Derivatives Pricing and Financial Modelling

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Tutorial 10: Solutions

1. (a)

$$X(T) = \int_0^T \tilde{W}_t dt = \int_0^T \int_0^t d\tilde{W}_s \, dt = \int_0^T \int_s^T dt \, d\tilde{W}_s = \int_0^T (T-s) d\tilde{W}_s$$

Hence $X(T)$ is Normal (under Q) with mean 0 and variance $\int_0^T (T-s)^2 ds = \frac{1}{3}T^3$.

(b) Hence $\exp(-X(T))$ is log-normal with

$$E_Q\left[e^{-X(T)}\right] = e^{\frac{1}{6}T^3}$$

(c)

$$\begin{aligned}
\text{For } u > t \quad r(u) &= r(t) + \int_t^u \theta(s)ds + \sigma(\tilde{W}_u - \tilde{W}_t) \\
\Rightarrow \int_t^T r(u)du &= r(t)(T-t) + \int_t^T \int_t^u \theta(s)ds \, du + \sigma \int_t^T \int_t^u d\tilde{W}_s \, du \\
&= r(t)(T-t) + \int_t^T \int_s^T du\theta(s)ds + \sigma \int_t^T \int_s^T du \, d\tilde{W}_s \\
&= r(t)(T-t) + \int_t^T (T-s)\theta(s)ds + \int_t^T \sigma(T-s) \, d\tilde{W}_s
\end{aligned}$$

Hence $\int_t^T r(u)du | \mathcal{F}_t$ is Normal under Q with mean $r(t)(T-t) + \int_t^T (T-s)\theta(s)ds$ and variance $\frac{\sigma^2}{3}(T-t)^3$.

$$\begin{aligned}
P(t,T) &= E_Q\left[e^{-\int_t^T r(u)du} \mid \mathcal{F}_t\right] \\
&= \exp\left[-r(t)(T-t) - \int_t^T (T-s)\theta(s)ds + \frac{\sigma^2}{6}(T-t)^3\right]
\end{aligned}$$

(d) In particular,

$$\begin{aligned}
P(0,T) &= \exp\left[-r(0)T - \int_0^T (T-s)\theta(s)ds + \frac{\sigma^2}{6}T^3\right] \\
\Rightarrow f(0,T) &= -\frac{\partial}{\partial T} \log P(0,T) \\
&= r(0) + \frac{\partial}{\partial T} \int_0^T \int_0^u \theta(s)ds \, du - \frac{1}{2}\sigma^2 T^2 \\
&= r(0) + \int_0^T \theta(s)ds - \frac{1}{2}\sigma^2 T^2
\end{aligned}$$

Differentiate again with respect to T:

$$\begin{aligned} \Rightarrow \frac{\partial}{\partial T} f(0,T) &= \theta(T) - \sigma^2 T \\ \Rightarrow \theta(T) &= \frac{\partial}{\partial T} f(0,T) + \sigma^2 T \end{aligned}$$

2. (a) We have $\sigma^2 \equiv \sigma^2(\alpha) = 0.0008\alpha$.

$$\begin{aligned} \mu(t) &= \frac{1}{\alpha}\frac{\partial}{\partial t} f(0,t) + f(0,t) + \frac{\sigma^2}{2\alpha^2}\left(1 - e^{-2\alpha t}\right) \\ &= \frac{1}{\alpha}\left(-0.002 e^{-0.2t}\right) + 0.06 + 0.01 e^{-0.2t} + 0.0004 \frac{(1 - e^{-2\alpha t})}{\alpha} \end{aligned}$$

Numerical experiments can be found on the Excel spreadsheet which is available at `http://www.ma.hw.ac.uk/~andrewc/msc/`.

(b)

$$\mu(0) = \mu(\infty) \Rightarrow 0.07 - \frac{0.002}{\alpha} = 0.06 + \frac{0.004}{\alpha} \Rightarrow \hat{\alpha} = 0.24$$

3. (a) The price of the call option is:

$$\begin{aligned} & 100\left[P(0,10)\Phi(h) - XP(0,0.25)\Phi(h - \sigma_p)\right] \\ \text{where } \; P(0,t) &= \exp\left[-\int_0^t f(0,s)ds\right] \\ &= \exp\left[-0.06t - 0.01\frac{(1 - e^{-0.2t})}{0.2}\right] \\ \Rightarrow P(0,10) &= 0.52559 \\ P(0,0.25) &= 0.98271 \\ \\ X &= 0.53500 \\ \sigma_p &= \frac{\sigma}{\alpha}\left(1 - e^{-\alpha(10-0.25)}\right)\left(\frac{1 - e^{-2\alpha \times 0.25}}{2\alpha}\right)^{1/2} \\ &= \frac{0.02}{0.24}\left(1 - e^{-2.34}\right)\left(\frac{1 - e^{-0.12}}{0.48}\right)^{1/2} \\ &= 0.03655 \\ h &= \frac{1}{\sigma_p}\log\frac{P(0,10}{XP(0,0.25)} + \frac{\sigma_p}{2} \\ &= 0.01 \\ \Phi(h) &= 0.5040 \\ \Phi(h - \sigma_p) &= 0.4894 \\ \Rightarrow \text{price} &= 0.7595 \text{ or } £0.76 \end{aligned}$$

(b) The minimum information required is $P(0,10)$, $P(0,0.25)$, X, σ, α, $T=0.25$ (exercise date), $U=10$ (maturity date of underlying).

4. (a) The model is arbitrage free if there exists some previsible process $\gamma(t)$ such that

$$\begin{aligned} \alpha(t,T) &= \sigma(t,T)(\gamma(t)-S(t,T)) \\ \text{where } S(t,T) &= -\int_t^T \sigma(t,u) \end{aligned}$$

(b) "Weak" definition:

If $\alpha(t,T)$ and $\sigma(t,T)$ are functions of t, T and the whole of $f(t,s)$ for all $s>t$ (and not on any aspect of the history $\mathcal{F}_t$ given $f(t,s)$) then the process is Markov. However, it is necessary for us to know the whole of the forward-rate curve at time t rather than a finite number of observations.

"Strong" definition:

The same as the weak definition but in addition it must be possible to reconstruct the whole of the forward-rate curve at time t given

i. the whole of the forward-rate curve at time 0 (call this $\mathcal{F}_0$)

ii. a *finite* number of observations at time t (call this $X(t)$ where $X(t) = (X_1(t),\ldots,X_m(t))$ with $m<\infty$).

Equivalently, if $X(t)$ is some finite dimensional Ito process with $r(t)=g(X(t))$ for some function g, then the term-structure model is (strong-definition) Markov.

(c) i. Ho and Lee:

$$\begin{aligned} r(t) &= f(0,t)+\int_0^t \sigma(s,t)d\tilde{W}_s - \int_0^t \sigma(s,t)S(s,t)ds \\ &= f(0,t)+\frac{1}{2}\sigma^2 t^2 + \sigma\tilde{W}_t \end{aligned}$$

$\Rightarrow$ Markov

ii. Or more generally $\sigma(t,T)=\sigma(t)$ for all t,T where $\sigma(t)$ is deterministic:

$$\begin{aligned} S(s,t) &= -\int_s^t \sigma(s,u)du = -(t-s)\sigma(s) \\ \Rightarrow -\int_0^t \sigma(s,t)S(s,t)ds &= \int_0^t \sigma(s)^2(t-s)ds \\ &= \int_0^t\int_s^t \sigma(s)^2 du\ ds \\ &= \int_0^t\int_0^u \sigma(s)^2 ds\ du \\ \Rightarrow \frac{d}{dt}\left(-\int_0^t \sigma(s,t)S(s,t)ds\right) &= \int_0^t \sigma(s)^2 ds \end{aligned}$$

$$\begin{aligned} \text{Also } \int_0^t \sigma(s,t) d\tilde{W}_s &= \int_0^t \sigma(s) d\tilde{W}_s \\ \Rightarrow d\left[\int_0^t \sigma(s,t) d\tilde{W}_s\right] &= \sigma(t) d\tilde{W}_t. \\ \text{Hence } dr(t) &= \frac{\partial}{\partial t} f(0,t)dt + \left(\int_0^t \sigma(s)^2 ds\right) dt + \sigma(t) d\tilde{W}_t \end{aligned}$$

$\Rightarrow$ Markov

iii. The same argument as in (ii) applies
$\Rightarrow$ Markov

iv. We concentrate on

$$\int_0^t \sigma(s,t) d\tilde{W}_s = \int_0^t \frac{\sigma}{t-s+\delta} d\tilde{W}_s$$

We use the following Lemma:

$$d\left[\int_0^t f(s,t) d\tilde{W}_s\right] = f(s,t) d\tilde{W}_t + \left(\int_0^t \frac{\partial}{\partial t} f(s,t) d\tilde{W}_s\right) dt$$

Thus:

$$\begin{aligned} d\left[\int_0^t \frac{\sigma}{t-s+\delta} d\tilde{W}_s\right] &= \frac{\sigma}{t-s+\delta} d\tilde{W}_t + \left(\int_0^t \frac{-\sigma}{(t-s-\delta)^2} d\tilde{W}_s\right) dt \\ &= \frac{\sigma}{t-s+\delta} d\tilde{W}_t + X(t)dt \\ \text{But } X(t) &= \int_0^t \frac{-\sigma}{(t-s-\delta)^2} d\tilde{W}_s \end{aligned}$$

is another stochastic integral. Thus we can say that $r(t)$ is Markov given $X(t)$. But $X(t)$ is only Markov given another stochastic integral *etc.*
So we cannot say that $r(t)$ is unconditionally Markov.

v. Hull and White:
Let $X(t) = \int_0^t \sigma(s,t) d\tilde{W}_s = \sigma \int_0^t e^{-\alpha(t-s)} d\tilde{W}_s$.
The we know that $dX(t) - \alpha X(t)dt + \sigma d\tilde{W}_t$ (Ornstein-Uhlenbeck). So $X(t)$ is Markov.
Hence $r(t)$ is Markov.

vi. $r(t) = f(0,t) + \int_0^t \sigma(s,t) d\tilde{W}_s - \sigma(s,t) S(s,t) ds$.
If $\sigma(s,t)$ is deterministic then $r(t)$ is Markov if and only if $\int_0^t \sigma(s,t) d\tilde{W}_s = X(t)$ is Markov.
Here

$$\begin{aligned} X(t) &= X_1(t) + X_2(t) \\ \text{where } X_i(t) &= \sigma_i \int_0^t e^{-\alpha_i(t-s)} d\tilde{W}_s \\ \text{But } dX_i(t) &= -\alpha_i X_i(t) dt + \sigma_i d\tilde{W}_t \end{aligned}$$

So $X_i(t)$ is Markov for $i = 1, 2$. Hence $X(t)$ is Markov in the sense that if we known $X_1(t)$ and $X_2(t)$ it is Markov.
The values for $X_1(t)$ and $X_2(t)$ can be inferred from, for example, two points on the forward-rate curve: for example, $r(t) = f(t,t)$ and $f(t, t+1)$; or $r(t)$ and $\frac{\partial}{\partial T} f(t,T)|_{T=t}$.
[Note: since $X_1(t)$ and $X_2(t)$ both depend upon $\tilde{W}_t$ they are not independent but nor are they perfectly correlated given $\mathcal{F}_0$.]

5. $\sigma(t,T) = \sigma e^{-\alpha(T-t)}$ implies Hull and White. Hence

$$\begin{aligned}
dr(t) &= \alpha(\mu(t) - r(t))dt + \sigma d\tilde{Z}(t) \\
\text{where } \mu(t) &= \frac{1}{\alpha}\frac{\partial}{\partial t} f(0,t) + f(0,t) + \frac{\sigma^2}{2\alpha^2}(1 - e^{-2\alpha t}) \\
&= \frac{1}{\alpha}\left(-\alpha\lambda_1 e^{-\alpha t} - \frac{\sigma^2}{\alpha^2}(1 - e^{-\alpha t})\alpha e^{-\alpha t}\right) + \lambda_0 + \lambda_1 e^{-\alpha t} \\
&\qquad - \frac{\sigma^2}{2\alpha^2}(1 - e^{-\alpha t})^2 + \frac{\sigma^2}{2\alpha^2}(1 - e^{-2\alpha t}) \\
&= \lambda_0
\end{aligned}$$

From Section 8.1 of the lecture notes:

$$\begin{aligned}
r(t) &= e^{-\alpha t} r(0) + \alpha \int_0^t e^{-\alpha(t-s)}\mu(s)ds + \sigma \int_0^t e^{-\alpha(t-s)} d\tilde{Z}(s) \\
&= e^{-\alpha t} r(0) + (1 - e^{-\alpha t})\lambda_0 + \sigma \int_0^t e^{-\alpha(t-s)} d\tilde{Z}(s) \\
&= g(t, r(0)) + \int_0^t h(s,t) d\tilde{Z}(s) \\
\text{where } g(t,r) &= \lambda_0 + (r - \lambda_0)e^{-\alpha t} \\
h(s,t) &= \sigma e^{-\alpha(t-s)}
\end{aligned}$$

This is called the Vasicek model!

6. (a) Arbitrage-free $\Rightarrow \alpha(t,T) = \sigma(t,T)(\gamma(t) - S(t,T))$ where $\gamma(t)$ is the market price of risk.
We know that:

$$f(t,T) = f(0,T) - \int_0^t \sigma(u,T)S(u,T)du + \int_0^t \sigma(u,T)\gamma(u)du + \int_0^t \sigma(u,T)dZ(u)$$

If $\gamma(t)$ is deterministic then $f(t,T)$ would be Gaussian. However, it is only necessary that $\gamma(t)$ is a previsible process. If $\gamma(t)$ is stochastic then $f(t,T)$ might not be Gaussian since the term $\int_0^t \sigma(u,T)\gamma(u)du$ may not be Gaussian.

(b) We exploit the result that if $X(t) = a(t) + \int_0^t b(s)dZ(s)$ where $a(t)$ and $b(t)$ are deterministic functions, then $X(t)$ is Normal with mean $a(t)$ and variance $\int_0^t b(s)^2 ds$.

If $\gamma(t)$ is determinstic then $f(t,T)$ is Normal with mean

$$f(0,T) - \int_0^t \sigma(u,T)S(u,T)du + \int_0^t \sigma(u,T)\gamma(u)du$$

and variance

$$\int_0^t \sigma(u,T)^2 du$$

(c)

$$\begin{aligned} P(t,T) &= \exp\left[-\int_t^T f(t,u)du\right] \\ \int_t^T f(t,u)du &= \int_t^T f(0,s)ds - \int_t^T \int_0^t \sigma(u,s)S(u,s)du\ ds + \int_t^T \int_0^t \sigma(u,s)d\tilde{Z}(u)\ ds \\ &= \int_t^T f(0,s)ds - \int_t^T \int_0^t \sigma(u,s)S(u,s)du\ ds + \int_0^t \int_t^T \sigma(u,s)ds\ d\tilde{Z}(u) \end{aligned}$$

Hence $-\int_t^T f(t,s)ds$ is Normal under Q with mean

$$\int_t^T f(0,s)ds - \int_t^T \int_0^t \sigma(u,s)S(u,s)du\ ds$$

and variance

$$\int_0^t \left(\int_t^T \sigma(u,s)ds\right)^2 du$$

Hence $P(t,T)$ will be log-normal with the same parameters.

7. (a) Let $V(t)$ be the price at t for g payable continuously until time T plus 100 payable at T.

Consider this as the price for coupons at the rate of g between t and $t+dt$ plus the price $\tilde{V}(t)$ for the payments after time $t+dt$. Thus:

$$\begin{aligned} V(t) &= g.dt + g\int_{t+dt}^T P(t,u)du + 100P(t,T) + o(dt) \\ &= g.dt + \tilde{V}(t) + o(dt) \\ \text{where}\quad \tilde{V}(t) &= g\int_{t+dt}^T P(t,u)du + 100P(t,T). \\ \text{Now}\quad d\tilde{V}(t) &= g\int_{t+dt}^T dP(t,u)du + 100dP(t,T) \\ &= g\int_{t+dt}^T P(t,u)(r(t)dt + S(t,u)d\tilde{Z}(t)) + 100P(t,T)(r(t)dt + S(t,T)d\tilde{Z}(t)) \\ &= \tilde{V}(t)r(t)dt + \left(g\int_{t+dt}^T P(t,u)S(t,u)du + 100P(t,T)S(t,T)\right)d\tilde{Z}(t) \\ &= V(t)r(t)dt + \left(g\int_t^T P(t,u)S(t,u)du + 100P(t,T)S(t,T)\right)d\tilde{Z}(t) + o(dt) \end{aligned}$$

Hence

$$\begin{aligned} b_v(t, r(t)) &= \left(g \int_t^T P(t,u)S(t,u)du + 100P(t,T)S(t,T) \right) \\ \text{and} \quad a_v(t, r(t)) &= V(t)r(t)dt - g.dt. \end{aligned}$$

(b) i.

$$\begin{aligned} V(0) &= 10 \int_0^T e^{-0.1u} du + 100e^{-0.1T} \\ &= 100 \end{aligned}$$

ii.

$$\begin{aligned} \sigma(t,u) &= \sigma e^{-0.1(u-t)} \\ \Rightarrow S(0,u) &= -\int_0^u \sigma(0,v)dv \\ &= -\sigma \int_0^u e^{-0.1v} dv \\ &= -10\sigma(1 - e^{-0.1u}) \\ \Rightarrow dV(0) &= -100\sigma \int_0^T e^{-0.1u}(1 - e^{-0.1u})du + 1000\sigma e^{-0.1T}(1 - e^{-0.1T}) \\ &= -500\sigma(1 - e^{-0.2T}) \\ \Rightarrow -\frac{dV(0)}{V(0)} &= -5\sigma(1 - e^{-0.2T}) \end{aligned}$$

iii. $-dV(0)/V(0)$ increases from 0 to 5σ as T goes from 0 to ∞, so the maximum occurs when $T = \infty$: that is we get the maximum volatility with an irredeemable coupon bond.

iv. Zero-coupon bonds have a volatility of $10\sigma(1 - e^{-0.1T})$. For example with $T = 16.094$ this is 8σ which is higher than that for irredeemable coupon bond.
It follows that a bond with a term of about 16 years to maturity and with a sufficiently low coupon rate will have a volatility of between 5σ and 8σ.

Not used in 2004

8. (a)

$$V(t) = E_Q\left[e^{-\int_t^T r(s)ds}\max\left(r(T) - r_c, 0\right) \mid \mathcal{F}_t\right]$$

(b) i. From the notes we use the change of measure drift $\gamma(t) = -S(t,T)$ but

$$\begin{aligned} S(t,T) &= -\int_t^T \sigma(t,u)du \\ \text{and } \sigma(t,u) &= \sigma e^{-\alpha(u-t)} \text{ under Vasicek} \\ \Rightarrow S(t,T) &= -\sigma\frac{\left(1 - e^{-\alpha(T-t)}\right)}{\alpha} \\ \gamma(t) &= \frac{\sigma}{\alpha}\left(1 - e^{-\alpha(T-t)}\right) \end{aligned}$$

ii. We have $d\hat{W}_t = d\tilde{W}_t + \gamma(t)dt$ and

$$\begin{aligned} dr(t) &= \alpha(\mu - r(t))dt + \sigma d\tilde{W}_t \\ &= \alpha(\mu - r(t))dt + \sigma(d\hat{W}_t - \gamma(t)dt) \\ &= \alpha(\mu(t) - r(t))dt + \sigma d\hat{W}_t \\ \text{where } \mu(t) &= \mu - \frac{\sigma^2}{\alpha^2}\left(1 - e^{-\alpha(T-t)}\right) \end{aligned}$$

$\mu(t)$ is a deterministic function.
Suppose that $X(s) = \sigma\int_t^s e^{-\alpha(s-u)}d\hat{W}_u$ for $s > t$.
$\Rightarrow dX(s) = -\alpha X(s)ds + \sigma d\hat{W}_s$.
Suppose also that $r(s) = e(s) + X(s)$ for some deterministic function $e(s)$.
Then $dr(s) = de(s) + dX(s)$

$$\begin{aligned} \Rightarrow \alpha(\mu(s) - r(s))ds + \sigma d\hat{W}_s &= \frac{de}{ds}ds + (-\alpha X(s)ds + \sigma d\hat{W}_s) \\ \Rightarrow \alpha(\mu(s) - e(s) - X(s))ds &= e'(s)ds - \alpha X(s)ds \\ \Rightarrow e'(s) + \alpha e(s) &= \mu(s) \\ \Rightarrow e(s) &= e^{-\alpha(s-t)}r(t) + \left(\mu - \frac{\sigma^2}{\alpha^2}\right)\left(1 - e^{-\alpha(s-t)}\right) \\ &\quad + \frac{\sigma^2}{2\alpha^2}e^{-\alpha(T-s)}\left(1 - e^{-2\alpha(s-t)}\right) \end{aligned}$$

Now $X(s)$ is an Ornstein-Uhlenbeck process under P_T so it is normally distributed.
$\Rightarrow r(s)|\mathcal{F}_t$ is normal with:

$$E_{P_T}\left[r(s)|\mathcal{F}_t\right] = e(s)$$

$$\begin{aligned} Var_{P_T}[r(s)|\mathcal{F}_t] &= Var_{P_T}[X(s)|\mathcal{F}_t] \\ &= \int_t^s \sigma^2 e^{-2\alpha(s-u)} du \\ &= \sigma^2 \frac{\left(1 - e^{-2\alpha(s-t)}\right)}{2\alpha} \end{aligned}$$

iii. Suppose that $X \sim N(\mu, \sigma^2)$. Then

$$E[(X - x_0)_+] = \sigma E\left[\left(\frac{X-\mu}{\sigma} - \frac{x_0 - \mu}{\sigma}\right)_+\right] = \sigma E[(Z - z_0)_+]$$

where $z_0 = (x_0 - \mu)/\sigma$ and $Z \sim N(0, 1)$.
Hence

$$\begin{aligned} E[(X - x_0)_+ &= \frac{\sigma}{\sqrt{2\pi}} \int_{z_0}^{\infty} (z - z_0) e^{-\frac{1}{2}z^2} dz \\ &= \frac{\sigma}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_0-\mu}{\sigma}\right)\right] - \sigma \frac{x_0 - \mu}{\sigma}\left(1 - \Phi\left(\frac{x_0-\mu}{\sigma}\right)\right) \\ \Rightarrow E_{P_T}[(r(T) - r_c)_+|\mathcal{F}_t] &= \frac{\sigma_r}{\sqrt{2\pi}} e^{-\frac{1}{2}z_0^2} - \sigma_r z_0 (1 - \Phi(z_0)) \\ z_0 &= \frac{r_c - \mu_r}{\sigma_r} \\ \text{where } \mu_r &= e^{-\alpha(T-t)} r(t) + \left(\mu - \frac{\sigma^2}{\alpha^2}\right)\left(1 - e^{-\alpha(T-t)}\right) + \frac{\sigma_r^2}{\alpha} \end{aligned}$$

iv.

$$\begin{aligned} V(t) &= P(t,T) E_{P_T}[(r(T) - r_c)_+|\mathcal{F}_t] \\ &= P(t,T)\left[\frac{\sigma_r}{\sqrt{2\pi}} e^{-\frac{1}{2}z_0^2} - \sigma_r z_0 (1 - \Phi(z_0))\right] \\ \text{where } P(t,T) &= e^{A(t,T) - B(t,T) r(t)} \quad \text{in the usual way} \end{aligned}$$