Derivatives Pricing and Financial Modelling

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Tutorial 7

1. (*) In a particular 1-period bond-pricing model, 2 bonds are available which mature at times 1 and 2. Their prices at time 0 are 0.9 and 0.81 respectively. At time 1 there will be one of three outcomes ω_1, ω_2 and ω_3. The prices of the outstanding bond for each outcome are given in the following table.

	ω_1	ω_2	ω_3
$P(1,2)$	0.88	0.9	0.92

   (a) Is this market complete (give theoretical reasons)?

   (b) Give an example of a derivative which illustrates your answer to 1a.

2. Suppose that $P(0,T) = \exp(-0.08T)$ for all T. Furthermore $r(t) = 0.08$ for $0 \leq t < 1$. No trading is possible between times 0 and 1.

   At time 1 the spot-rate curve will be either:

$$\begin{aligned} R(1,s) &= 0.08 + u(s) \quad \text{for all } s \\ \text{or} \quad R(1,s) &= 0.08 - d(s) \quad \text{for all } s \end{aligned}$$

   for some curves $u(s)$ and $d(s)$ (both of which are strictly positive).

   (a) Suppose that $u(s)$ for $s \geq 2$ and $d(2)$ are given. The prices of all zero-coupon bonds maturing after time 1 evolve in an arbitrage-free way. Thus determine the form of $d(s)$ for all $s \geq 2$ in terms of $d(2)$ and $u(s)$. What do you notice about $d(s)$ as $s \to \infty$?

   (b) Suppose instead that $d(s) = 0.01$ for all $s \geq 2$ and $u(2) = 0.01$. Show that it is not possible to derive values for $u(s)$ for all s which keep the model arbitrage free.

3. (*) Suppose that the risk-free rate of interest $r(t)$ is governed by the stochastic differential equation:

$$dr(t) = \mu(t, r(t))dt + \sigma(t, r(t))dZ(t)$$

   where $\mu(t,r)$ and $\sigma(t,r)$ satisfy the usual conditions for the existence and uniqueness of $r(t)$.

Let the value of the cash account at t be denoted by $B(t)$ with $dB(t) = r(t)B(t)dt$. Show that

$$B(t) = B(0)\exp\left[\int_0^t r(s)ds\right].$$

4. (*) Suppose that:

$$r(t) = \begin{cases} r_0 & \text{for } 0 \leq t < 1 \\ r_0 + \epsilon & \text{for } 1 \leq t \end{cases}$$

where ϵ is a positive-valued random variable with probability density function $f(\epsilon) > 0$ $(0 \leq \epsilon < \infty)$ under the risk-neutral measure Q. Define $P(t,T) = E_Q\left[\exp(-\int_t^T r(u)du) \mid \mathcal{F}_t\right]$.

   (a) Prove that $l(0) = \lim_{T\to\infty} R(0,T) = r_0$.

   (b) An arbitrage opportunity is present at time 0 if there exists n, maturity dates $T_1 < T_2 < \ldots < T_n$, $T < T_1$, and $x_1, \ldots, x_n$, such that:

   - $\sum_{i=1}^n x_i P(0,T_i) = 0$;
   - $Pr_P\left[\sum_{i=1}^n x_i P(T,T_i) \geq 0\right] = 1$; and
   - $Pr_P\left[\sum_{i=1}^n x_i P(T,T_i) > 0\right] > 0$.

   Prove from first principles that no arbitrage opportunity exists at time 0 under this model.

5. (a) Suppose that U has an exponential distribution under Q with mean $1/\lambda$, but is not observable until time U, and that:

$$r(t) = \begin{cases} r_0 & \text{for } 0 \leq t < U \\ r_1 & \text{for } U \leq t \end{cases}$$

   where r_1 is equal to $r_0 + \epsilon$ with probability p under Q, or r_0 with probability $1-p$, and ϵ is some known positive constant.

   What is $l(0) = \lim_{T\to\infty} R(0,T) = \lim_{T\to\infty} f(0,T)$?

   Discuss, using general reasoning, the shape of $f(0,T)$ for $0 \leq T < \infty$.

   (b) Suppose instead that for the same λ, ϵ and p, $Pr_Q(U \geq \tau) = (1-p) + p\exp(-\lambda\tau)$ and that

$$r(t) = \begin{cases} r_0 & \text{for } 0 \leq t < U \\ r_0 + \epsilon & \text{for } U \leq t. \end{cases}$$

   Is $f(0,T)$ the same here as in part (a) of this question?

   (c) Why are the forward-rate curves, $f(t,T)$ in parts (a) and (b) not equal in general for $t > 0$?