Derivatives Pricing and Financial Modelling

Andrew Cairns: room M3.08

E-mail: A.Cairns@ma.hw.ac.uk

Tutorial 8: Solutions

1. See the MSc web page `http://www.ma.hw.ac.uk/~andrewc/msc` for an Excel spreadsheet which deals with this problem.

   The model is:

$$P(t+1,T+t) = \begin{cases} u(T)\frac{P(t,t+T)}{P(t,t+1)} & \text{if up} \\ d(T)\frac{P(t,t+T)}{P(t,t+1)} & \text{if down} \end{cases}$$

   Recombining tree implies that:

$$\begin{aligned} u(T) &= \frac{1}{(1-\pi)k^{T-1}+\pi} \\ d(T) &= \frac{k^{T-1}}{(1-\pi)k^{T-1}+\pi} \\ \text{where } \pi &= Pr_Q(up) = \frac{1-d(2)}{u(2)-d(2)} = 0.8 \\ \text{and } k &= \frac{d(2)}{u(2)} = 0.950495 \end{aligned}$$

   (a) Hence

T	2	3	4	5	6	7
$u(T)$	1.01	1.0197	1.0291	1.0382	1.0469	1.0554
$d(T)$	0.96	0.9212	0.8837	0.8474	0.8122	0.7783

   (b) $\pi = 0.8$

   (c)

$$\begin{aligned} P(0,3) &= 0.87 \\ P(t,3,i) &= \text{Price at } t \text{ after } i \text{ down steps} \\ P(1,3,0) &= u(3)P(0,3)/P(0,1) = 0.9241 \\ P(1,3,1) &= d(3)P(0,3)/P(0,1) = 0.8348 \\ P(1,2,0) &= u(2)P(0,2)/P(0,1) = 0.9679 \\ P(1,2,1) &= d(2)P(0,2)/P(0,1) = 0.92 \\ \Rightarrow P(2,3,0) &= u(2)P(1,3,0)/P(1,2,0) = 0.9643 \\ P(2,3,1) &= u(2)P(1,3,1)/P(1,2,1) \\ &= d(2)P(1,3,0)/P(1,2,0) = 0.9165 \\ P(2,3,2) &= d(2)P(1,3,1)/P(1,2,1) = 0.8711 \end{aligned}$$

Cash accumulation is:

$$\begin{aligned}
B(t+1,i) &= \begin{cases} B(t,i)/P(t,t+1,i) & \text{after an up} \\ B(t,i-1)/P(t,t+1,i-1) & \text{after a down} \end{cases} \\
\Rightarrow B(1,i) &= 1/0.96 \quad i=0,1 \\
B(2,0) &= 1/0.96 \times 0.9679 \\
B(2,1) &= \begin{cases} 1/0.96 \times 0.9679 & \text{if up-down} \\ 1/0.96 \times 0.92 & \text{if down-up} \end{cases} \\
B(2,2) &= 1/0.96 \times 0.92
\end{aligned}$$

Hence

outcome	Put payoff, $X = V(2)$	$B(2)$	Probability
up-up	0	1.0762	0.64
up-down	0.0435	1.0762	0.16
down-up	0.0435	1.1322	0.16
down-down	0.0889	1.1322	0.04

(d) Hence the price at time 0 is:

$$V(0,0) = \sum_{\omega} \frac{X(\omega)}{B(2)(\omega)} Pr_Q(\{\omega\}) = 0.0158$$

Alternatively:

$$\begin{aligned}
V(1,0) &= P(1,2,0)\left(\pi V(2,0) + (1-\pi)V(2,1)\right) = 0.0084 \\
V(1,1) &= P(1,2,1)\left(\pi V(2,1) + (1-\pi)V(2,2)\right) = 0.0484 \\
V(0,0) &= P(0,1,0)\left(\pi V(1,0) + (1-\pi)V(1,1)\right) = 0.0157
\end{aligned}$$

(e) i. At $(t,i) = (1,0)$ hold an amount $a(1,0)$ in cash plus $b(1,0)$ units of $P(1,3,0)$. Then

$$\begin{aligned}
\text{up} \;\Rightarrow \frac{a(1,0)}{P(1,2,0)} + b(1,0)P(2,3,0) &= 0 \\
\text{down} \;\Rightarrow \frac{a(1,0)}{P(1,2,0)} + b(1,0)P(2,3,1) &= 0.0435 \\
\Rightarrow b(1,0) &= -0.9100 \\
a(1,0) &= 0.8493 \\
\Rightarrow \text{Value at } (t,i) = (1,0), V(1,0) &= a(1,0) + b(1,0)P(1,3,0) \\
&= 0.0084
\end{aligned}$$

At $(t,i) = (1,1)$ hold $a(1,1)$ cash plus $b(1,1)$ units of $P(1,3,0)$. Then

$$\text{up} \;\Rightarrow \frac{a(1,1)}{P(1,2,1)} + b(1,1)P(2,3,1) = 0.0435$$

$$\begin{aligned}
\text{down} \quad \Rightarrow \frac{a(1,1)}{P(1,2,1)} + b(1,1)P(2,3,2) &= 0.0880 \\
\Rightarrow b(1,1) &= -1 \\
a(1,1) &= 0.8832 \\
\Rightarrow \text{Value at } (t,i) = (1,1), V(1,1) &= a(1,1) + b(1,1)P(1,3,1) \\
&= 0.0484
\end{aligned}$$

At time 0 hold $a(0,0)$ cash plus $b(0,0)$ units of $P(0,3)$. Then

$$\begin{aligned}
\text{up} \quad \Rightarrow \frac{a(0,0)}{P(0,1)} + b(0,0)P(1,3,0) &= V(1,0) \\
\text{down} \quad \Rightarrow \frac{a(0,0)}{P(0,1)} + b(0,0)P(1,3,1) &= V(1,1) \\
\Rightarrow b(0,0) &= -0.4479 \\
a(0,0) &= 0.4054 \\
\Rightarrow \text{Value at } (t,i) = (0,0), V(0,0) &= a(0,0) + b(0,0)P(0,3) \\
&= 0.0157
\end{aligned}$$

(This is slightly different to the previous solution due to rounding errors.)

ii. The prices of $P(t,6)$ in the tree are:

$$\begin{array}{ccc}
 & & 0.853891 \\
 & 0.796115 & \\
0.73 & & 0.696951 \\
 & 0.617625 & \\
 & & 0.568855
\end{array}$$

Then:

$$\begin{aligned}
b(1,0) &= -0.27692 \\
a(1,0) &= 0.22887 \\
\Rightarrow V(1,0) &= 0.00841 \\
b(1,1) &= -0.35427 \\
a(1,1) &= 0.26714 \\
\Rightarrow V(1,1) &= 0.04833 \\
b(0,0) &= -0.22365 \\
a(0,0) &= 0.17900
\end{aligned}$$

$$\Rightarrow V(0,0) = 0.01574$$

Comment: the holdings in $P(t,6)$ are always closer to zero than those in $P(t,3)$. This is because the longer-dated bonds have more volatile prices.

2. (a) See the MSc web page `http://www.ma.hw.ac.uk/~andrewc/msc` for an Excel spreadsheet which deals with this problem.

$t =$	0	1	2	3	4
State j	$r(t,j)$				
0	0.06	0.065	0.07	0.075	0.08
1		0.055	0.06	0.065	0.07
2			0.05	0.055	0.06
3				0.045	0.05
4					0.04
$P(t,1,j)$					
0	0.941765	1			
1		1			
2					
3					
4					
$P(t,2,j)$					
0	0.887818	0.937067	1		
1		0.946485	1		
2			1		
3					
4					
$P(t,3,j)$					
0	0.837830	0.878985	0.932394	1	
1		0.896741	0.941765	1	
2			0.951229	1	
3				1	
4					
$P(t,4,j)$					
0	0.791495	0.825356	0.870238	0.927743	1
1		0.850492	0.887818	0.937067	1
2			0.905754	0.946485	1
3				0.955997	1
4					1

In the table prices are calculated using the backwards recursion formula:

$$
\begin{aligned}
P(t,T) &= P(t,t+1)E_Q[P(t+1,T) \mid \mathcal{F}_t] \\
\text{e.g. } P(0,2) &= P(0,1)E_Q[\hat{p}P(1,2,0) + (1-\hat{p})P(1,2,1)] \\
&= e^{-0.06}(0.4 \times e^{-0.065} + 0.6 \times e^{-0.055}) \\
&= 0.887818
\end{aligned}
$$

(b) Solve:

$$\begin{aligned} 1 &= \rho_4(P(0,1)+P(0,2)+P(0,3)+P(0,4))+P(0,4) \\ \Rightarrow \rho_4 &= 0.060281 \end{aligned}$$

(c) **Terminology:** *The rate* $R_n = \exp(r_n) - 1$ *would normally be what is called 12-month LIBOR in the market-place.*

Consider the following hedge for A:

- Enter into one swap contract receiving $R_n = \exp(r_n) - 1$ from B and paying R^* to B.
- At time 0 buy 1 unit of the par bond at a total cost of R^*/ρ_4.
- At time 0 borrow cash of 1. The interest on the outstanding capital on the loan payable at the end of each year is at the rate of $R_n = \exp(r_n) - 1$ per annum. The loan will be fully repaid at time 4.
- At time 1 the net cashflow to A is 0: that is, R^* from the par bond minus R_1 to service the loan plus R_1 from B minus R^* payable to B.
  The capital outstanding on the loan is still 1.
- At time 2 the net cashflow to A is 0: that is, R^* from the par bond minus R_2 to service the loan plus R_2 from B minus R^* payable to B.
  The capital outstanding on the loan is still 1.
- At time 3 the net cashflow to A is 0: that is, R^* from the par bond minus R_3 to service the loan plus R_3 from B minus R^* payable to B.
  The capital outstanding on the loan is still 1.
- At time 4 the net cashflow to A is 0: that is, $1 + R^*$ from the par bond, minus R_4 to service the loan minus 1 to repay the loan, plus R_4 from B, minus R^* payable to B.
- There are no further cashflows.

Since the future cashflows are all 0 with certainty, the law of one price dictates that this portfolio must have zero value at time 0.

Thus the fair swap rate R^* to have zero value to A and B at time 0 is the 4-year par yield ρ_4.

The above hedge but with $R^* = 0.06$ will leave A with surplus income of $\rho_4 - 0.06$ at times 1, 2, 3 and 4. Hence the value of the contract to A will be $(0.060281 - 0.06)(P(0,1) + P(0,2) + P(0,3) + P(0,4)) = 0.000972$.

(d) A will enter into the contract at time 1 if the 3-year par yield at time 1 is greater than R^*.

The value of the contract is then:

$$\max\{0, (\rho_4 - R^*)\sum_{s=2}^{4} P(1,s)\} = \begin{cases} \max\{0, +0.016159\} = 0.016159 & \text{if up} \\ \max\{0, -0.012115\} = 0 & \text{if down} \end{cases}$$

The value of the contract at time 0 is:

$$V(0) = P(0,1)\left(\hat{p} \times 0.016159 + (1-\hat{p}) \times 0\right) = 0.006087$$

The swaption can never have a negative value as the option will only be exercised if there is a positive profit for A.

(e) The value at time 2 of the convertible bond is:

$$V(2,i) = \max\{1.06P(2,4,i), P(2,3,i)\} = \begin{cases} 0.932394 & i=0 \\ 0.941765 & i=1 \\ 0.960099 & i=2 \text{ (conversion exercised)} \end{cases}$$

Hence

$$\begin{aligned} V(1,i) &= P(1,2,i)(\hat{p}V(2,i) + (1-\hat{p}V(2,i+1)) \\ \Rightarrow V(1,0) &= 0.878985 = P(1,3,0) \\ V(1,1) &= 0.901778 > P(1,3,1) \\ V(0,0) &= P(0,1,0)(\hat{p}V(1,0) + (1-\hat{p})V(1,1)) \\ &= 0.840676 > P(0,3) \end{aligned}$$

3. (a) From remark 3.2.1 and Corollary 3.2.2 we have:

$$\begin{aligned} r(t) &= F(0,t,t+1) - \log u(t+1) - D_t \log k \\ \text{where } D_t &= \sum_{s=1}^{t} I_s \\ u(t+1) &= \frac{1}{(1-q)k^t + q} \\ k &< 1 \\ \text{Also } r(t) &= r(0) + \delta(2D_t - t) \text{(the equilibrium model)} \\ \Rightarrow 2\delta &= -\log k \\ \Rightarrow k &= e^{-2\delta} \\ \Rightarrow F(0,t,t+1) &= r(0) - \delta t + \log u(t+1) \\ &= r(0) - \delta t - \log\left[(1-q)e^{-2\delta t} + q\right] \\ \Rightarrow \sum_{t=0}^{T-1} F(0,t,t+1) &= r(0)T - \delta\frac{1}{2}T(T-1) + \sum_{t=0}^{T-1} \log u(t+1) \\ \Rightarrow P(0,T) &= \exp\left[-\sum_{t=0}^{T-1} F(0,t,t+1)\right] \\ &= \exp\left[-r(0)T + \delta\frac{1}{2}T(T-1) - \sum_{t=0}^{T-1} \log u(t+1)\right] \\ \text{or } P(t,T,r(t)) &= \exp\left[-r(t)(T-t) + \delta\frac{1}{2}(T-t)(T-t-1) - \sum_{s=0}^{T-t-1} \log u(s+1)\right] \end{aligned}$$

(b) Given T we can see that, trivially, the result is true for $t = T-1$.
Suppose the result is true for $t = \tau+1, \ldots, T-1$. Then:

$$\begin{aligned}
& P(\tau, T, r(\tau)) \\
= \; & e^{-r(\tau)} \left[(1-q)P(\tau+1, T, r(\tau)+\delta) + qP(\tau+1, T, r(\tau)-\delta)\right] \\
= \; & e^{-r(\tau)} \left[(1-q)\exp\left(-(r(\tau)+\delta)(T-\tau-1) + \frac{1}{2}\delta(T-\tau-1)(T-\tau-2) - \sum_{s=0}^{T-\tau-2} \log u(s+1)\right)\right. \\
& \left. +q\exp\left(-(r(\tau)-\delta)(T-\tau-1) + \frac{1}{2}\delta(T-\tau-1)(T-\tau-2) - \sum_{s=0}^{T-\tau-2} \log u(s+1)\right)\right] \\
= \; & \exp\left[-r(\tau)(T-\tau) - \frac{1}{2}\delta(T-\tau)(T-\tau-1) - \sum_{s=0}^{T-\tau-2} \log u(s+1)\right] \\
& \times \left\{(1-q)e^{-2\delta(T-\tau-1)} + q\right\}
\end{aligned}$$

But:

$$\begin{aligned}
(1-q)e^{-2\delta(T-\tau-1)} + q &= (1-q)k^{T-\tau-1} + q \\
&= \frac{1}{u(T-\tau)} \\
\Rightarrow P(\tau, T, r(\tau)) &= \exp\left[-r(\tau)(T-\tau) - \frac{1}{2}\delta(T-\tau)(T-\tau-1) - \sum_{s=0}^{T-\tau-1} \log u(s+1)\right]
\end{aligned}$$

So the result is true for $t = \tau$.
By induction the result is true for $t - T - 1, T-2, \ldots, 1, 0$: that is:

$$P(0, T) \;=\; \exp\left[-r(0)T + \delta\frac{1}{2}T(T-1) - \sum_{t=0}^{T-1} \log u(t+1)\right]$$

Not used in 2001-2004.

4.

$$\begin{aligned}
l_F(t) &= \lim_{T\to\infty} F(t, T-1, T) \\
F(t, T-1, T) &= F(0, T-1, T) + \log \frac{u(T-t)}{u(T)} - D_t \log k \\
\text{now} \quad \log \frac{u(T-t)}{u(T)} &= \log \frac{(1-q)k^{T-1}+q}{(1-q)k^{T-t-1}+q} \\
&\to \log \frac{q}{q} \quad \text{as } T \to \infty \\
&= 0
\end{aligned}$$

since $0 < k < 1$.

Hence $l_F(t) = l_F(0) - D_t \log k$.

Since $\log k < 0$ and D_t is increasing, $l_F(t)$ is increasing as required.

Not used in 2004.

5. Let $U_t = \sum_{s=1}^{t} I_s$ be the number of up steps in prices up to time t.

Notation: $P(t, T, U_t) = P(t, T)$ given U_t up steps.

$$\begin{aligned} k &= \frac{d(2)}{u(2)} \\ \pi &= \frac{1 - d(2)}{u(2) - d(2)} \\ u(T) &= \frac{1}{(1-\pi)k^{T-1} + \pi} \\ d(T) &= \frac{k^{T-1}}{(1-\pi)k^{T-1} + \pi} \\ \frac{u(T)}{d(T)} &= k^{-(T-1)} \end{aligned}$$

Consider $t = 0$.

$$P(0, T) = \frac{P(0, T)}{P(0, 0)} \exp\left[-\sum_{s=1}^{T} \log \frac{d(s)}{d(s)} - (T - t)U_0\right]$$

so the result is true for $t = 0$.

Suppose the result is true for $t > 0$.

We can write:

$$\begin{aligned} P(t+1, T, U_{t+1}) &= \begin{cases} u(T-t)P(t, T, U_t)/P(t, t+1, U_t) & \text{if up} \\ d(T-t)P(t, T, U_t)/P(t, t+1, U_t) & \text{if down} \end{cases} \\ &= d(T-t)k^{-(T-t-1)I_{t+1}} P(t, T, U_t)/P(t, t+1, U_t) \\ &= \frac{d(T-t)}{k^{(T-t-1)I_{t+1}}} \frac{\exp\left[-\sum_{s=t+1}^{T} \log \frac{d(s-t)}{d(s)} - (T-t)U_t \log k\right] \frac{P(0,T)}{P(0,t)}}{\exp\left[-\sum_{s=t+1}^{t+1} \log \frac{d(s-t)}{d(s)} - U_t \log k\right] \frac{P(0,t+1)}{P(0,t)}} \\ &= \frac{P(0, T)}{P(0, t+1)} \exp\left[\log d(T-t) - \sum_{t+1}^{T} \log d(s-t) + \sum_{s=t+1}^{T} \log d(s)\right. \\ &\qquad \left. - \log d(t+1) + \log d(1) - (T-t-1)I_{t+1} \log k - (T-t-1)U_t \log k\right] \\ &= \frac{P(0, T)}{P(0, t+1)} \exp\left[-\sum_{s=t+2}^{T} \log \frac{d(s-t-1)}{d(s)} - (T-t-1)U_{t+1} \log k\right] \end{aligned}$$

as required.

So the result follows by induction.