Derivatives Pricing and Financial Modelling

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Tutorial 10

1. (Ho-Lee)

   Let $X(T) = \int_0^T \tilde{W}_t dt$.

   (a) What is the distribution of $X(T)$?

   (b) Find $E[\exp(-X(T))]$.

   (c) Let $r(T) = r(0) + \int_0^T \theta(t)dt + \sigma\tilde{W}_t$ for some deterministic function $\theta(t)$. Find an expression for $P(t,T)$.

   (d) Hence show that if the initial forward-rate curve $f(0,T)$ is given, then

   $$\theta(T) = \frac{\partial}{\partial T} f(0,T) + \sigma^2 T$$

   (e) Show that there exists a similar expression to that under the Vasicek model for the price of a European call option on a zero-coupon bond.

2. Suppose that $f(0,t) = 0.06 + 0.01\exp(-0.2t)$.

   Consider the Hull & White model.

   Suppose that we know that

   $$\lim_{t\to\infty} Var[r(t)] = \frac{\sigma^2}{2\alpha} = 0.02^2$$

   is fixed.

   (a) Investigate the form of $\mu(t)$ in the Hull & White model for various choices of α.

   (b) For what value of α does $\mu(0) = \mu(\infty)$?

3. (*) Under the Hull and White model suppose that $\alpha = 0.24$, $\sigma = 0.02$ and $f(0,t) = 0.06 + 0.01e^{-0.2t}$.

   (a) Calculate the price of a 3-month European call option written on a zero-coupon bond which will mature in 10 years time with a nominal value of £100 and a strike price of £53.50.

(b) What is the minimum amount of information required to make the calculation in (a)?

4. Under the Heath-Jarrow-Morton framework we have

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t$$

(a) Under what circumstances is such a model arbitrage free?

(b) Under what circumstances is this model Markov? (You should identify separately an "impractical" and a "practical/useful" definition.)

(c) Which of the following models are Markov under the equivalent martingale measure Q for a suitable drift term $\alpha(t,T)$:

i. $\sigma(t,T) = \sigma$ for all t, T;

ii. $\sigma(t,T) = (1 - e^{-\delta t})\sigma$ for all t, T;

iii. $\sigma(t,T) = \sigma(t)$ for all t, T where $\sigma(t)$ is an Ito process satisfying the SDE $d\sigma(t) = a(m - \sigma(t))dt + b\sqrt{\sigma(t)}d\hat{W}_t$ and $\hat{W}_t$ is a brownian motion under Q which is independent of $\tilde{W}_t$;

iv. $\sigma(t,T) = \sigma/(T - t + \delta)$ for all t, T and for some $\delta > 0$;

v. $\sigma(t,T) = \sigma e^{-\alpha(T-t)}$ for all t, T;

vi. $\sigma(t,T) = \sigma_1 e^{-\alpha_1(T-t)} + \sigma_2 e^{-\alpha_2(T-t)}$ for all t, T?

5. (*) Suppose:

$$\begin{aligned}
f(0,T) &= \lambda_0 + \lambda_1 e^{-\alpha T} - \frac{\sigma^2}{2\alpha^2}\left(1 - e^{-\alpha T}\right)^2 \\
\sigma(t,T) &= \sigma e^{-\alpha(T-t)} \\
df(t,T) &= \theta(t,T)dt + \sigma(t,T)d\tilde{W}(t) \\
\text{where } \theta(t,T) &= -\sigma(t,T)S(t,T) \\
S(t,T) &= -\int_t^T \sigma(t,u)du
\end{aligned}$$

Derive a formula for $r(t)$ of the form:

$$r(t) = g(t, r(0)) + \int_0^t h(s,t)d\tilde{W}(s)$$

for suitable deterministic functions g and h.

What name is given to this model?

6. (*) Suppose that the model $df(t,T) = \alpha(t,T)dt + \sigma(t,T)dZ(t)$ where $Z(t)$ is a Brownian Motion under the real-world measure P, is arbitrage free and where $\sigma(t,T)$ is deterministic. The initial forward-rate curve $f(0,u)$ is given.

   (a) Why is $f(t,T)$ not necessarily Gaussian?

   (b) Suppose that the market price of risk $\gamma(t)$ is deterministic. Prove that $f(t,T)$ is now Gaussian.

   (c) Under the equivalent martingale measure Q we have

   $$df(t,T) = -\sigma(t,T)S(t,T)dt + \sigma(t,T)d\tilde{Z}(t)$$

   where $\tilde{Z}(t)$ is a Brownian motion under Q and $S(t,T) = -\int_t^T \sigma(t,u)du$.
   Given $P(0,\tau)$ for all τ, show that for any $0 < t < T < \infty$ $P(t,T)$ is log-normally distributed under Q.

7. The dynamics of zero-coupon prices are defined by

   $$dP(t,T) = P(t,T)\left(r(t)dt + S(t,T)d\tilde{Z}(t)\right)$$

   for all T, where $\tilde{Z}(t)$ is Brownian motion under the equivalent martingale measure Q.

   A coupon bond pays a coupon rate of g per annum continuously until the maturity date T when the nominal capital of 100 is repaid. The price at time t of this bond is denoted by $V(t)$.

   (a) Show that for some functions a_v and b_v:

   $$dV(t) = a_V(t, r(t), V(t))dt + b_V(t, r(t), \mathcal{P}(t))d\tilde{Z}(t)$$

   where $\mathcal{P}(t) = \{P(t,u) : t \leq u \leq T\}$.

   (b) Suppose that

   $$\begin{aligned} P(0,u) &= e^{-0.1u} \quad \text{for all } u \\ S(t,u) &= -10\sigma\left(1 - e^{-0.1(u-t)}\right) \quad \text{for all } t,\ u \\ g &= 10 \end{aligned}$$

   i. What is $V(0)$ as a function of T?

   ii. What is the volatility of $V(t)$ at time 0 (that is, the $d\tilde{Z}$ component of $dV(t)/V(t)$?

   iii. Hence deduce that the irredeemable bond ($T = \infty$) has the highest volatility amongst all bonds with a coupon of 10%.

   iv. Give an example of a bond which has a higher volatility than the irredeemable 10% coupon bond.

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8. (*) An interest-rate cap is a derivative that guarantees that the rate of interest on a loan at any given time will be the lesser of the prevailing rate, $r(t)$, and the cap rate, r_c. Thus over the outstanding term of the loan from t to T the effective payoff at s on the derivative relative to an uncapped loan is $\max\{r(s) - r_c, 0\}.ds$.

   This is effectively a collection of interest-rate call options where the payoff at s is $\max\{r(s) - r_c, 0\}$.

   (a) Write down an expression for the value at time t of such an interest-rate call option using the equivalent martingale measure Q.

   (b) Suppose that under Q the risk-free rate $r(t)$ follows the Vasicek model:

$$dr(t) = \alpha\left(\mu - r(t)\right)dt + \sigma d\tilde{W}_t$$

   i. Find the forward measure P_T under which $Y(t,U) = P(t,U).P(0,T)/P(t,T)$ is a martingale and find the change of measure drift $\gamma(s)$.

   ii. Show that under P_T, $r(s)$ given $\mathcal{F}_t$ has a normal distribution with

$$\begin{aligned} E_{P_T}\left[r(s) \mid \mathcal{F}_t\right] &= e^{-\alpha(s-t)}r(t) + \left(\mu - \frac{\sigma^2}{\alpha^2}\right)\left(1 - e^{-\alpha(s-t)}\right) \\ &\quad + \frac{\sigma^2}{2\alpha^2}e^{-\alpha(T-s)}\left(1 - e^{-2\alpha(s-t)}\right) \\ \text{and } Var_{P_T}\left[r(s) \mid \mathcal{F}_t\right] &= \frac{\sigma^2}{2\alpha}\left(1 - e^{-2\alpha(s-t)}\right) \end{aligned}$$

   iii. Hence find an expression for

$$E_{P_T}\left[\left(r(T) - r_c\right)_+ \mid \mathcal{F}_t\right]$$

   iv. Finally find an expression for the price of this option.

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9. Suppose that the risk-free rate of interest $r(t)$ follows the Hull & White interest-rate model

$$dr(t) = \alpha(\theta(t) - r(t))dt + \sigma d\tilde{W}_t$$

where $\tilde{W}_t$ is standard Brownian motion under the *risk-neutral measure* Q, and $\theta(t)$ is a deterministic function which is determined by observed bond prices $P(0, T)$ at time 0.

A non-dividend-paying stock has price $R(t)$ at time t with

$$dR(t) = R(t)\left(\mu(t)dt + \sigma_R dW_t^R\right)$$

and W_t^R is a standard Brownian motion under the *real-world measure* P. We denote $\tilde{W}_t^R$ for the equivalent Brownian motion to W_t^R driving stock prices under Q. $\tilde{W}_t^R$ and $\tilde{W}_t$ are independent.

A binary call option on the stock pays £1 at time T if $R(T) \geq K$ and £0 otherwise.

(a) Use the forward-measure approach to determine a value for this option at time $t < T$.

(b) Without developing formulae, discuss briefly how you would hedge this option in order to replicate the payoff.

(c) Suppose now that $\alpha = 0.5$, $\sigma = 0.03$, $\theta(t) = \theta = 0.06$, $r(0) = 0.03$, $\mu(t) = r(t) + 0.04$, $\sigma_R = 0.25$, $R(0) = 95$, $K = 100$ and $T = 0.5$.

Calculate the price at time 0 of the binary option:

i. using the formula derived in part (a);

ii. using the standard Black-Scholes model with a constant deterministic risk-free rate of $r = -(\log P(0, T))/T$;

and compare the results.

(d) Discuss whether or not the comparison in part (c) would be different if $R(0) = 105$.