Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 5 Solutions

1. (a) Consider ${}_{t+dt}p_x^{00}$, and condition on the state occupied at time $t$:

$$\begin{aligned} {}_{t+dt}p_x^{00} &= {}_tp_x^{00}\,{}_{dt}p_{x+t}^{00} + {}_tp_x^{01}\,{}_{dt}p_{x+t}^{10} \\ &= {}_tp_x^{00}\,(1 - {}_tp_x^{01} - {}_tp_x^{02}) + {}_tp_x^{01}\,{}_{dt}p_{x+t}^{10} \\ &= {}_tp_x^{00}\,(1 - \mu_{x+t}^{01} - \mu_{x+t}^{02} + o(dt)) + {}_tp_x^{01}\,(\mu_{x+t}^{10} + o(dt)). \end{aligned}$$

Therefore:

$$\frac{{}_{t+dt}p_x^{00} - {}_tp_x^{00}}{dt} = {}_tp_x^{01}\,\mu_{x+t}^{10} - {}_tp_x^{00}\,(\mu_{x+t}^{01} + \mu_{x+t}^{02}) + \frac{o(dt)}{dt}$$

and on letting $t \to 0$ we have:

$$\frac{d}{dt}{}_tp_x^{00} = {}_tp_x^{01}\,\mu_{x+t}^{10} - {}_tp_x^{00}\,(\mu_{x+t}^{01} + \mu_{x+t}^{02}).$$

Similarly, we can show that:

$$\frac{d}{dt}{}_tp_x^{01} = {}_tp_x^{00}\,\mu_{x+t}^{01} - {}_tp_x^{01}\,(\mu_{x+t}^{10} + \mu_{x+t}^{12}).$$

(b) Thiele's equations are:

$$\begin{aligned} \frac{d}{dt}V^0(t) &= V^0(t)\cdot\delta + \bar{P} - \mu_{x+t}^{01}(V^1(t) - V^0(t)) - \mu_{x+t}^{02}(100 - V^0(t)) \\ \frac{d}{dt}V^1(t) &= V^1(t)\cdot\delta - 1 - \mu_{x+t}^{10}(V^0(t) - V^1(t)) - \mu_{x+t}^{12}(100 - V^1(t)) \\ \frac{d}{dt}V^2(t) &= 0. \end{aligned}$$

(c) In this case you cannot solve Thiele's equations forwards, because $V^1(0)$ is not known, even if you assumed that $V^0(0) = 0$.. However you know that $V^i(n) = 0$ for all three states, so you would solve the equations backwards from there.

2. (a) The Kolmogorov equations assuming presence in state 0 at age $x$ are:

$$\begin{aligned} \frac{\partial}{\partial t}{}_tp_x^{00} &= {}_tp_x^{01}\,\mu_{x+t}^{10} - {}_tp_x^{00}\,\mu_{x+t}^{01} \\ \frac{\partial}{\partial t}{}_tp_x^{01} &= {}_tp_x^{00}\,\mu_{x+t}^{01} - {}_tp_x^{01}\,\mu_{x+t}^{10} \end{aligned}$$

but one of these is redundant since ${}_tp_x^{01} = 1 - {}_tp_x^{00}$. There is a similar pair of equations assuming presence in state 1 at age $x$:

$$\begin{aligned}
\frac{\partial}{\partial t}{}_tp_x^{11} &= {}_tp_x^{10}\,\mu_{x+t}^{01} - {}_tp_x^{11}\,\mu_{x+t}^{10} \\
\frac{\partial}{\partial t}{}_tp_x^{10} &= {}_tp_x^{11}\,\mu_{x+t}^{10} - {}_tp_x^{10}\,\mu_{x+t}^{01}.
\end{aligned}$$

(b) An example of an Excel worksheet (`tut5_q2.xls`) for solving this problem can be downloaded from the course web page at:

`www/ma.hw.ac.uk/~andrea/f79af`.

(c) Thiele's equations with annual rate of premium $P$ are:

$$\begin{aligned}
\frac{d}{dt}V^0(t) &= V^0(t)\,\delta + P - \mu_{x+t}^{01}(V^1(t) - V^0(t)) \\
\frac{d}{dt}V^1(t) &= V^1(t)\,\delta - 1 - \mu_{x+t}^{10}(V^0(t) - V^1(t)).
\end{aligned}$$

(d) See `tut5_q2.xls`. The premium rate 0.124379 per annum gives $V^0(0) = 0.000001$.

(e) With the rate of premium in (d), the policy value $V^0(t)$ is negative for about the last four years of the term. Negative policy values are generally to be avoided. They mean that the life office is treating the policy as an asset instead of as a liability. This may turn out to be correct if the policy runs for its full term, but if (for example) the policyholder decides to cancel the policy sometime in the last four years, the negative policy value would lead the life office to make a loss.

3. (a) We can list the life histories in the form of a table:

| No. of Transitions | $(T_1, S_1)$ | $(T_2, S_2)$ | $(T_3, S_3)$ | $(T_i, S_i)$ for $i \geq 4$ |
|---|---|---|---|---|
| 0 | $(\infty, -1)$ | $(\infty, -1)$ | $(\infty, -1)$ | $(\infty, -1)$ |
| 1 | $(T_1, 1)$ | $(\infty, -1)$ | $(\infty, -1)$ | $(\infty, -1)$ |
| 2 | $(T_1, 1)$ | $(T_2, 0)$ | $(\infty, -1)$ | $(\infty, -1)$ |
| 3 | $(T_1, 1)$ | $(T_2, 0)$ | $(T_3, 1)$ | $(\infty, -1)$. |

(b) Likewise given the annual rate of premium $P$ we can tabulate the required present values.

| No. of Transitions | Present Value in Integral Form | Present Value in Annuity Form |
|---|---|---|
| 0 | $P \int_0^{10} e^{-\delta t} dt$ | $P\,\bar{a}_{\overline{10}\rceil}$ |
| 1 | $P \int_0^{T_1} e^{-\delta t} dt$ | $P\,\bar{a}_{\overline{T_1}\rceil}$ |
| 2 | $P \left( \int_0^{T_1} e^{-\delta t} dt + \int_{T_2}^{10} e^{-\delta t} dt \right)$ | $P\,(\bar{a}_{\overline{T_1}\rceil} + {}_{T_2}\vert\bar{a}_{\overline{10-T_2}\rceil})$ |
| 3 | $P \left( \int_0^{T_1} e^{-\delta t} dt + \int_{T_2}^{T_3} e^{-\delta t} dt \right)$ | $P\,(\bar{a}_{\overline{T_1}\rceil} + {}_{T_2}\vert\bar{a}_{\overline{T_3-T_2}\rceil})$. |

(c) Each term in the above table contibutes a terms to an infinite series which is the expression for the EPV of the premiums, if we formulate the model using random event times. Note that the probability that the life history has exactly $n$ transitions ($n \geq 0$), denoted $H(n)$, is:

$$H(n) = \mathrm{P}[T_1 < T_2 \ldots < T_n < 10, T_{n+1} = T_{n+2} = \ldots = \infty].$$

Hence the EPV of premium payments is:

$$\begin{aligned} \text{EPV} &= P \big( H(0)\,\bar{a}_{\overline{10}\rceil} + H(1)\,\mathrm{E}[\bar{a}_{\overline{T_1}\rceil}] \\ &\quad + H(2)\,\mathrm{E}[\bar{a}_{\overline{T_1}\rceil} + {}_{T_2}\vert\bar{a}_{\overline{10-T_2}\rceil}] + H(3)\,\mathrm{E}[\bar{a}_{\overline{T_1}\rceil} + {}_{T_2}\vert\bar{a}_{\overline{T_3-T_2}\rceil}] + \ldots \big). \end{aligned}$$

(d) The presence of the reversible transition means that the EPV of the premium payment is an infinite series. Moreover, the $n$th term of the series involves $n$ random event times so the expected value that appears in the $n$th term involves the distribution of this $n$-dimensional random variable, i.e. it will require the evaluation of an $n$-dimensional integral. Compared with solving a system of linear ODEs, this is very difficult and will quickly overpower even quite a capable computer.

4. (a) Consider the probability of staying in state 1 for time $t + dt$:

$$\begin{aligned} {}_{t+dt}p_x^{\overline{11}} &= {}_tp_x^{\overline{11}}\;{}_{dt}p_x^{11} \qquad (\text{since here } {}_{dt}p_x^{\overline{11}} = {}_{dt}p_x^{11}) \\ &= {}_tp_x^{\overline{11}}\,\left\{1 - (\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14})\,dt\right\} + \mathrm{o}(dt) \end{aligned}$$

Therefore:

$$\frac{{}_{t+dt}p_x^{\overline{11}} - {}_tp_x^{\overline{11}}}{dt} = -(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14})\,{}_tp_x^{\overline{11}} + \frac{\mathrm{o}(dt)}{dt}$$

and on letting $dt \to 0$ we get:

$$\frac{d}{dt}{}_tp_x^{\overline{11}} = -(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14})\,{}_tp_x^{\overline{11}}.$$

(b) Since:

$$\exp\left\{-\int_0^0 \mu_{x+r}^{12} + \mu_{x+r}^{13} + \mu_{x+r}^{14} dr\right\} = \exp(0) = 1$$

and:

$$\begin{aligned}
&\frac{d}{dt}\exp\left\{-\int_0^t \mu_{x+r}^{12} + \mu_{x+r}^{13} + \mu_{x+r}^{14} dr\right\} \\
= \;& -(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14}) \times \exp\left\{-\int_0^t \mu_{x+r}^{12} + \mu_{x+r}^{13} + \mu_{x+r}^{14} dr\right\} \\
= \;& -(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14})\, {}_tp_x^{\overline{11}}
\end{aligned}$$

we have found a solution as required.

5. The equation of value is:

$$\begin{aligned}
0.92\,\bar{P}\int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{11}\, dt \;=\; & \frac{\bar{B}}{2}\int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{12}\, dt \\
& + \bar{B}\int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{13}\, dt
\end{aligned}$$

Putting $\delta = 0.05$ gives the desired result.

Now:

$$\begin{aligned}
\int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{11}\, dt &= \int_0^{40}\left[\frac{100-60-t}{100-60}\right]^3 dt = \left[-\frac{1}{4}\frac{(40-t)^4}{40^3}\right]_0^{40} \\
&= (0) - (-10) = 10.
\end{aligned}$$

and:

$$\begin{aligned}
\int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{12}\, dt &= \int_0^{40} \frac{t\,(100-60-t)}{4000}\, dt \\
&= \left[\frac{1}{4000}\left(20\,t^2 - \frac{t^3}{3}\right)\right]_0^{40} \\
&= \frac{1}{4000}\left[\left(20(40^2) - \frac{40^3}{3}\right) - (0)\right] \\
&= \frac{8}{3}\ (= 2.66667)
\end{aligned}$$

and:

$$\int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{13}\, dt = \frac{16}{3} \quad \text{since} \quad {}_tp_{60}^{13} = 2 \times {}_tp_{60}^{12}$$

hence:

$$0.92\,\bar{P}\,(10) = 5,000\,\left(\frac{8}{3}\right) + 10,000\,\left(\frac{16}{3}\right)$$

Which gives: $\bar{P} = \dfrac{66,666.667}{0.92 \times 10} = 7,246.38$ to 2 d.p..

6. (a) Consider the two possible routes of getting to state 2 in time $t + dt$:

$$\begin{aligned} {}_{t+dt}p_x^{12} &= {}_tp_x^{11}\;{}_{dt}p_x^{12} + {}_tp_x^{12}\;{}_{dt}p_x^{22} \\ &= {}_tp_x^{11}\,\mu_{x+t}^{12}\,dt + {}_tp_x^{12}\,(1 - (\mu_{x+t}^{23} + \mu_{x+t}^{24})\,dt) + \mathrm{o}(dt). \end{aligned}$$

Rearranging gives:

$$\frac{{}_{t+dt}p_x^{12} - {}_tp_x^{12}}{dt} = {}_tp_x^{11}\,\mu_{x+t}^{12} - {}_tp_x^{12}\,(\mu_{x+t}^{23} + \mu_{x+t}^{24}) + \frac{\mathrm{o}(dt)}{dt}$$

and on letting $dt \to 0$ we get:

$$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11}\,\mu_{x+t}^{12} - {}_tp_x^{11}\,(\mu_{x+t}^{23} + \mu_{x+t}^{24}).$$

(b) Using Euler's method we get:

$$\begin{aligned} {}_sp_x^{12} &\approx {}_0p_x^{12} + s\,\left.\frac{d}{dt}{}_tp_x^{12}\right|_{t=0} \\ &= 0 + s\,({}_0p_x^{11}\,\mu_x^{12} - {}_0p_x^{12}\,(\mu_x^{23} + \mu_x^{24}) \\ &= s\,(\mu_x^{12} - 0) = s\,\mu_x^{12}. \end{aligned}$$

Hence, using stepsize $s = 1$, we get ${}_1p_x^{12} \approx 1 \times \mu_x^{12} = 0.025$.

(c) We can take another Euler step by starting from the result of the first step: Hence:

$$\begin{aligned} {}_{2s}p_x^{12} &\approx {}_sp_x^{12} + s\,\left.\frac{d}{dt}{}_tp_x^{12}\right|_{t=s} \\ &= s\,\mu_{x+s}^{12} + s\,\left[{}_sp_x^{11}\,\mu_{x+s}^{12} - {}_sp_x^{12}\,(\mu_{x+s}^{23} + \mu_{x+s}^{24})\right] \\ &= s\,\mu_{x+s}^{12}\,\left[1 + \mathrm{e}^{-(\mu_{x+s}^{12} + \mu_{x+s}^{14})\,t} - s\,(\mu_{x+s}^{23} + \mu_{x+s}^{24})\right]. \end{aligned}$$

So, using a stepsize of $s = 0.5$, we get:

$$\begin{aligned} {}_1p_x^{12} &\approx 0.5(0.025)\,(1 + \mathrm{e}^{-0.5\,(0.025+0.01)} - 0.5(0.05 + 0.02)) \\ &= 0.024346 \;\text{ to 6 d.p..} \end{aligned}$$

(d) Using the formula given we get:

$$
\begin{aligned}
{}_1p_x^{12} &= \frac{0.025}{0.025 + 0.01 - 0.05 - 0.02} \left(e^{-1\,(0.05+0.02)} - e^{-1\,(0.025+0.01)}\right) \\
&= 0.023723 \quad \text{to 6 d.p..}
\end{aligned}
$$

Both answers from (b) and (c) are quite close to the actual value, but both overstate it. Answer (c) is closer — as expected since it uses a smaller stepsize. To get a more accurate answer using Euler's method, use a smaller stepsize.