Risk Management

10: Interest Rate Risk Management

- Hull (Risk Management): Chapter 8
- Crouhy: Chapter 6
- Sweeting: Chapter 16.3
- Hull (Options Futures and Other Derivatives, 9th edition)
    - Chapters 4, 6, 7 (revision of key market concepts)
    - Chapter 31: Short-rate models (31.1-31.3)
    - Chapter 32: The HJM model (32.1)
- plus other topics in these slides

Outline

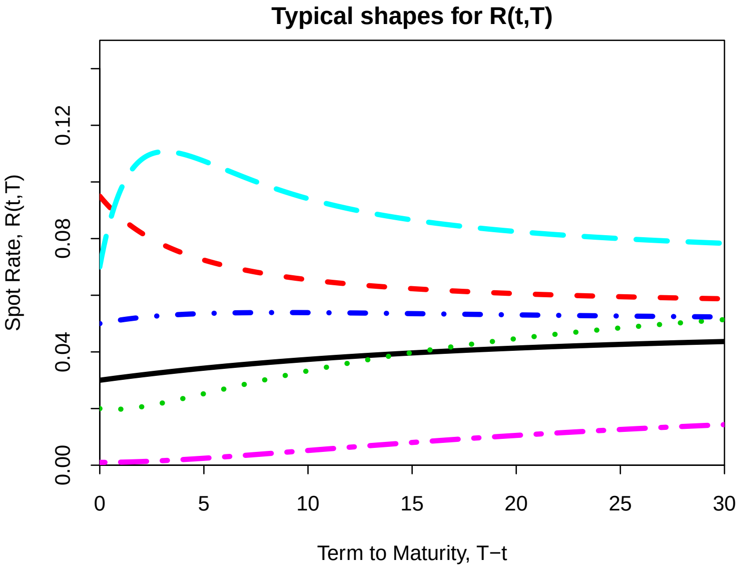
- Unit 10.1: Introduction and Redington Immunization
- Unit 10.2: Arbitrage-free stochastic models – short-rate models
- Unit 10.3: More general multi-factor models
- Unit 10.4: Stochastic liabilities and instruments for hedging interest-rate risk

Unit 10.1: Introduction and Redington Immunization

Basic Notation and Terminology

- Assume all cashflows at $t = 1, \ldots, n$
- Zero-coupon bonds: $P(t, T) =$ price at t for £1 at T
- Spot rate: $R(t, T) = -(T - t)^{-1} \log P(t, T)$
- History $\Rightarrow$ rising yield curves are more common [Long bonds $\Rightarrow$ more risky in short term $\Rightarrow$ risk premium over cash]

Typical Yield Curves



Coupon bonds

- Bond i pays $c_i(s)$ at $s = 1, \ldots, n$
- No arbitrage $\Rightarrow$ current price

$$B_i(t) = \sum_{s=t+1}^{n} c_i(s) P(t, s)$$

- Yield to maturity $\Rightarrow$ unique solution, $y_i(t)$ to

$$B_i(t) = \sum_{s=t+1}^{n} c_i(s) e^{-y_i(t)(s-t)}$$

Liabilities

- Fixed liabilities: $L(s)$ payable at time $s = 1, \ldots, n$
- Market consistent value: $V_L(t) = \sum_{s=t+1}^{n} L(s)P(t,s)$
- Portfolio of assets: u_i units of bond $B_i(t)$

$$\Rightarrow V_A(t) = \sum_i u_i \sum_{s=t+1}^{n} c_i(s)P(t,s) = \sum_{s=t+1}^{n} P(t,s) \sum_i u_i c_i(s)$$

Hence just use zero coupon bonds for convenience.

- Interest rate risk: uncertainty in $V_A(t)$, $V_L(t)$ associated with uncertainty in the *term structure of interest rates*.

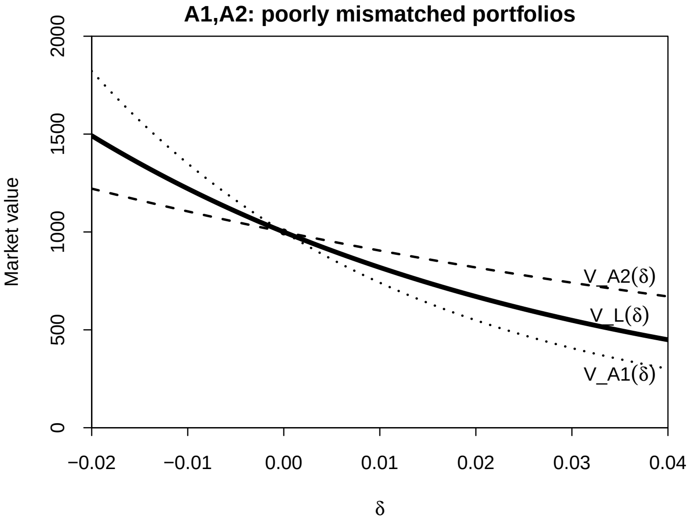
Asset-Liability Matching

- Hold $u_i = L(i)$ units of $P(t, i)$
- Then any change in the *term structure of interest rates*, $R(t, T)$, will still result in $V_A(t) = V_L(t)$
  $\Rightarrow$ perfectly hedged
- BUT: investment in all n zero-coupon bonds might not be possible/practical or expensive
  $\Rightarrow$ mismatching

Parallel shifts in the spot-rate curve

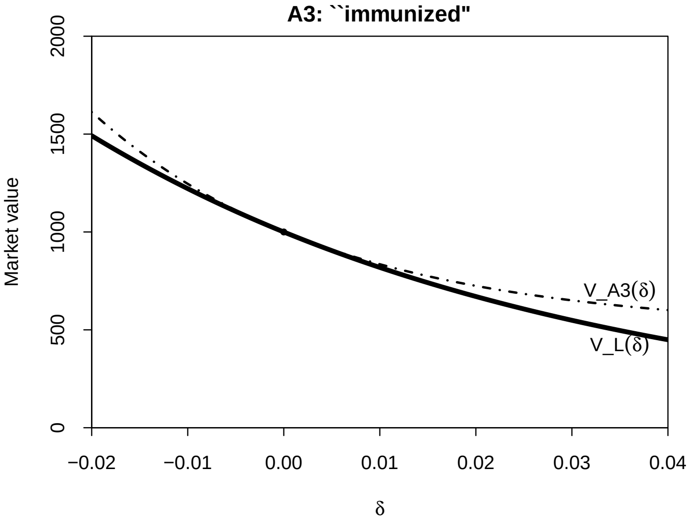
$R(t, T) \longrightarrow \tilde{R}(t, T) = R(t, T) + \delta$ where δ is unknown.

Hence $\tilde{P}(t, T) = P(t, T)e^{-\delta(T-t)}$



$A_1 \Rightarrow$ loss if $R(t,T)$ rises $(\delta > 0)$

$A_2 \Rightarrow$ loss if $R(t,T)$ falls $(\delta < 0)$



$A_3 \Rightarrow$ small profits if $R(t, T)$ falls or rises ($\delta < 0$)

Redington's Theory of Immunization

- Notation: $V_L(\delta) \equiv V_L(t;\delta)$
- Duration ("Macaulay Duration"):

$$
\begin{aligned}
\tau_L &= -\frac{1}{V_L(\delta)} \left.\frac{\partial V_L}{\partial \delta}(\delta)\right|_{\delta=0} \\
&= -\frac{\partial}{\partial \delta} \log V_L(\delta)|_{\delta=0} \\
&= \frac{\sum_{s=t+1}^{n} (s-t)L(s)P(t,s)}{\sum_{s=t+1}^{n} L(s)P(t,s)} \\
&= \text{weighted average of payment dates}
\end{aligned}
$$

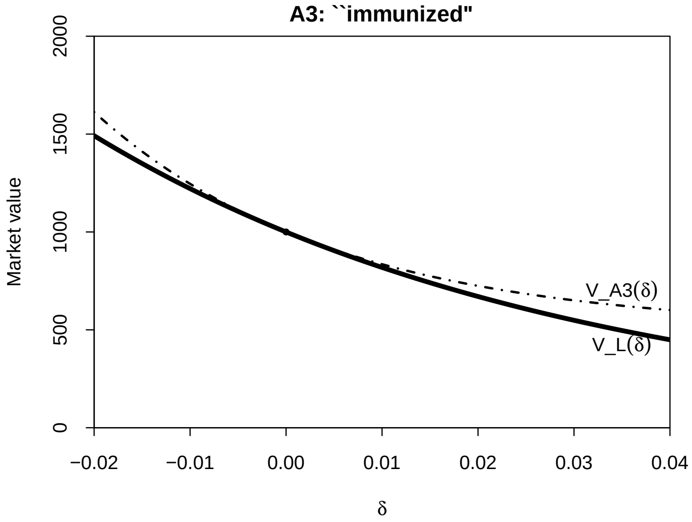
- Convexity:

$$C_L = \frac{1}{V_L(\delta)} \frac{\partial^2 V_L}{\partial \delta^2}(\delta)\bigg|_{\delta=0} = \frac{\sum_{s=t+1}^{n}(s-t)^2 L(s)P(t,s)}{\sum_{s=t+1}^{n} L(s)P(t,s)}$$

Conditions for immunization at time $t = 0$:

1	P.V. Matching	$V_A(0,0) = V_L(0,0)$
2	Duration Matching	$\tau_A = \tau_L$
3	Convexity Condition	$C_A > C_L$

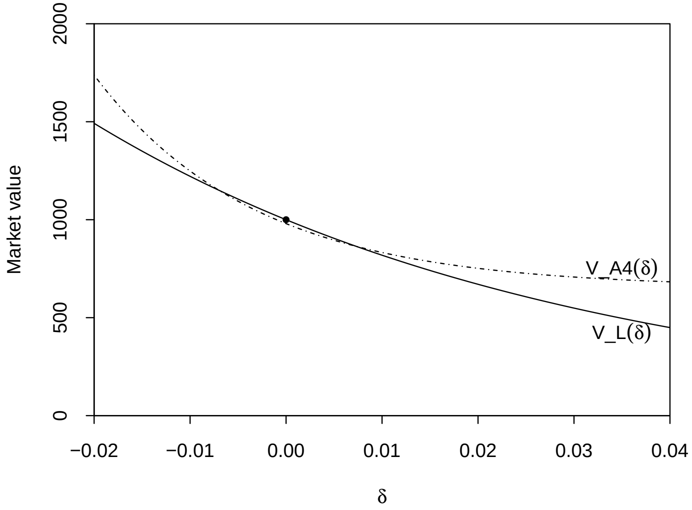
If conditions 1, 2 and 3 are satisfied then the portfolio is said to be immunized against parallel shifts in the yield curve.



BUT this looks like arbitrage. So what is happening?

Why does arbitrage not arise?

1. We get more than just parallel shifts in the yield curve
   - level; slope; curvature
2. Time dimension $\Rightarrow$ change happens between t and $t + \Delta t$ and not instantaneously at t.
   $\Rightarrow$ for arbitrage free models, if $V_A = V_L$ at t:



e.g. small change in δ $(+/-)$ $\Rightarrow$ small loss $V_A(\delta) - V_L(\delta)$

Duration matching

Duration matching is a form of *Delta Hedging*
$\Rightarrow$ hedging against small changes in specific risk factors

Summary

- Know the key alternative building blocks and terminology underpinning the term structure of interest rates
- Understand how the market value of a set of deterministic liabilities responds to changes in the term structure of interest rates
- Demonstrate how Redington's theory of immunization works
- Understand how duration matching is a form of delta hedging

Unit 10.2: Arbitrage-free stochastic models
Short-rate models

Arbitrage Free Stochastic Models

Key concept:

- $r(t)$ = instantaneous risk free rate of interest (short rate)
- $\$1$ at time t invested in a cash account
- grows to

$$\$1 + r(t)dt$$

between t and $t + dt$ where dt is very small
- Cash account $C(t)$ at time t

$$\begin{aligned} dC(t) &= r(t)C(t)dt \\ \Rightarrow C(t) &= C(0)\exp\left[\int_0^t r(s)ds\right] \end{aligned}$$

(even if $r(t)$ is stochastic)
- Seen before in courses on compound interest where $r(t)$ is deterministic and sometimes referred to as the force of interest

Arbitrage Free Stochastic Models (cont.)

Example: Vasicek Model (1977) – 1 risk factor

- $r(t)$ = instantaneous risk free rate of interest (short rate)
- Vasicek: $dr(t) = \alpha(\mu - r(t))dt + \sigma d\tilde{W}(t)$

$$
\begin{aligned}
\text{where} \quad \tilde{W}(t) &= \text{Brownian Motion under risk-neutral } Q. \\
\Rightarrow P(t,T) &= E_Q\left[e^{-\int_t^T r(s)ds} \;\middle|\; r(t) \right] \\
&= \exp[A(T-t) - B(T-t)r(t)] \\
\text{where} \quad B(s) &= \left(1 - e^{-\alpha s}\right)/\alpha \\
A(s) &= (B(s) - s)\left(\mu - \frac{\sigma^2}{2\alpha^2}\right) - \frac{\sigma^2}{4\alpha}B(s)^2
\end{aligned}
$$

General short rate models

- $X(t) = (X_1(t), \ldots, X_m(t))' = m$-factor Markov diffusion process:
  $dX(t) = a\big(t, X(t)\big)dt \; + \; b\big(t, X(t)\big)d\tilde{W}(t)$
- $r(t) = f(X(t)) =$ short rate

$$\Rightarrow P(t, T) = E_Q\left[e^{-\int_t^T r(s)ds} \;\middle|\; X(t)\right]$$

  where Q is the risk-neutral pricing measure
- Well known models: Vasicek; Cox-Ingersoll-Ross; Black-Karasinski; Hull-White (several); Ho-Lee

Vasicek again

$$
\begin{aligned}
R(t,T) &= -\frac{1}{T-t}\log P(t,T) \\
&= -\frac{A(T-t)}{(T-t)} + \frac{B(T-t)}{(T-t)} r(t)
\end{aligned}
$$

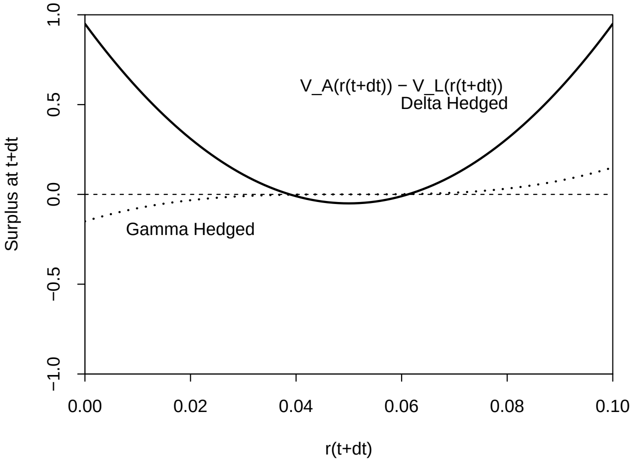
Now: $r(t)$ is the uncertain component,
so *unanticipated* changes in $r(t)$ result in
unanticipated shifts in $R(t, t+u)$ proportional to
$B(u)/u = (1 - e^{-\alpha u})/\alpha u$.

Delta hedging under Vasicek

- Redington immunization $\Rightarrow$ Delta hedge against parallel shifts in $R(t, t+u)$ $\Rightarrow$ match durations
- Vasicek Delta hedging $\Rightarrow$ Delta hedge against "level" shifts in $R(t, t+u)$ proportional to $B(u)/u = (1 - e^{-\alpha u})/\alpha u$.

At t: $V_A\Big(t, r(t)\Big) = V_L\Big(t, r(t)\Big)$

At $t + dt$:



Summary

- Understand how compounding works in a continuous-time setting
- Understand the ideas underpinning the Vasicek and other short-rate models

Unit 10.3: More general multi-factor models

More general approach

$$t \rightarrow t + \Delta t$$

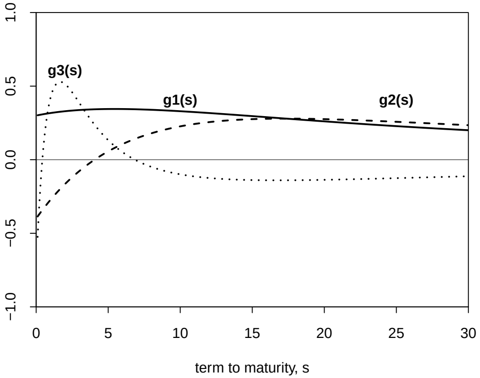
$$R(t,T) \rightarrow R(t,T) + \sum_{i=1}^{m} x_i g_i(T-t)$$

$$x_i \quad \text{uncertain, } i = 1, \ldots, m\text{: } x = (x_1, \ldots, x_m)'$$

$$g_i(T-t) \quad \text{known}$$

e.g. $m = 3$:

- $g_1(s)$ = changes in level
- $g_2(s)$ = changes in slope
- $g_3(s)$ = changes in curvature



- PV matching $\Rightarrow V_A(x)|_{x=0} = V_L(x)|_{x=0}$
- Delta hedging $\Rightarrow \partial V_A(x)/\partial x_i|_{x=0} = \partial V_L(x)/\partial x_i|_{x=0}$ for $i = 1, 2, 3$
- Convexity condition: exercise!

Heath-Jarrow-Morton model

Work with instantaneous forward rates rather than $R(t,T)$:

$$
\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T}\log P(t,T) \\
\Rightarrow P(t,T) &= \exp\left[-\int_t^T f(t,u)du\right] \\
\text{and} \quad R(t,T) &= \frac{1}{T-t}\int_t^T f(t,u)du
\end{aligned}
$$

Heath-Jarrow-Morton model

$$df(t,T) = \alpha(t,T)dt + \sum_{i=1}^{m} \sigma_i(t,T)dW_i(t)$$

- $\alpha(t,T)$ is a drift term that ensures the model is arbitrage free
- the $\sigma_i(t,T)$ are volatilities
- the $W_i(t)$ are independent, standard Brownian motions
- $\alpha(t,T)$ and the $\sigma_i(t,T)$ possibly depend on the current curve, $f(t,T)$.

Summary

- Understand how a generalised Redington-type model with multiple factors can be used to manage interest-rate risk
- Describe the Heath-Jarrow-Morton model for the forward-rate curve

Unit 7.4: Stochastic liabilities and instruments for hedging interest rate risk

Stochastic liabilities

Some liabilities are not fixed, but are sensitive to interest rates and other factors at t.

Case Study: Equitable Life and GAO's

- GAO= Guaranteed Annuity Option
- $a(t)$ = price at t for £1 per annum from age 65
  = function of $\{R(t,T) : T > t\}$ and life expectancy at t
- $X(s)$ = pension account at s
- Pension, $\pi(t) = \max\left\{\frac{X(t)}{g}\ ,\ \frac{X(t)}{a(t)}\right\}$ per annum
- Value at t of pension:

$$\pi(t)a(t) = X(t) + \frac{X(t)}{g}\max\{a(t) - g, 0\}$$

How do we manage this risk?

Need to hedge against:

- $a(t) \Rightarrow$ interest rate risk; longevity risk
- $X(t) \Rightarrow$ market risk

$\Rightarrow$ need a mixture of traded/OTC

- Equity derivatives
- Interest rate derivatives
- Longevity derivatives

OR a well-designed OTC derivative ($\Rightarrow$ complex + expensive)

Instruments for hedging interest rate risk

- Zero-coupon bonds
- Coupon bonds
- Interest-rate futures
- Interest-rate swaps
- Forward LIBOR contracts
- Simple interest-rate or bond options
- Swaptions
- Exotic options

Exchange traded or OTC?

Exchange traded:

- standardised
- simple
- highly liquid
- margin requirements
- less well suited to your own requirements

OTC:

- better tailored to your own requirements
- sometimes with some standardised components
- or more complex or non-standard
- less liquid
- counterparty credit risk (or collateral requirements)
- more expensive

Futures contracts

e.g. ICE Futures Europe – Long Gilt Future

- zero initial cost
- margin calls
  (value reset to zero $\Rightarrow$ daily profit/loss)
- cash settlement or physical delivery
- highly liquid, cheap $\Rightarrow$ hedging is not too costly
- Complications: e.g. long bond futures have option features
  - delivery of one out of a choice of three gilts
  - delivery at any time during the delivery month (seller's choice)
- price dynamics similar to underlying bond prices

LIBOR and forward LIBOR

Example: The 3-month LIBOR rate is 4.4% $\Rightarrow$

- Borrow \$ 1 now
- Repay \$ $\left(1 + \frac{0.044 \times 3}{12}\right) = \$1.011$ in 3 months

Forward LIBOR: $L(t, T, T + \tau)$

- Contract at t (zero cost)
- Borrow K at T
- Repay $K\,(1 + L(t, T, T + \tau)\tau)$ at $T + \tau$
- $\tau =$ "Tenor"

Interest rate swaps

- Fixed for floating; tenor $= \tau$
- A pays B: fixed τK at $t = \tau, 2\tau, \ldots, T$
- B pays A: floating $\tau L(t-\tau, t-\tau, t)$ at
  $t = \tau, 2\tau, \ldots, T$
  [Floating rate at t is known at $t - \tau$.]
- OTC but usually on standardised terms (ISDA)
- Choose K so that the initial value is zero.

Immunization and hedging with futures and swaps???

- Futures have their value reset to zero each day
- Swaps start with zero value

- Value $= 0 \Rightarrow$ duration $= \infty$
- Hence *duration* matching not possible
- Replace duration matching condition with

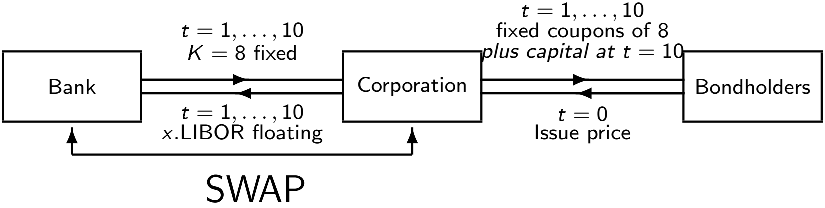
$$\left.\frac{\partial V_A}{\partial x_i}\right|_{x=0} = \left.\frac{\partial V_L}{\partial x_i}\right|_{x=0}$$

in multifactor models for $i = 1, \ldots, m$ (slides 28-29).

Case Study

A corporation has issued a 10-year bond paying a fixed coupon of 8% per annum.

It can "transform" this into "short-term debt" by arranging an interest-rate swap with a bank.



Options

- "Caplet" $\Rightarrow$ call option on LIBOR
  Payoff $= \max\{L(t,t,t+\tau) - c,\ 0\}$ at time $t+\tau$.
  $c =$ cap rate
  Premium $= C(0,t,t+\tau)$
- "Interest rate cap"
  $\Rightarrow$ collection of caplets with regular payment dates
- Floorlet $\Rightarrow \max\{c - L(t,t,t+\tau),\ 0\}$ at $t+\tau$
- Floor $=$ collection of floorlets

Case Study

A corporation has borrowed £100 M from a bank

- Repay capital in 10 years
- Quarterly interest: 3-month LIBOR plus 270 b.p.'s
- 3-month LIBOR currently 2.5%

Problem:

If quarterly interest exceeds 7% p.a. then risk of default

Solutions:

A: Swap floating for fixed

B: Pay a premium NOW to buy a 10-year cap with quarterly payments and $c = 4.3\%$ $[4.3 + (270/100) = 7.0]$

Swaption (Swap Option)

- Underlying swap:
  - Starts at T_0
  - Payments at $t = T_0 + 1, \ldots, T_1$
  - Fixed K for floating $L(t-1, t-1, t)$ at time t
- K is fixed at $t = 0$
- $K(T_0, T_1)$ = At-The-Money swap rate determined at T_0

- Option $\Rightarrow$
  the right but not the obligation to enter into the swap at T_0
- "Pay" fixed rate $\Rightarrow$ "Payer Swaption"
  Exercise a payer swaption at T_0 if $K < K(T_0, T_1)$
- "Receive" fixed rate $\Rightarrow$ "Receiver Swaption"

Hedging interest-rate-sensitive liabilities

- Frequent dynamic hedging using liquid traded derivatives
- Buy-and-hold hedges with occasional rebalancing using standard OTC instruments (e.g. swaps or swaptions)
    - E.g. Attempt to Delta and Gamma hedge by identifying standard instruments that have a similar sensitivity to interest-rate changes
- Full customised OTC hedge with no rebalancing

Summary

- Be able to formulate accurately the payoff function of an interest-rate-sensitive liability
- Understand the range of interest rate derivative contracts that could be used for hedging stochastic liabilities
- Discuss how to use these derivatives for a variety of interest-rate sensitive liabilities