

Derivatives Pricing and Financial Modelling

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Tutorial 6 Solutions

1. (a) Treasury bills
- (b) strips
- (c) Treasury notes (< 10 years at time of issue)
 Treasury bonds (\geq 10 years at issue)
- (d) Treasury notes
- (e) Treasury strips

2. We solve $P = \sum_{j=1}^n c_j \exp(-\delta t_j) = \hat{P}(\delta)$ in δ where $c_j > 0$ and $t_j > 0$ for all j and $P > 0$.

$c_j \exp(-\delta t_j)$ is strictly decreasing as δ increases.

$\Rightarrow \hat{P}(\delta)$ is strictly decreasing as δ increases.

Also, $\hat{P}(\delta)$ tends to $+\infty$ as $\delta \rightarrow -\infty$, $\hat{P}(\delta)$ tends to zero as $\delta \rightarrow +\infty$ and $\hat{P}(\delta)$ is continuous.

Therefore there exists a $\hat{\delta}$ such that $\hat{P}(\hat{\delta}) = P$.

Since $\hat{P}(\delta)$ is strictly decreasing this solution is unique.

3. (a) Suppose

$$\begin{aligned} f(t, t+s) &< 0 \quad \text{for } T < t+s < S \\ \Rightarrow F(t, t+T, t+S) &= \frac{1}{S-T} \int_T^S f(t, t+s) ds < 0 \\ \Rightarrow P(t, t+S) &= P(t, t+T) \exp[-(S-T)F(t, t+T, t+S)] \\ &> P(t, t+T) \end{aligned}$$

Now consider the following portfolio strategy:

- At t , buy one unit of the $t+T$ -bond at a price of $P(t, t+T)$
- At t , sell $P(t, t+T)/P(t, t+S) < 1$ units of the $t+S$ -bond receiving $P(t, t+T)$ in cash for the sale.
- At time $t+T$ we receive 1 from the redemption of the $t+T$ bond.
Put this cash under the mattress until time $t+S$.
- At time $t+S$ we then have 1 in cash but must pay out $P(t, t+T)/P(t, t+S) < 1$ to close off the short position in the $t+S$ -bond. This leaves us an arbitrage-free profit of $1 - P(t, t+T)/P(t, t+S) > 0$.

The assumption of no arbitrage therefore means that we must have $P(t, t+T) \geq P(t, t+S)$.

(b) $f(t, t+s) > 0$ for all $s \Rightarrow P(t, t+s)$ is decreasing for all s .

4. The government has the right to redeem the bond at any time between 2010 and 2014.

\Rightarrow the option is in favour of the government

\Rightarrow lower value for investor

\Rightarrow conventional bond 8% 2010 has the higher value

More rigorously: let c_1, c_2, \dots, c_n be the coupon payments payable up to the first possible redemption time in 2010 due at times T_1, \dots, T_n . Let T_n be the first redemption time.

Let $\pi_1(t)$ be the price of the conventional bond at time t , and $\pi_2(t)$ be the price of the callable bond. Assume that coupons occur just before any T_j (that is, T_j^-) and that redemption occurs just after T_n (that is, T_n^+). Then

$$\begin{aligned}\pi_1(T_n) &= 100 \\ \pi_2(t) &= 100 \text{ if exercised at } T_n^+ \\ &> 100 \text{ ???} \\ &< 100 \text{ ???}\end{aligned}$$

If $\pi_2(T_n) > 100$ then the government would in fact redeem the bond immediately, paying 100 which means that the price cannot be maintained above 100. So $\pi_2(T_n) \leq 100$ with probability 1.

$$\begin{aligned}\pi_2(t) &= \sum_{j=1}^n c_j P(t, T_j) + E_Q \left[e^{-\int_t^{T_n} r(s) ds} \pi_2(T_n) \mid \mathcal{F}_t \right] \\ &\leq \sum_{j=1}^n c_j P(t, T_j) + 100 P(t, T_n) \\ &= \pi_1(t)\end{aligned}$$

5. Similar to question 4, but the option is in favour of the investor. Let

$$\begin{aligned}\pi_1(t) &= \text{price at } t \text{ of Tr. 8\% 2010} \\ \pi_2(t) &= \text{price at } t \text{ of Conv 8\% 2010} \\ \pi_3(t) &= \text{price at } t \text{ of Tr. } 8\frac{3}{4}\% \text{ 2017}\end{aligned}$$

Assume coupons c_1, \dots, c_{n-1} at times $t < T_1 < \dots < T_{n-1}$. Conversion can take place at an appointed time T_n which falls after T_{n-1} but before the subsequent coupon payment.

At time T_n the value of Conv. 8% 2010 is

$$\begin{aligned}\pi_2(T_n) &= \max\{\pi_1(T_n), \pi_3(T_n)\} \geq \pi_1(T_n) \\ \pi_2(t) &= \sum_{j=1}^{T_{n-1}} c_j P(t, T_j) + E_Q \left[e^{-\int_t^{T_n} r(s) ds} \pi_2(T_n) \mid \mathcal{F}_t \right] \\ &\geq \sum_{j=1}^{T_{n-1}} c_j P(t, T_j) + E_Q \left[e^{-\int_t^{T_n} r(s) ds} \pi_1(T_n) \mid \mathcal{F}_t \right] \\ &= \pi_1(t)\end{aligned}$$

T	1	2	3
$P(0, T)$	0.941765	0.878095	0.810584
$\sum_{t=1}^T P(0, t)$	0.941765	1.819860	2.630444

$$F(0, 1, 2) = \log P(0, 1)/P(0, 2) = 0.07$$

$$F(0, 1, 3) = \frac{1}{2} \log P(0, 1)/P(0, 3) = 0.075$$

$$F(0, 2, 3) = \log P(0, 2)/P(0, 3) = 0.08$$

Par yields: $\rho(0, T) = (1 - P(0, T))/\sum_{t=1}^T P(0, t)$. Hence:

T	1	2	3
$\rho(0, T)$	6.18%	6.70%	7.20%

7. Take $T_1 = 5, T_2 = 10, T_3 = 15$, and $P_i = \exp(-0.08T_i)$.

Let x_i be the number of units of bond i .

Take $x_2 = -1$.

Solve:

$$\begin{aligned}\sum_{i=1}^3 x_i P_i &= 0 \Rightarrow \text{zero cost at time 0} \\ \sum_{i=1}^3 x_i T_i P_i &= 0 \\ \Rightarrow x_1 e^{-0.4} + x_3 e^{-1.2} &= e^{-0.8} \\ \text{and } 5x_1 e^{-0.4} + 15x_3 e^{-1.2} &= 10e^{-0.8} \\ \Rightarrow x_3 &= 0.74591 \\ x_1 &= 0.33516\end{aligned}$$

If $\epsilon = +0.02$ then $V(1) = 0.00203$.

If $\epsilon = -0.02$ then $V(1) = 0.00292$.

Therefore we have created an arbitrage.

8. (a) Let $q_i = Pr_Q(\{\omega_i\})$ where Q is the equivalent martingale measure.

Clearly we require $\sum_{i=1}^3 q_i = 1$.

From the prices of $P(t, 2)$ and $P(t, 3)$ we can establish the values of q_1, q_2 and q_3 by exploiting the martingale requirement:

$$\begin{aligned} \frac{P(0, T)}{B(0)} &= E_Q \left[\frac{P(1, T)}{B(1)} \mid \mathcal{F}_0 \right] \quad \text{for } T = 2, 3 \\ B(0) &= 1 \\ B(1) &= \frac{1}{P(0, 1)} = 1/0.9 \\ \Rightarrow \text{for each } T \quad \frac{P(0, T)}{P(0, 1)} &= \sum_i q_i P(1, T)(\omega_i) \\ \Rightarrow \text{Solve } 0.88q_1 + 0.9q_2 + 0.92(1 - q_1 - q_2) &= \frac{0.81}{0.9} \\ \text{and } 0.77q_1 + 0.805q_2 + 0.86(1 - q_1 - q_2) &= \frac{0.729}{0.9} \\ \Rightarrow q_1 &= 0.25 \\ q_2 &= 0.5 \\ q_3 &= 0.25 \\ \text{Also } \frac{P(0, 4)}{P(0, 1)} &= \sum_i q_i P(1, 4)(\omega_i) \\ \Rightarrow \text{Solve } 0.7q_1 + 0.75q_2 + xq_3 &= \frac{0.684}{0.9} \\ \Rightarrow x &= 0.84 \end{aligned}$$

- (b) Yes the market is complete because we have established that Q is unique and the market is arbitrage free.

Alternatively we can note that we have a sufficient number of tradeable assets available to allow us to replicate any derivative payoff at time 1.

- (c) Use cash plus $P(t, 2)$ and $P(t, 3)$ to replicate $P(1, 4)$.

Hence buy x_T units at time 0 of $P(0, T)$ for $T = 1, 2, 3$.

At time 1 we have three possible outcomes. Under each outcome and with the right choice of the x_T the portfolio should have the same value as $P(1, 4)$.

Hence:

$$\begin{aligned} x_1 + 0.88x_2 + 0.77x_3 &= 0.7 \\ x_1 + 0.9x_2 + 0.805x_3 &= 0.75 \\ x_1 + 0.92x_2 + 0.86x_3 &= 0.81 \\ \Rightarrow x_1 &= -1.115 \\ x_2 &= 1.625 \\ x_3 &= 0.5 \end{aligned}$$

The total cost of this portfolio at time 0 is

$$\sum_T x_T P(0, T) = 0.67725 < P(0, 4)$$

For no arbitrage we should have $\sum_T x_T P(0, T) = P(0, 4)$ since the portfolio has the same payoff at time 1 as the $P(1, 4)$.