $F(t, T-1, T) = F(0, T-1, T) + \log[u(T-t)/u(T)] - D_t \log k$.

   Let $l_F(t) = \lim_{T\to\infty} F(t, T-1, T)$.

   Derive a formula for $l_F(t)$ in terms of $l_F(0)$, $D_t$ and $k$.

   Hence deduce that $l_F(t)$ satisfies the necessary condition in the Dybvig-Ingersoll-Ross result for an arbitrage-free term structure model.