Derivatives Pricing and Financial Modelling

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Tutorial 9: Solutions

1. We have:

$$\begin{aligned}
P(t,t+s) &= \exp[A(t,t+s) - B(t,t+s)r(t)] \\
\text{where } B(t,t+s) &= \frac{1-e^{-\alpha s}}{\alpha} \\
A(t,t+s) &= (B(t,t+s)-s)\left(\mu - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha}B(t,t+s)^2 \\
f(t,t+s) &= -\frac{\partial}{\partial s}\log P(t,t+s) \\
&= \left(\mu - \frac{\sigma^2}{2\alpha^2}\right)\left(1-e^{-\alpha s}\right) + \frac{\sigma^2}{2\alpha}\left(\frac{1-e^{-\alpha s}}{\alpha}\right)e^{-\alpha s} + e^{-\alpha s}r(t) \\
&= \left(\mu - \frac{\sigma^2}{2\alpha^2}\right) + (r(t)-\mu)e^{-\alpha s} + \frac{\sigma^2}{\alpha^2}e^{-\alpha s} - \frac{\sigma^2}{2\alpha^2}e^{-2\alpha s} \\
&= \left(\mu - \frac{\sigma^2}{2\alpha^2}\right) + (r(t)-\mu+\frac{\sigma^2}{2\alpha^2})e^{-\alpha s} + \frac{\sigma^2}{2\alpha^2}\left(1-e^{-\alpha s}\right)e^{-\alpha s}
\end{aligned}$$

In the final form we see that the third term has the potential to produce a small hump in the forward-rate curve.

2. (a)

$$\begin{aligned}
-\omega Y - \nu X &\sim N(0, \omega^2+\nu^2+2\omega\nu\rho) \\
\Rightarrow E_Q\left[e^{-\omega Y-\nu X}\right] &= e^{\frac{1}{2}\omega^2+\frac{1}{2}\nu^2+\omega\nu\rho}
\end{aligned}$$

(b)

$$\begin{aligned}
E_Q\left[e^{-\omega Y-\nu X}\right] &= E_Q\left[e^{-\omega Y}\right]E_Q\left[e^{-\nu X}\frac{e^{-\omega Y}}{E_Q\left[e^{-\omega Y}\right]}\right] \\
&= E_Q\left[e^{-\omega Y}\right]E_Q\left[e^{-\nu X}\frac{dP}{dQ}\right] \\
&= E_Q\left[e^{-\omega Y}\right]E_P\left[e^{-\nu X}\right]
\end{aligned}$$

(c) Hence:

$$e^{\frac{1}{2}\omega^2+\frac{1}{2}\nu^2+\omega\nu\rho} = e^{\frac{1}{2}\omega^2}E_P\left[e^{-\nu X}\right]$$

$$
\begin{aligned}
\Rightarrow E_P\left[e^{-\nu X}\right] &= e^{\frac{1}{2}\nu^2+\omega\nu\rho} \\
\Rightarrow X &\sim N(-\omega\rho, 1) \quad \text{under } P
\end{aligned}
$$

(d) Now assume $\omega = 1$. Then:

$$
\begin{aligned}
E_Q\left[e^{-Y} I(X<x)\right] &= E_Q\left[e^{-Y}\right] E_P[I(X<x)] \\
\text{where } X &\sim N(-\rho, 1) \ \text{ under } P \\
\Rightarrow E_Q\left[e^{-Y} I(X<x)\right] &= e^{1/2} Pr_P(X<x) \\
&= e^{1/2}\Phi(x+\rho)
\end{aligned}
$$

(e) Under Q, $Y|X \sim N(\rho X, (1-\rho^2))$.

$$
\begin{aligned}
E_Q\left[e^{-Y} I(X<x)\right] &= E_Q\left[I(X<x)\ E_Q\left(e^{-Y} \mid X\right)\ \right] \\
&= E_Q\left[I(X<x) e^{-\rho X+\frac{1}{2}(1-\rho^2)}\right] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\rho u+\frac{1}{2}(1-\rho^2)-\frac{1}{2}u^2} du \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}(u+\rho)^2+\frac{1}{2}} du \\
&= e^{1/2}\Phi(x+\rho)
\end{aligned}
$$

3. (a) Bookwork!

(b)

$$
\begin{aligned}
X(T) &= \int_0^T r(u)du \\
&= \int_0^T \left(\mu + (r(0)-\mu)e^{-\alpha t} + \sigma \int_0^t e^{-\alpha(t-s)} d\tilde{W}_s\right) dt \\
&= \mu T + (r(0)-\mu)\frac{\left(1-e^{-\alpha T}\right)}{\alpha} + \sigma \int_0^T \int_s^T e^{-\alpha(t-s)} dt\ d\tilde{W}_s \quad \text{by B} \\
&= \mu T + (r(0)-\mu)\frac{\left(1-e^{-\alpha T}\right)}{\alpha} + \sigma \int_0^T \frac{\left(1-e^{-\alpha(T-s)}\right)}{\alpha} d\tilde{W}_s
\end{aligned}
$$

(c) From result A, $r(T)$ and $X(T)$ have a bivariate normal distribution with:

$$
\begin{aligned}
E_Q[r(T)] &= \mu + (r(0)-\mu)e^{-\alpha T} \\
E_Q[X(T)] &= \mu T + (r(0)-\mu)\frac{\left(1-e^{-\alpha T}\right)}{\alpha} \\
Var[r(T)] &= Var_Q\left[\sigma \int_0^T e^{-\alpha(T-s)} d\tilde{W}_s\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma^2 \int_0^T e^{-2\alpha(T-s)} ds \\
&= \sigma^2 \frac{1-e^{-2\alpha T}}{2\alpha} \\
Var_Q[X(T)] &= Var_Q\left[\sigma \int_0^T \frac{\left(1-e^{-\alpha(T-s)}\right)}{\alpha} d\tilde{W}_s\right] \\
&= \sigma^2 \int_0^T \frac{\left(1-e^{-\alpha(T-s)}\right)^2}{\alpha^2} ds \\
&= \frac{\sigma^2}{\alpha^2}\left[T - \frac{2\left(1-e^{-\alpha T}\right)}{\alpha} + \frac{\left(1-e^{-2\alpha T}\right)}{2\alpha}\right] \\
Cov_Q[r(T), X(T)] &= E_Q\left[\left(\sigma \int_0^T e^{-\alpha(T-s)} d\tilde{W}_s\right)\left(\sigma \int_0^T \frac{\left(1-e^{-\alpha(T-s)}\right)}{\alpha} d\tilde{W}_s\right)\right] \\
&= \frac{\sigma^2}{\alpha} \int_0^T e^{-\alpha(T-s)}\left(1-e^{-\alpha(T-s)}\right) ds \\
&= \sigma^2\left[\frac{\left(1-e^{-\alpha T}\right)}{\alpha^2} - \frac{\left(1-e^{-2\alpha T}\right)}{2\alpha^2}\right]
\end{aligned}
$$

(d) Since the model is Markov and time homogeneous the distribution of $X(T)$ given $r(t) = r$ is the same as that for $X(T-t)$ given $r(0) = r$.
Since $X(T)|r(t) = r$ is normally distributed under Q:

$$
\begin{aligned}
P(t,T) &= \exp\left(-E_Q[X(T) - X(t) \mid r(t) = r] + \frac{1}{2} Var_Q[X(T) - X(t) \mid r(t) = r]\right) \\
&= \exp\Bigg\{ -\mu(T-t) - (r(t) - \mu)\frac{\left(1-e^{-\alpha(T-t)}\right)}{\alpha} \\
&\qquad + \frac{1}{2}\frac{\sigma^2}{\alpha^2}\left[T - t - \frac{2\left(1-e^{-\alpha(T-t)}\right)}{\alpha} + \frac{\left(1-e^{-2\alpha(T-t)}\right)}{2\alpha}\right]\Bigg\} \\
&= \exp[A(t,T) - B(t,T)r(t)] \\
\text{where } B(t,T) &= \frac{1-e^{-\alpha(T-t)}}{\alpha} \\
\text{and } A(t,T) &= (T-t)\left(-\mu + \frac{\sigma^2}{2\alpha^2}\right) + \left(\mu - \frac{\sigma^2}{2\alpha^2}\right)\left(\frac{1-e^{-\alpha(T-t)}}{\alpha}\right) \\
&\qquad + \frac{\sigma^2}{2\alpha^2}\left(\frac{1-e^{-\alpha(T-t)}}{\alpha}\right) - \frac{\sigma^2}{\alpha^2}\left(\frac{1-e^{-\alpha(T-t)}}{\alpha}\right) \\
&\qquad\quad + \frac{\sigma^2}{2\alpha^2}\left(\frac{1-e^{-2\alpha(T-t)}}{2\alpha}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (B(t,T) - (T-t))\left(\mu - \frac{\sigma^2}{2\alpha^2}\right) \\
&\quad + \frac{\sigma^2}{4\alpha^3}\left(-1 + 2e^{-\alpha(T-t)} - e^{-2\alpha(T-t)}\right) \\
&= (B(t,T) - (T-t))\left(\mu - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha}B(t,T)^2
\end{aligned}
$$

(e) $(r(T), X(S))$ has a bivariate normal distribution under Q with:

$$
\begin{aligned}
E_Q[r(T)|r(0)=r] &= \mu + (r-\mu)e^{-\alpha T} \\
&= \mu_1 \\
E_Q[X(S)|r(0)=r] &= \mu S + (r-\mu)\frac{1-e^{-\alpha S}}{\alpha} \\
&= \mu_2(S) \\
Var_Q[r(T)|r(0)=r] &= \sigma^2\frac{1-e^{-2\alpha T}}{2\alpha} \\
&= \sigma_1^2 \\
Var_Q[X(S)|r(0)=r] &= \frac{\sigma^2}{\alpha^2}\left[S - \frac{2\left(1-e^{-\alpha S}\right)}{\alpha} + \frac{\left(1-e^{-2\alpha S}\right)}{2\alpha}\right] \\
&= \sigma_2^2(S) \\
Cov_Q[r(T), X(S) \mid r(0)=r] &= \int_0^T \sigma e^{-\alpha(T-u)} \times \sigma\frac{\left(1-e^{-\alpha(S-u)}\right)}{\alpha}du \\
&= \frac{\sigma^2}{\alpha^2}\left[\frac{1-e^{-\alpha T}}{\alpha} - \frac{e^{-\alpha(S-T)}\left(1-e^{-2\alpha T}\right)}{2\alpha}\right] \\
&= \rho(S)\sigma_1\sigma_2(S)
\end{aligned}
$$

(f) Let $X_1 = r(T)$ and $X_2 = X(\tau)$. Let:

$$
\frac{dP}{dQ} = \frac{e^{-\omega X_2}}{E_Q\left[e^{-\omega X_2}\right]}
$$

Consider:

$$
\begin{aligned}
E_Q\left[e^{-\omega X_2 - \nu X_1}\right] &= E_Q\left[e^{-\omega X_2}\right]E_P\left[e^{-\nu X_1}\right] \\
\Rightarrow E_P\left[e^{-\nu X_1}\right] &= \exp\left[-\nu(\mu_1 - \rho(\tau)\omega\sigma_1\sigma_2(\tau)) + \frac{1}{2}\nu\sigma_1^2\right]
\end{aligned}
$$

Thus, X_1 has a normal distribution under P with mean $\mu_1 - \rho(\tau)\omega\sigma_1\sigma_2(S)$ and variance σ_1^2. (**)

Now consider

$$\begin{aligned} C(0) &= E_Q\left[e^{-X(T)}(P(T,S)-K)_+\right] \\ &= E_Q\left[e^{-X(T)}\left(E_Q\left[e^{-(X(S)-X(T))}\;\middle|\; r(T)\right]-K\right)I\Big(r(T)<r^*\Big)\right] \\ \text{where}\;\; r^* &= \frac{A(T,S)-\log K}{B(T,S)} \\ \Rightarrow C(0) &= E_Q\left[e^{-X(S)}I(r(T)<r^*)\right]-KE_Q\left[e^{-X(T)}I(r(T)<r^*)\right] \\ &= E_Q\left[e^{-X(S)}\right]E_{P_1}\left[I(r(T)<r^*)\right]-KE_Q\left[e^{-X(T)}\right]E_{P_2}\left[I(r(T)<r^*)\right] \\ &= E_Q\left[e^{-X(S)}\right]Pr_{P_1}(r(T)<r^*)-KE_Q\left[e^{-X(T)}\right]Pr_{P_2}(r(T)<r^*) \end{aligned}$$

Under P_2 (e.g. take $\omega=1$) $r(T)$ is normal with mean $r_2=\mu_1-\rho(T)\sigma_1\sigma_2(T)$ and variance σ_1^2.

$$\begin{aligned} \Rightarrow Pr_{P_2}(r(T)<r^*) &= \Phi(d_2) \\ \text{where}\;\; d_2 &= \frac{r^*-r_2}{\sigma_1} \\ \text{Now}\;\; r^* &= \frac{A(T,S)-\log K}{B(T,S)} \\ \text{but}\;\; P(0,S) &= E_Q\left[e^{-X(T)}\right]E_{P_2}\left[P(T,S,r(T))\right] \\ &= P(0,T)E_{P_2}\left[e^{A(T,S)-B(T,S)r(T)}\right] \\ \Rightarrow \frac{P(0,S)}{P(0,T)} &= e^{A(T,S)-B(T,S)r_2+\frac{1}{2}B(T,S)^2\sigma_1^2} \\ \Rightarrow \log\frac{P(0,S)}{KP(0,T)} &= A(T,S)-B(T,S)r_2+\frac{1}{2}B(T,S)^2\sigma_1^2-\log K \\ \Rightarrow r^* &= \frac{1}{B(T,S)}\log\frac{P(0,S)}{KP(0,T)}+r_2-\frac{1}{2}B(T,S)\sigma_1^2 \\ \Rightarrow d_2 &= \frac{r^*-r_2}{\sigma_1} \\ &= \frac{1}{\sigma_1 B(T,S)}\log\frac{P(0,S)}{KP(0,T)}-\frac{1}{2}B(T,S)\sigma_1 \\ &= \frac{1}{\sigma_P}\log\frac{P(0,S)}{KP(0,T)}-\frac{1}{2}\sigma_P \\ \text{where}\;\; \sigma_P &= \sigma_1 B(T,S) \\ &= \sigma\frac{\left(1-e^{-\alpha(S-T)}\right)}{\alpha}\sqrt{\frac{1-e^{-2\alpha T}}{2\alpha}} \end{aligned}$$

Under P_1 (with $\omega=1$) $r(T)$ is normal with mean $r_1=\mu_1-\rho(S)\sigma_1\sigma_2(S)$ and variance σ_1^2. Thus:

$$Pr(r(T)<r^*) = \Phi(d_1)$$

$$\begin{aligned}
\text{where } d_1 &= \frac{r^* - r_1}{\sigma_1} \\
\text{Now } d_1 - d_2 &= \frac{r_2 - r_1}{\sigma_1} \\
&= \frac{\rho(S)\sigma_1\sigma_2(S) - \rho(T)\sigma_1\sigma_2(T)}{\sigma_1} \\
&= \frac{\sigma^2}{\alpha^2\sigma_1}\left[\left(1 - e^{-\alpha T}\right) - \frac{1}{2}e^{-\alpha(S-T)}\left(1 - e^{-2\alpha T}\right)\right] \\
&\qquad - \frac{\sigma^2}{\alpha^2\sigma_1}\left[\left(1 - e^{-\alpha T}\right) - \frac{1}{2}\left(1 - e^{-2\alpha T}\right)\right] \\
&= \sigma\frac{\left(1 - e^{-\alpha(S-T)}\right)}{\alpha}\sqrt{\frac{1 - e^{-2\alpha T}}{2\alpha}} \\
&= \sigma_P \\
\Rightarrow d_1 &= d_2 + \sigma_P \\
&= \frac{1}{\sigma_P}\log\frac{P(0,S)}{KP(0,T)} + \frac{1}{2}\sigma_P
\end{aligned}$$

4. (a)

$$\begin{aligned}
dr(t) &= \alpha(\mu - r(t))dt + \sigma\sqrt{r(t)}d\tilde{Z}_t \\
X(t) &= \log r(t) \\
\text{By Ito} & \\
dX(t) &= \frac{1}{r}dr - \frac{1}{2}\frac{1}{r^2}(dr)^2 \\
&= \left[(\alpha\mu - \frac{1}{2}\sigma^2)e^{-X} - \alpha\right] + \sigma e^{-\frac{1}{2}X}d\tilde{Z}
\end{aligned}$$

If $\alpha\mu > \frac{1}{2}\sigma^2$ then the drift term will become strongly positive if $X(t)$ gets too far below zero. Thus the process will be autoregressive and stationary. In particular it will not explode to minus infinity.
If $\alpha\mu \leq \frac{1}{2}\sigma^2$ then the drift term is negative for all values of $X(t)$. Therefore the process will tend to $-\infty$.
If $\alpha\mu < \frac{1}{2}\sigma^2$ then the drift term will tend to $-\infty$ very fast. The result (without a rigorous analysis) is that $X(t)$ may tend to $-\infty$ in finite time. Hence $r(t)$ would hit zero in finite time.

(b) Let $X(t) = r(t) - \beta$. Then:

$$\begin{aligned}
dX(t) &= \alpha(\mu' - X(t))dt + \sigma\sqrt{X(t)}d\tilde{Z}(t) \\
\text{where } \mu' &= \mu - \beta \\
\text{Now } P(t,T) &= E_Q\left[e^{-\int_t^T (X(s)+\beta)ds} \mid \mathcal{F}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= e^{-\beta(T-t)} E_Q \left[e^{-\int_t^T X(s)ds} \mid \mathcal{F}_t \right] \\
&= \exp\left[-\beta(T-t) + A'(t,T) - B'(t,T)(r(t)-\beta)\right] \\
\text{where } A'(t,T) &= A(t,T)(\mu', \sigma, \alpha) \\
B'(t,T) &= B(t,T)(\mu', \sigma, \alpha)
\end{aligned}
$$

and $A(t,T)$ and $B(t,T)$ are the standard functions employed in the CIR pricing formula.

(c) No additional constraints are required as the volatility of $r(t)$ tends to zero much more quickly under the this model than the CIR model. This allows the positive drift $\alpha(\mu - r(t))$ a better chance of getting $r(t)$ away from zero.
Alternatively consider $X(t) = \log r(t)$. Then

$$
dX(t) = \left[\alpha\mu e^{-X} - \alpha - \frac{1}{2}\sigma^2\right] + \sigma d\tilde{Z}
$$

Provided $\alpha > 0$, $\mu > 0$ and $\sigma > 0$ the drift always becomes positive as $X \to -\infty$.

5. (a)

$$
\begin{aligned}
P(0,1) &= 0.905156 \\
P(0,11) &= 0.361580
\end{aligned}
$$

(b)

$$
\begin{aligned}
Pr(Y < y) &= \chi^2(1, \lambda; y) \\
&= Pr(-\sqrt{y} < Z + \sqrt{\lambda} < \sqrt{y}) \\
&= Pr(-\sqrt{y} - \sqrt{\lambda} < Z < \sqrt{y} - \sqrt{\lambda} \\
&= \Phi(\sqrt{y} - \sqrt{\lambda}) - \Phi(-\sqrt{y} - \sqrt{\lambda})
\end{aligned}
$$

(c)

$$
\begin{aligned}
C(0) &= P(0,11,r(0))\chi^2(d, \lambda_2; y_2) - KP(0,1,r(0))\chi^2(1, \lambda_1; y_1) \\
\text{where } K &= 0.449329 \\
d &= \frac{4\alpha\mu}{\sigma^2} \\
&= 1 \\
\gamma &= \sqrt{\alpha^2 + 2\sigma^2} \\
&= 0.0718070 \\
\lambda_1 &= \frac{8\gamma^2 e^{\gamma T} r}{\sigma^2 \left(e^{\gamma T} - 1\right)\left(2\gamma + (\gamma + \alpha)\left(e^{\gamma T} - 1\right)\right)} \\
&= 158.8701
\end{aligned}
$$

$$
\begin{aligned}
\lambda_2 &= \frac{8\gamma^2 e^{\gamma T} r}{\sigma^2 \left(e^{\gamma T}-1\right)\left(2\gamma + \left(\gamma+\alpha+\sigma^2 \bar{B}(U-T)\right)\left(e^{\gamma T}-1\right)\right)} \\
&= 157.1048 \\
r^* &= (\bar{A}(U-T) - \log K)/\bar{B}(U-T) \\
&= 0.0851575 \\
k_1 &= \frac{\sigma^2 \left(e^{\gamma T}-1\right)}{2\left(2\gamma + (\gamma+\alpha)\left(e^{\gamma T}-1\right)\right)} \\
&= 0.000620853 \\
k_2 &= \frac{\sigma^2 \left(e^{\gamma T}-1\right)}{2\left(2\gamma + \left(\gamma+\alpha+\sigma^2\bar{B}(U-T)\right)\left(e^{\gamma T}-1\right)\right)} \\
&= 0.000613955 \\
y_1 &= r^*/k_1 \\
&= 137.1622 \\
y_2 &= r^*/k_2 \\
&= 138.7033
\end{aligned}
$$

$$
\begin{aligned}
\text{Hence } \chi^2(1, \lambda_i; y_1) &= \Phi(\sqrt{y_i} - \sqrt{\lambda_i}) - \Phi(-\sqrt{y_i} - \sqrt{\lambda_i}) \\
&= \begin{cases} \Phi(-0.8927) - \Phi(-24.3160) = 0.186009 & \text{for } i = 1 \\ \Phi(-0.7569) - \Phi(-24.3114) = 0.224555 & \text{for } i = 2 \end{cases} \\
\Rightarrow C(0) &= 0.005546
\end{aligned}
$$

(The final value plus all intermediate values have been taken from computer calcluations. Manual calculations using rounded, intermediate values may introduce rounding errors in the final value of $C(0)$ (e.g. 0.00552) This is quite okay!)

6.

$$
\begin{aligned}
dX(t) &= \alpha(\mu - X(t))dt + \sigma dW_t \\
r(t) &= e^{X(t)} \\
\Rightarrow dr &= e^X dX + \frac{1}{2} e^X (dX)^2 \\
&= r(t)\left(\alpha(\mu - \log r(t)) + \sigma dW_t\right) + \frac{1}{2} r(t)\sigma^2 dt \\
&= \left(\alpha' r(t) - \gamma r(t)\log r(t)\right) dt + \sigma r(t) dW_t \\
\text{where } \alpha' &= \alpha\mu + \frac{1}{2}\sigma^2 \\
\gamma &= \alpha
\end{aligned}
$$

This is in the form of the Black-Karasinski model.

Thus we know that $r(t)$ if it comes from the BK model then $\log r(t)$ follows an Ornstein-Uhlenbeck process.

7. There are two methods to solve this problem.

Method A

Assume that $dF(t) = F(t)(\sigma_F(t)dW(t) + \mu_F(t)dt)$ where $\sigma_F(t)$ and $\mu_F(t)$ are previsible processes and $W(t)$ is a Brownian motion under the real-world measure P.

Step 1:

Establish the measure $\tilde{Q}$ equivalent to P under which $F(t)$ is a martingale. Thus, let $\gamma(t) = \mu_F(t)/\sigma_F(t)$ and take:

$$\frac{d\tilde{Q}}{dP} = e^{-\int_0^T \gamma(u)dW(u) - \frac{1}{2}\int_0^T \gamma(u)^2 du}$$

Then $dF(t) = F(t)\sigma_F(t)d\tilde{W}(t)$ where $\tilde{W}(t) = W(t) + \int_0^t \gamma(u)du$ is a B.M. under $\tilde{Q}$.

Step 2:

Define $D(t) = E_{\tilde{Q}}[B(T)^{-1}P(T,S) \mid \mathcal{F}_t]$. This a martingale under $\tilde{Q}$.

Step 3:

By the martingale representation theorem there exists a previsible process $\phi(t)$ such that $D(t) = D(0) + \int_0^t \phi(u)dF(u)$.

Step 4:

Implement the portfolio process which holds at time t, $B(t)\phi(t)$ futures contracts and $D(t)$ units of the risk-free bond $B(t)$. The value at t of this portfolio is $V(t) = B(t)D(t)$ since the futures contracts have zero value. Then the instantaneous change in the portfolio value is:

$$\begin{aligned} dV(t) &= B(t)dD(t) + D(t)dB(t) + dD(t).dB(t) \\ &= B(t)\phi(t)dF(t) + D(t)dB(t) + 0.dt \end{aligned}$$

which is equal to the instantaneous investment gain over the same period. This the portfolio process is self-financing.

Step 5:

Furthermore $V(T) = P(T,S)$ so the portfolio process is replicating. Hence $V(t) = B(t)E_{\tilde{Q}}[B(T)^{-1}P(T,S) \mid \mathcal{F}_t]$ is the unique no-arbitrage price for the bond at time t: that is, $P(t,T) = B(t)E_{\tilde{Q}}[B(T)^{-1}P(T,S) \mid \mathcal{F}_t]$.

But we know that $P(t,T) = B(t)E_Q[B(T)^{-1}P(T,S) \mid \mathcal{F}_t]$. So Q and $\tilde{Q}$ are the same and $F(t)$ is a martingale under Q. It follows that $F(t) = E_Q[F(T) \mid \mathcal{F}_t] = E_Q[P(T,S) \mid \mathcal{F}_t]$ as required.

Method B:

Let $\pi(t)$ be the accumulation in cash of the net cashflows arising from the futures contract. Thus:

$$d\pi(t) = r(t)\pi(t)dt + dF(t) \tag{1}$$

We also know that futures prices evolve in a way which ensures that the value at any time t for the futures contract is zero: that is, the total expected discounted cashflows are zero or

$$E_Q\left[\int_t^T \frac{B(t)}{B(s)} dF(s) \,\middle|\, \mathcal{F}_t\right] = 0 \tag{2}$$

Now define $\tilde{\pi}(t) = \pi(t)/B(t)$. Then:

$$\begin{aligned}
d\pi(t) &= B(t)d\tilde{\pi}(t) + \tilde{\pi}(t)dB(t) \\
&= B(t)d\tilde{\pi}(t) + r(t)B(t)\tilde{\pi}(t)dt \\
&= B(t)d\tilde{\pi}(t) + r(t)\pi(t)dt \\
\Rightarrow d\tilde{\pi}(t) &= \frac{d\pi(t) - r(t)\pi(t)dt}{B(t)} \\
&= \frac{dF(t)}{B(t)} \quad \text{by equation (1).} \\
\Rightarrow \tilde{\pi}(t) &= \tilde{\pi}(0) + \int_0^t \frac{dF(u)}{B(u)} = \int_0^t \frac{dF(u)}{B(u)} \\
\text{and} \quad \pi(t) &= \int_0^t \frac{B(t)}{B(u)} dF(u). \\
\text{However,} \quad E_Q[\tilde{\pi}(T) \mid \mathcal{F}_t] &= \int_0^t \frac{dF(u)}{B(u)} + B(t)^{-1} E_Q\left[\int_t^T \frac{B(t)}{B(s)} dF(s) \,\middle|\, \mathcal{F}_t\right] \\
&= \int_0^t \frac{dF(u)}{B(u)} + B(t)^{-1}.0 \quad \text{by equation (2)} \\
&= \tilde{\pi}(t)
\end{aligned}$$

Hence $\tilde{\pi}(t)$ is a Q-martingale. Since $\tilde{W}(t)$ is also a Q-martingale there exists, by the martingale representation theorem, a previsible process $\phi(t)$ such that:

$$\begin{aligned}
d\tilde{\pi}(t) &= \phi(t)d\tilde{W}(t) \\
\Rightarrow \frac{dF(t)}{B(t)} &= \phi(t)d\tilde{W}(t) \\
\Rightarrow dF(t) &= \phi(t)B(t)d\tilde{W}(t)
\end{aligned}$$

Therefore $F(t)$ is a Q-martingale (subject to verification of the usual technical requirements on $\phi(t)B(t)$). It follows that $F(t) = E_Q[F(T)|\mathcal{F}_t] = E_Q[P(T,S)|\mathcal{F}_t]$.