Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 7 Solutions

1. (a) $\ddot{a}_{70|70} = \ddot{a}^{f}_{70} - \ddot{a}_{70:70} = 3.168.$

   (b) $\ddot{a}_{65|64} = \ddot{a}^{f}_{64} - \ddot{a}_{65:64} = 3.116$

   (c) $\ddot{a}^{(12)}_{65|64} \approx \ddot{a}^{(12)f}_{64} - \ddot{a}^{(12)}_{65:64} = (\ddot{a}^{f}_{64} - 0.458) - (\ddot{a}_{65:64} - 0.458) = 3.116$

2. We note that after the first five years for which the annuity is guaranteed, we need to consider what happens in each of the three possible cases:

   (a) both husband and wife are alive, which has a probability ${}_5p_{65} \cdot {}_5p_{61}$,
   (b) only the husband is alive, which has probability ${}_5p_{65}(1 - {}_5p_{61})$, and
   (c) only the wife is alive which has probability $(1 - {}_5p_{65}){}_5p_{61}$.

   The equation of value is

$$P = 100 + 10060\ddot{a}^{(12)}_{\overline{5}|} + v^5 \left[\begin{array}{ll} & {}_5p_{65} \cdot {}_5p_{61} \left(10060\ddot{a}^{(12)}_{70} + 5060\ddot{a}^{(12)}_{70|66} \right) \\ + & {}_5p_{65} \cdot (1 - {}_5p_{61}) \cdot 10060\ddot{a}^{(12)}_{70} \\ + & (1 - {}_5p_{65}) \cdot {}_5p_{61} \cdot 5060\ddot{a}^{(12)}_{66} \end{array} \right].$$

   Now, substituting

$$\ddot{a}^{(12)}_{\overline{5}|} = 4.5477, \quad v^5 = 0.82193, \quad {}_5p_{65} = \frac{9238134}{9647797} = 0.95754, \quad {}_5p_{61} = \frac{9658285}{9828163} = 0.98272,$$

$$\ddot{a}^{(12)}_{70} = 11.562 - \frac{11}{24} = 11.104, \quad \ddot{a}^{(12)}_{66} = 14.494 - \frac{11}{24} = 14.036,$$

   and $\ddot{a}^{(12)}_{70|66} \approx 14.494 - 10.368 = 4.126.$ gives $P = 152,350$

3. (a) ${}_nq^{2}_{xy} = {}_nq_x - {}_nq^{1}_{xy}.$

   (b) First find ${}_np_x$:

$$
\begin{aligned}
{}_np_x &= \exp\left[-\int_0^n \mu_{x+t}dt\right] \\
&= \exp\left[-\int_0^n \frac{1}{90-x-t}dt\right] \\
&= \exp[[\log(90-x-t)]_{t=0}^{t=n}] \\
&= (90-x-n)/(90-x)
\end{aligned}
$$

So ${}_nq_x = n/(90-x)$. Next:

$$
\begin{aligned}
{}_nq^1_{xy} &= \int_0^n {}_tp_{xy}\mu_{x+t}dt \\
&= \int_0^n \frac{90-x-t}{90-x}\frac{90=y-t}{90-y}\frac{1}{90-x}dt \\
&= \frac{1}{(90-x)(90-y)}\int_0^n (90-y-t)dt \\
&= \frac{(90-y)n-n^2/2}{(90-x)(90-y)}
\end{aligned}
$$

Hence ${}_{10}q^2_{30:40} = \frac{1}{60}$.

4. (a) $\bar{A}^{\,1}_{40:40} = \frac{1}{2}\bar{A}_{40:40}$ (by symmetry) $= \frac{1}{2}(1-\delta\bar{a}_{40:40}) = 0.1754$.
   (b) $A_{\overline{40:40}} = (1-d\ddot{a}_{\overline{40:40}}) = 1-d(2\ddot{a}_{40}-\ddot{a}_{40:40}) = 0.2024$
   (c) $A^{\,2}_{40:40} = \frac{1}{2}A_{\overline{40:40}}$ (by symmetry) $= 0.1012$.
   (d) $\bar{A}_{\overline{40:40}:\overline{10}|} = \bar{A}^{\;\;1}_{\overline{40:40}:\overline{10}|} + A_{\overline{40:40}:\overline{10}|}^{\qquad\;\;1}$ where the first term is a temporary assurance payable on second death, and the second term is a pure endowment. (Note that the '1' is not over either life but over the whole 'status' $\overline{40:40}$; the sum assured is payable when the status fails. The next line shows a similar symbol with a first-death status 40 : 40.) We have:

$$
\begin{aligned}
\bar{A}^{\;\;1}_{\overline{40:40}:\overline{10}|} &= 2\bar{A}^{1}_{40:\overline{10}|} - \bar{A}^{\;\;1}_{40:40:\overline{10}|} \\
&= (1+i)^{1/2}(2A^{1}_{40:\overline{10}|} - A^{\;\;1}_{40:40:\overline{10}|}) \\
&= (1+i)^{1/2}\left(2\left(A_{40}-\frac{D_{50}}{D_{40}}A_{50}\right) - (A_{40:40} - v^{10}{}_{10}p_{40:40}A_{50:50})\right) \\
&= 0.000477
\end{aligned}
$$

(using $A_{xx} = 1-d\ddot{a}_{xx}$). Then, the pure endowment benefit will be payable as long as both are not dead:

$$A_{\overline{40:40}:\overline{10}|}^{\qquad\;\;1} = v^{10}(1-{}_{10}q_{40}{}_{10}q_{40}) = 0.675107$$

so $\bar{A}_{\overline{40:40}:\overline{10}|} = 0.67558$.

5. (a) ${}_{\infty}q^{1}_{xy}$ is the probability that (x) will die before (y).

$$ {}_{\infty}q^{1}_{xy} = \int_0^\infty {}_tp_{xy}\mu_{x+t}dt. $$

(b) $\bar{A}^2_{xy}$ is the EPV of an assurance of 1 payable immediately on the death of (x), provided (y) is then dead.

$$ \bar{A}^2_{xy} = \int_0^\infty v^t {}_tp_x(1 - {}_tp_y)\mu_{x+t}dt. $$

(c) $\bar{A}^{1}_{xy:\overline{n}|}$ is the EPV of an assurance payable immediately on the death of (x) within n years, provided (y) is then alive.

$$ \bar{A}^{1}_{xy:\overline{n}|} = \int_0^n v^t {}_tp_{xy}\mu_{x+t}dt. $$

6. Let P be the annual premium. Then the equation of value is

$$ \begin{aligned} P\ddot{a}_{70:65} &= 5,000\bar{A}_{\overline{70:65}} + 10,000\left[\ddot{a}_{\overline{70:65}} - \ddot{a}_{70:65}\right] \\ &= 5,000\left[1 - \delta\bar{a}_{\overline{70:65}}\right] + 10,000\left[\ddot{a}_{70} + \ddot{a}_{65} - 2\ddot{a}_{70:65}\right] \end{aligned} $$

(Note that equivalently $P\ddot{a}_{70:65} = 5,000\bar{A}_{\overline{70:65}} + 10,000\left[\ddot{a}_{70|65} + \ddot{a}_{65|70}\right]$.)

$$ \ddot{a}_{70:65} = 10.494, \quad \bar{a}_{\overline{70:65}} \approx a_{\overline{70:65}} + 0.5 = 10.562 + 13.871 - 9.494 + 0.5 = 15.439, $$

and

$$ \ddot{a}_{70} + \ddot{a}_{65} - 2\ddot{a}_{70:65} = 11.562 + 14.871 - 2(10.494) = 5.445. $$

Substitution gives us

$$ 10.494P = 10,000(5.445) + 5,000(1 - 0.003922(15.439)). $$

Therefore $P = 5376.64$

7. Let P be the value of the single premium. The equation of value is

$$ 0.94P = 50,000\int_0^{15} v^t \cdot {}_tp_{60}\mu_{60+t} \cdot {}_tp_{50}dt $$

To use the three-eighths rule we need to evaluate the integrand at points $t = 0$, $t = 5$, $t = 10$ and $t = 15$. We set these out in the following table.

t	v^t	${}_tp_{60}$	μ_{60+t}	${}_tp_{50}$	Product
0	1	1	0.001918	1	0.001918
5	0.69655	0.9853	0.004332	0.9841	0.002926
10	0.48519	0.9537	0.009240	0.95630	0.004089
15	0.33797	0.8920	0.018414	0.9083	0.005042

Therefore we have

$$0.94P = 50,000\frac{15}{8}\left[0.001918 + 3(0.002926) + 3(0.004089) + 0.005042\right] = 50,000(0.0525)$$

such that $P = 2,792.90$.

8. Let P be the single premium. The equation of value is:

$$P = 50,000 \int_0^{10} v^t \, {}_tp_{55}^m \, {}_tp_{50}^f \, \mu_{50+t}^f \, dt.$$

The survival probabilities can also be expressed in terms of integrals, for example:

$${}_tp_{55}^m = \exp\left(-\int_0^t \mu_{55+s}^m \, ds\right).$$

The trapezium rule evaluates the integral of a function $f(x)$ by dividing the range of integration into steps of length h and approximating the function by a straight line over each step. Thus if the integral is from a to b and we use N steps of length $h = (b-a)/N$ we have:

$$\int_a^b f(t)\,dt \approx \sum_{k=1}^{N} h\,\frac{f(a+(k-1)h) + f(a+kh)}{2}.$$

Thus we can evaluate both survival probabilities and the EPV recursively. If we have computed:

$${}_{kh}p_{55}^m, \qquad {}_{kh}p_{50}^f \quad \text{and} \quad \bar{A}^{\;\;1}_{55:50:\overline{kh}|}$$

then we know the integrand in the EPV at time kh, and we have:

$$\begin{aligned}
{}_{(k+1)h}p_{55}^m &= \exp\left(-\int_0^{(k+1)h} \mu_{55+s}^m ds\right) \\
&= \exp\left(-\int_0^{kh} \mu_{55+s}^m ds\right) \exp\left(-\int_{kh}^{(k+1)h} \mu_{55+s}^m ds\right) \\
&\approx {}_{kh}p_{55}^m \times \exp\left(-h\,\frac{\mu_{55+kh}^m + \mu_{55+(k+1)h}^m}{2}\right)
\end{aligned}$$

and similarly for the other survival probability. Thus we can evaluate the integrand in the EPV at time $(k+1)h$, and another application of the trapezium rule gives us $\bar{A}^{\;\;1}_{55:50:\overline{(k+1)h}|}$. This can conveniently be set out in a few columns of a spreadsheet, see as an example:

`www.ma.hw.ac.uk/~andrea/f79af/worksheets/tut7_q7.xls`.

The answer, using step size $h = 0.01$ and the approximate (linearly interpolated) forces of mortality from `amf92_mu.xls`, is $P = £978.54$.