Risk Management
8: Market Risk Measurement and Management I

Reading:

- Study Note on Stochastic Volatility
- McNeil et al. Chapters 3 and 4
- Moody's Analytics: Model Validation
- Study Note on Model Validation

Overview

- Unit 8.1: Introduction
- Unit 8.2: Testing for independence
- Unit 8.3: Stochastic volatility and GARCH models

Unit 8.1: Introduction

Types of Market Risk

The risk of losses in positions arising from movements in market prices.

- Interest rate risk (Ch. 10)
- Factors driving equity prices
- Foreign exchange
- Commodity
- Credit spreads: credit risk risk premium, illiquidity premium (*)
- etc.

(*) $\Rightarrow$ excludes premium for expected credit losses

Key situations involving market risk

- Banking
  - derivatives desk
  - FX
  - interbank lending
- Insurer
  - investment of reserves
  - investment-linked savings contracts
  - contracts with guarantees
  - non-investment-linked contracts
- Investment trusts; Mutual funds; Hedge funds
  - terms of reference
  - risk tolerances, limits
  - basis risk for passive funds

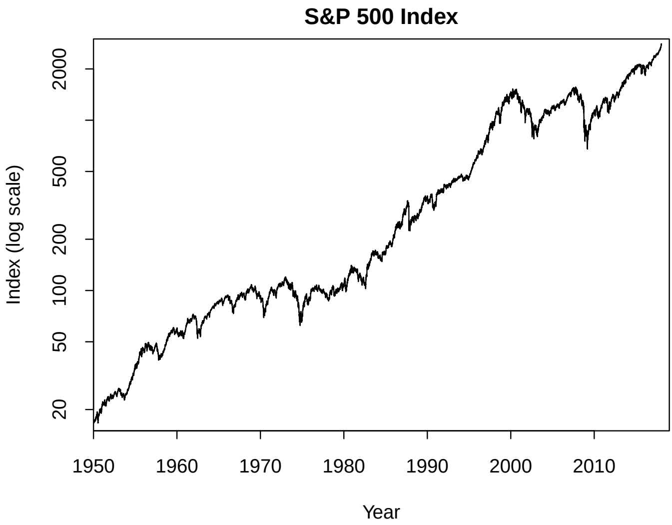
Example: Analysis of S&P500 Returns

- Time unit = 1 trading day (about 252 per year)
- $\delta(t)$ = log return on index from $t-1$ to t:

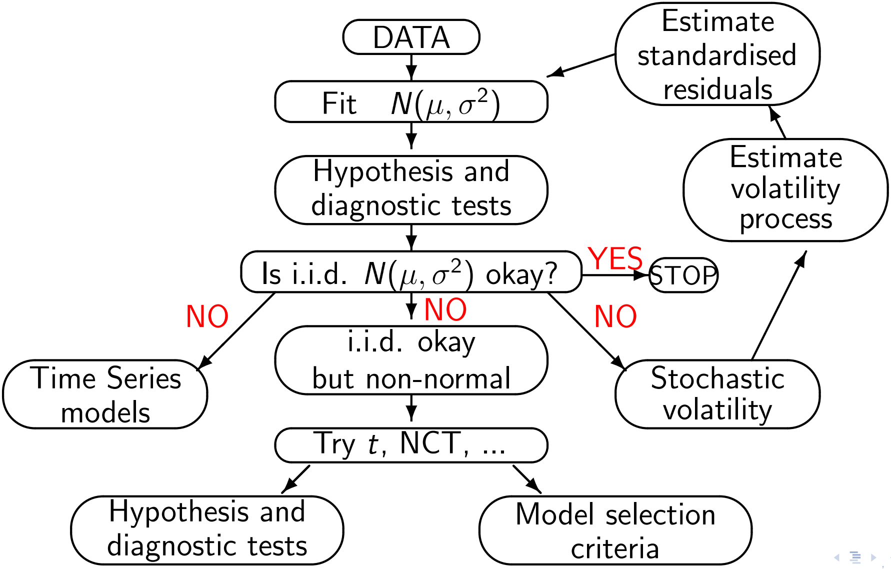
$$\Rightarrow \$1 \text{ at } t-1 \quad \longrightarrow \quad \$e^{\delta(t)} \text{ at } t.$$

- What model for $\delta(t)$?
- $R(t) = \delta(1) + \delta(2) + \ldots + \delta(t)$ = log return from 0 to t:
  \$ 1 at time 0 $\rightarrow$ \$ $e^{R(t)}$ at time t
- What model for $R(t)$?

Historical Data



Model building & selection flow chart



Stochastic Volatility

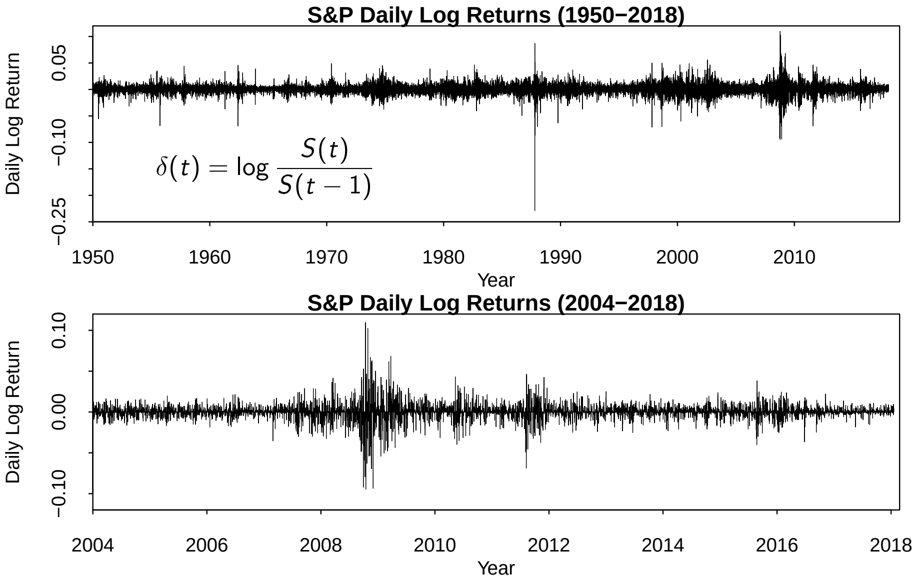
Stylised facts

1. Returns are leptokurtic and heavy tailed.
2. Returns are not i.i.d., although autocorrelation is low.
3. Squared returns have significant autocorrelation.
4. Volatility appears to vary over time.
5. Extreme returns appear in clusters.

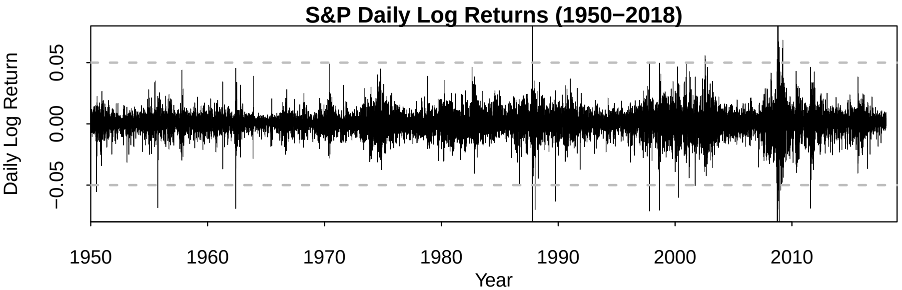
Hypothesis Tests and Graphical Diagnostics

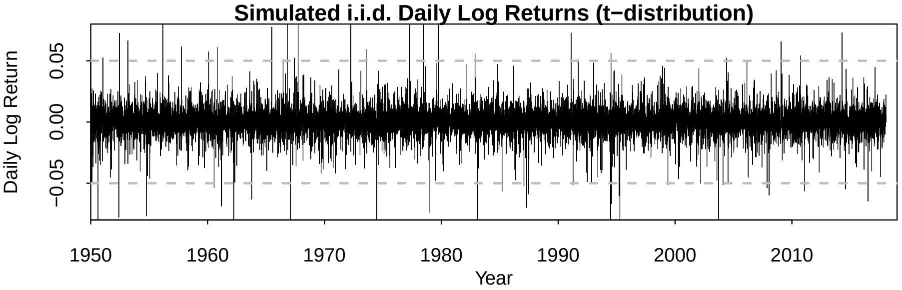
- Formal statistical analysis e.g.
  - Hypothesis tests
  - Numerical model selection criteria (e.g. Bayes Information Criterion)
- Graphical diagnostics
  - Range of graphics that explore the raw data and processed data under a particular hypothesis
  - Each graphic should exhibit certain characteristics if the hypothesis is true (e.g. if you have chosen a good model)
  - Wrong characteristics $\Rightarrow$ pointers to an alternative, better hypothesis or model
  - Graphical diagnostics will sometimes reveal issues that formal tests do not

Graphical Diagnostic: Daily Log Returns



Are the $\delta(t)$ i.i.d.? versus Clusters of Extremes





Summary

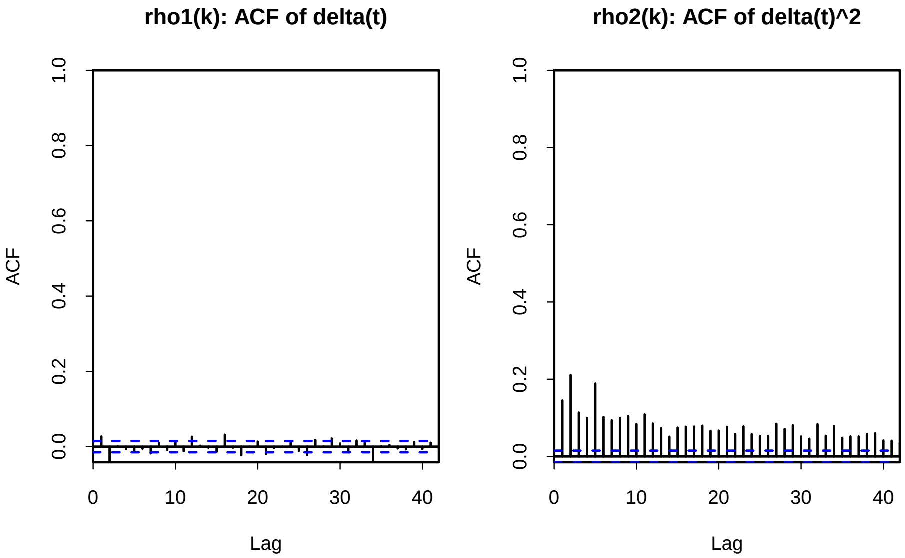
- Know the main types of market risk
- Show an understanding of the basic properties of a financial time series
- Know the stylised facts that apply to most financial time series
- Use a time series plot of daily log returns and know how to interpret it including visual evidence for stochastic volatility

Unit 8.2: Testing for Independence

Testing for non independence: The autocorrelation function

- Data: $\delta(t)$ for $t = 1, \ldots, n$
- $k =$lag
- Autocorrelation function: $\rho_1(k) \; = \; \text{cor}\Big(\delta(t), \; \delta(t+k) \; \Big)$ (unconditional correlation)
- If the $\delta(t)$ are i.i.d. then $\rho_1(k) = 0$ for all $k > 0$
- If empirical $\hat{\rho}_1(k) \approx 0 \Rightarrow$ *consistent* with i.i.d. hypothesis but does not prove i.i.d. hypothesis
- Also look at $\rho_2(k) \; = \; \text{cor}\Big(\delta(t)^2, \; \delta(t+k)^2 \; \Big)$

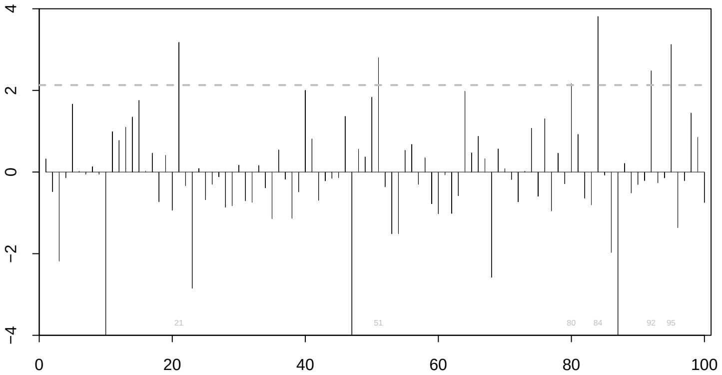
Graphical Diagnostic: Autocorrelation Function



Exceedances above a high threshold

- Data: $\delta(t)$ for $t = 1, \ldots, n$
- $q =$ empirical 95% quantile of the $\delta(t)$
- $T_1 =$ first time that $\delta(t) > q$
- $T_2 =$ second time ...
- ...
- $T_m =$ final time that $\delta(t) > q$

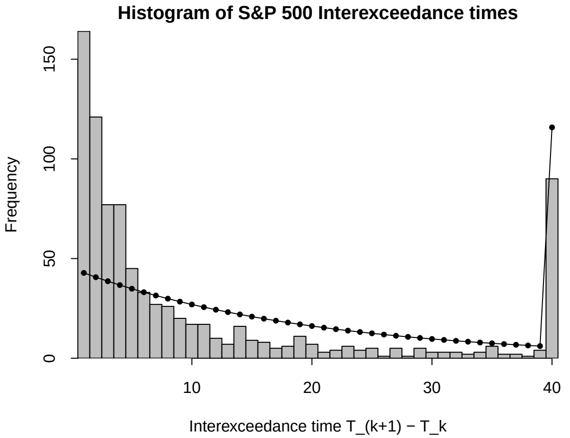
Example: Identification of Exceedance Times



Distribution of Interexceedance Times

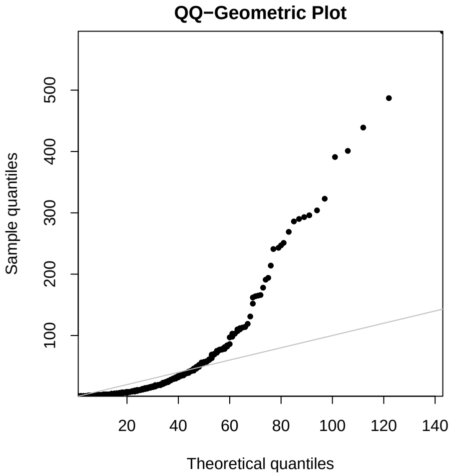
- $q = 95\%$ quantile implies
  - $\#$ exceedances $\approx n/20$
  - $E[T_k - T_{k-1}] \approx 20$
- H_0: $\delta(t)$ are i.i.d.
- H_1: $\delta(t)$ are not i.i.d.
- H_0 true $\Rightarrow (T_2 - T_1),\ (T_3 - T_2), \ldots$ are independent and
  $T_k - T_{k-1}$ has a Geometric distribution
- For $k = 1, \ldots, m-1$,
  $Pr(T_{k+1} - T_k \ = \ s) = (1-p)p^{s-1}$ for $s \geq 1$,
  $p = 0.95$

Graphical Diagnostic: Histogram



Threshold: Empirical 95% quantile
Bars: histogram of actual interexceedance times
Line: Expected frequencies for Geometric(0.05)

Graphical Diagnostic: QQ plot



Theoretical quantiles: from Geometric(0.05)

Digression: QQ plot construction

- Data: $X_i : i = 1, \ldots, n$
- Rank the data: $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$
- $F(x)$ = fitted cumulative distribution function
- For $i = 1, \ldots, n$ define

$$u_i = \frac{i - \frac{1}{2}}{n}$$

  an evenly spaced sequence filling up $(0, 1)$.
- For $i = 1, \ldots, n$ define

$$Y_{(i)} = F^{(-1)}(u_i)$$

  the u_i quantile of the random variable with CDF $F(x)$.
- The QQ plot is a scatterplot of $\left(Y_{(i)}, X_{(i)}\right)$ for $i = 1, \ldots, n$.
- If the CDF $F(x)$ represents a good fit, then the QQ plot should look reasonably linear along the diagonal $y = x$.

Preliminary Conclusions

- Very strong evidence that daily log returns are not i.i.d.
- Graphical diagnostics (several)
  - do not support i.i.d. hypothesis
  - extended periods of high and low volatility
  - clusters of extremes (+ve and −ve)
- Possibly stochastic volatility
- Next steps: find a potential model for stochastic volatility that fits the observed pattern

Summary

- Show how the autocorrelation function can be used to test for independence or not
- Show how analysis of interexceedance times can be used to test for independence or not
- Summarise all of the potential graphical diagnostics for independence of returns
- Show how autocorrelation plots and analysis of interexceedance times leads to the conclusion that financial returns exhibit volatility clustering

Unit 8.3: Stochastic Volatility and GARCH Models

Stochastic Volatility: GARCH Models

Generalised Auto-Regressive Conditionally Heteroscedastic

- $\delta(t)$ = daily log return with mean μ
- Define $X(t) = \delta(t) - \mu$.
- Let $Z(1), Z(2), \ldots$ i.i.d., $E[Z(t)] = 0$, $Var[Z(t)] = 1$
- GARCH(p, q) model $\Rightarrow$

$$\begin{aligned} X(t) &= \sigma(t)Z(t) \\ \sigma(t)^2 &= \alpha_0 + \sum_{i=1}^{p} \alpha_i X(t-i)^2 + \sum_{j=1}^{q} \beta_j \sigma(t-j)^2 \end{aligned}$$

- $\alpha_0 > 0$
- $\alpha_i \geq 0$ for $i = 1, \ldots, p$
- $\beta_j \geq 0$ for $j = 1, \ldots, q$.

GARCH(1,1)

$$\begin{aligned} X(t) &= \sigma(t)Z(t) \\ \sigma(t)^2 &= \alpha_0 + \alpha_1 X(t-1)^2 + \beta_1 \sigma(t-1)^2 \\ &= \alpha_0 + \Big(\alpha_1 Z(t-1)^2 + \beta_1\Big)\sigma(t-1)^2 \end{aligned}$$

- $E[\log(\alpha_1 Z(t-1)^2 + \beta_1)] < 0 \Rightarrow$ strictly stationary.
- $\alpha_1 + \beta_1 < 1 \Rightarrow$ covariance stationary
  $Var[X(t)] = \alpha_0/(1 - \alpha_1 - \beta_1)$.

Maximum Likelihood Estimation

- θ = parameter vector
  - $\sigma(1)$
  - $\alpha_0, \alpha_1, \beta_1, \mu$
  - ϕ = density of $Z(t)$ parameters (s.t. mean 0, Var. 1)
- $\delta(t)$ i.i.d. $\Rightarrow$ Likelihood is

$$L(\theta; \delta) = \prod_{t=1}^{T} f(\delta(t); \theta)$$

Likelihood for non-i.i.d. $\delta(t)$

$$
\begin{aligned}
L(\theta;\delta) &= f(\delta(1),\ldots,\delta(T)|\theta) \\
&= f(\delta(1)|\theta) \times f(\delta(2)|\delta(1);\theta) \times \ldots \\
&\qquad \times f(\delta(T)|\delta(1),\ldots,\delta(T-1),\theta)
\end{aligned}
$$

Additionally:

$$
f(\delta(t)|\delta(1),\ldots,\delta(t-1),\theta) \equiv f(\delta(t)|\sigma(t),\mu,\phi)
$$

and $\sigma(t)$ depends on $\sigma(1),\alpha_0,\alpha_1,\beta_1,\mu$ and $\delta(1),\ldots,\delta(t-1)$

Full MLE

- $\sigma(1)$ is a parameter to be estimated
- $\sigma(t)^2 = \alpha_0 + (\alpha_1 Z(t-1)^2 + \beta_1)\sigma(t-1)^2$

$$\begin{aligned} L(\theta;\delta) &\equiv f(\delta(1)|\sigma(1)) \times f(\delta(2)|\sigma(2)) \times \dots \\ &\quad \times f(\delta(T)|\sigma(T)) \end{aligned}$$

- What is the density of $\delta(t)|\sigma(t)$?

Recursive calculations:

- $\psi = (\sigma(1), \alpha_0, \alpha_1, \beta_1, \mu)$
- $Z_\psi(1) = (\delta(1) - \mu)/\sigma_\psi(1)$
- For $t = 2, 3, \dots, n$:
  - $\sigma_\psi(t)^2 = \alpha_0 + (\alpha_1 Z_\psi(t-1)^2 + \beta_1)\sigma_\psi(t-1)^2$
  - $Z_\psi(t) = (\delta(t) - \mu)/\sigma_\psi(t)$

Full MLE

$$L(\theta;\delta) \;=\; \prod_{t=1}^{T} \frac{1}{\sigma_\psi(t)} h\left(\frac{\delta(t)-\mu}{\sigma_\psi(t)};\phi\right)$$

- $h(z) =$ density of i.i.d. $Z(t)$
- $\phi =$ parameters of $h(z)$ subject to $E[Z] = 0$, $Var[Z] = 1$
- $\psi = (\sigma(1), \alpha_0, \alpha_1, \beta_1, \mu)$
- $\theta = (\psi, \phi)$

Quasi Maximum Likelihood

Five steps:

- Estimate $\alpha_0, \alpha_1, \beta_1$ (and $\mu, \sigma(1)$) as if $Z(t) \sim N(0,1)$
- Extract $\hat{\mu}$ and $\hat{\sigma}_\psi(t)$ for $t = 1, 2, \ldots$
- Calculate the volatility standardised residuals, $Z(t)$
- Find the "correct" distribution for $Z(t)$ $\Rightarrow$ correct density $h(z)$
- Then carry out full MLE with the updated $h(z)$

S&P500 Data from 1/1980 to 1/2018

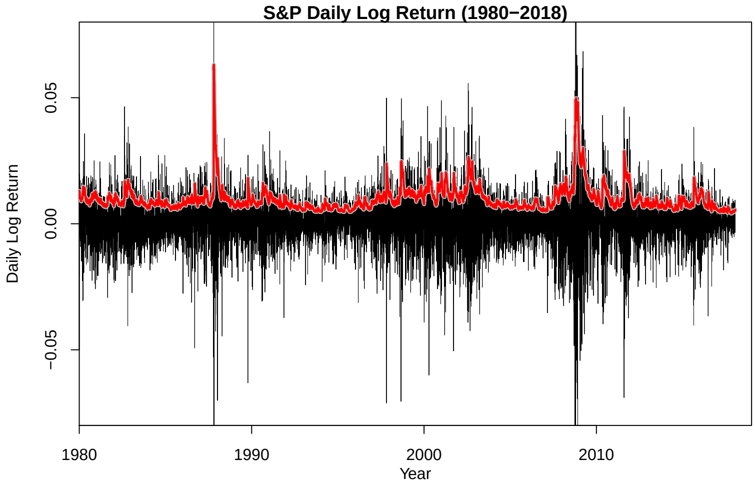
- 9597 observations of $\delta(t)$
- Pre-whitening:
  Quasi maximum likelihood $\Rightarrow Z(1), \ldots, Z(T)$.
  QML: estimated GARCH(1,1) parameters:

$$\hat{\alpha}_0 = 1.442 \times 10^{-6}, \quad \hat{\alpha}_1 = 0.0839, \quad \hat{\beta}_1 = 0.9045.$$

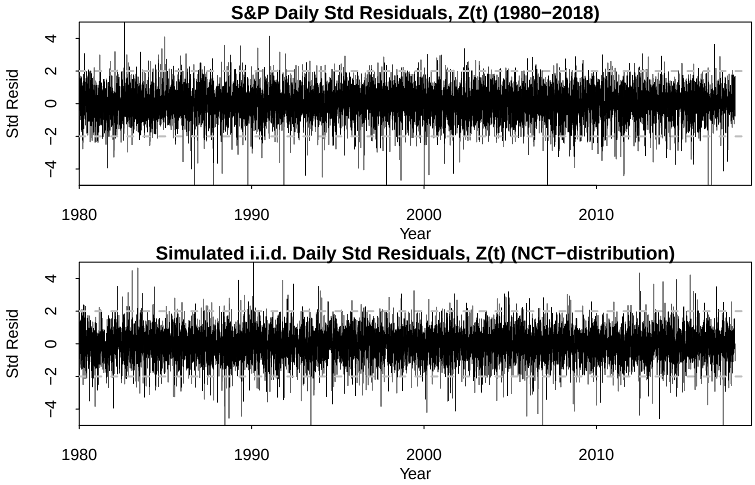
$$\hat{\sigma}(1) = 0.01083, \quad \hat{\mu} = 0.0005715.$$

- $\sqrt{E[\sigma(t)^2]} = \sqrt{\hat{\alpha}_0/(1 - \hat{\alpha}_1 - \hat{\beta}_1)} = 0.01115$
  (per trading day)

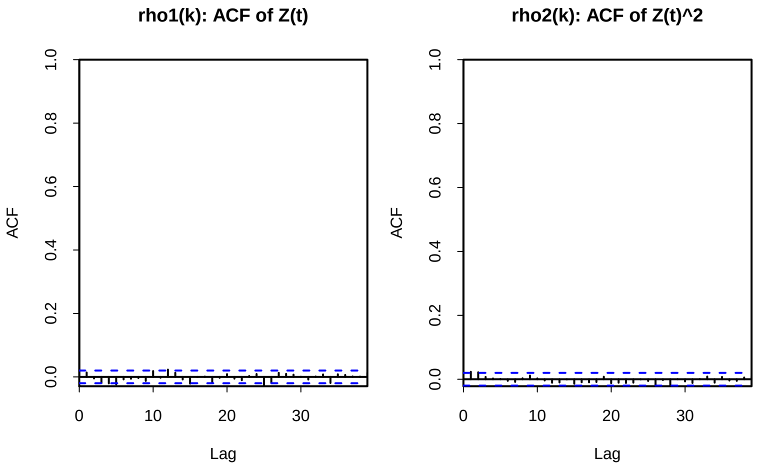
The Fitted Volatility $\sigma(t)$



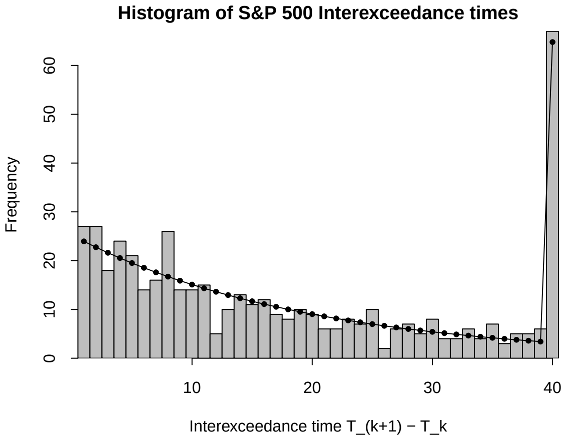
Standardised Residuals, $Z(t)$



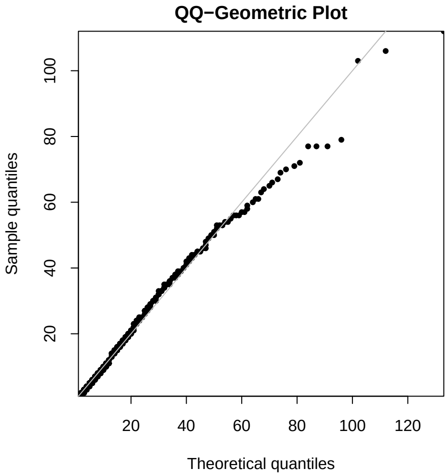
Autocorrelation Functions for $Z(t)$



Distribution of Interexceedance Times



QQ Plot of Interexceedance Times



Conclusions

- Daily log returns and many other financial series exhibit periods of high and low volatility.
- The GARCH(1,1) model does a good job of modelling the volatility as a stochastic process
- Next steps: develop a model for the i.i.d. random innovations, $Z(t)$
  - Fat tails
  - Skewed
  - Use extreme value theory for extreme tail probabilities

Conclusions (cont.)

- Key point:
  Failure to use (a) a stochastic volatility and (b) fat-tailed distributions and/or extreme value theory will result in hopelessly inaccurate estimates of the probability of bad events. In particular, you will seriously underestimate the probability of bad events if you are already in the midst of a turbulent period.

Postcript: model validation and other issues

Modelling and model selection also requires

- checks that the data are correct
- checks of programming: likelihood and optimisation code; forecasting model
- robustness checks
- independent model validation
- audit trail and documentation

Reading: Study Note on Model Validation

Summary

- Define the GARCH(1,1) model for stochastic volatility
- Know the conditions for a GARCH(1,1) model to be covariance stationary
- Derive the likelihood function for a GARCH(1,1) model
- Understand how quasi maximum likelihood works
- Use a GARCH(1,1) to analyse financial returns data

Summary (cont.)

- Apply diagnostic tests to verify if GARCH(1,1) standardised residuals are independent and identically distributed