Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 4 Solutions

1. See notes.

2. (a) The variance of the present value of the future loss is:

$$\begin{aligned}
& \operatorname{Var}\left[v^{\min(K_{35}+1,n)} - P_{35:\overline{25|}}.\ddot{a}_{\overline{\min(K_{35}+1,n)|}}\right] \\
= & \operatorname{Var}\left[v^{\min(K_{35}+1,n)} - P_{35:\overline{25|}} - \frac{1-v^{\min(K_{35}+1,n)}}{d}\right] \\
= & \left(1+\frac{P_{35:\overline{25|}}}{d}\right)^2 \operatorname{Var}\left[v^{\min(K_{35}+1,n)}\right] \\
= & \left(1+\frac{P_{35:\overline{25|}}}{d}\right)^2 \left({}^2A_{35:\overline{25|}} - A^2_{35:\overline{25|}}\right)
\end{aligned}$$

where ${}^2A_{35:\overline{25|}}$ is calculated at rate $i^2+2i = 10.25\%$. The answer is 0.008737.

(b) The net premium, by definition, sets the EPV of the future loss at outset to zero. Let L be the total loss on all the policies sold. If N policies are sold, then by the Central Limit Theorem:

$$\frac{L}{\sqrt{0.008737N}} \longrightarrow X$$

where X is a random variable with a Normal(0,1) distribution. The 95^{th} percentile of the Normal(0,1) distribution is 1.645 so approximately:

$$\mathrm{P}\left[\frac{L}{\sqrt{0.008737N}} > 1.645\right] = \mathrm{P}\left[L > \sqrt{0.008737N}\times 1.645\right] = 0.05.$$

So the answer is $\sqrt{0.008737N}\times 1.645$. With $N = 10, 100$ or $1,000$ this is 0.48624, 1.53761 or 4.86236 respectively. Expressed as a percentage of one year's premium income, $N \times P_{35:\overline{25|}} = 0.02143N$, they are 227%, 72% and 23% respectively.

(c) The 75^{th} percentile of the Normal(0,1) distribution is 0.674 so approximately:

$$\mathrm{P}\left[\frac{L}{\sqrt{0.008737N}} > 0.674\right] = \mathrm{P}\left[L > \sqrt{0.008737N}\times 0.674\right] = 0.25.$$

So the answer is $\sqrt{0.008737N} \times 0.674$. With $N = 10, 100$ or $1,000$ this is 0.19922, 0.63000 or 1.99224 respectively. Expressed as a percentage of one year's premium income, $N \times P_{35:\overline{25}|} = 0.02143N$, they are 93%, 29% or 9%.

3. (a) From Thiele's differential equations we get:

$$\frac{d}{dt}V(t) = V(t)\,\delta + \pi - (S - V(t))\mu_{x+t} = V(t)\,\delta + V(t)\,\delta^{'} - V(t)\,\delta^{'} + \pi - (S - V(t))\mu_{x+t} \tag{1}$$

and

$$\frac{d}{dt}V^{'}(t) = V^{'}(t)\,\delta^{'} + \pi^{'} - (S - V^{'}(t))\mu_{x+t}. \tag{2}$$

Subtracting equation 1 from equation 2 gives us:

$$\begin{aligned}
\frac{d}{dt}\left(V^{'}(t) - V(t)\right) &= V^{'}(t)(\delta^{'} + \mu_{x+t}) + \pi^{'} - S\mu_{x+t} \qquad (3)\\
&\quad - \left[V(t)(\delta^{'} + \mu_{x+t}) + \pi - V(t)(\delta^{'} - \delta) - S\mu_{x+t}\right]\\
&= \left(\delta^{'} + \mu_{x+t}\right)\left(V^{'}(t) - V(t)\right) + (\pi^{'} - \pi) + V(t)(\delta^{'} - \delta)\\
&= \left(\delta^{'} + \mu_{x+t}\right)\left(V^{'}(t) - V(t)\right) + c_t \qquad (4)
\end{aligned}$$

where $c_t = (\pi^{'} - \pi) + V(t)(\delta^{'} - \delta)$ as required.
Now we note that, by the chain rule of differentiation:

$$\frac{d}{dt}\left[G(t)\left(V^{'}(t) - V(t)\right)\right] = \left(V^{'}(t) - V(t)\right)\frac{d}{dt}G(t) + G(t)\frac{d}{dt}\left(V^{'}(t) - V(t)\right) \tag{5}$$

But:

$$\begin{aligned}
\frac{d}{dt}G(t) &= \frac{d}{dt}\exp\left[-\int_0^t \left(\mu_{x+r} + \delta^{'}\right) dr\right]\\
&= \exp\left[-\int_0^t \left(\mu_{x+r} + \delta^{'}\right) dr\right]\frac{d}{dt}\left[-\int_0^t \left(\mu_{x+r} + \delta^{'}\right) dr\right]\\
&= -G(t)\left(\mu_{x+t} + \delta^{'}\right) \qquad (6)
\end{aligned}$$

Substituting equations 6 and 3 into equation 5, then:

$$\begin{aligned}
\frac{d}{dt}\left[G(t)\left(V^{'}(t)-V(t)\right)\right] &= \left(V^{'}(t)-V(t)\right)\frac{d}{dt}G(t) \\
&\quad +G(t)\frac{d}{dt}\left(V^{'}(t)-V(t)\right) \\
&= \left(V^{'}(t)-V(t)\right)\left[-G(t)\left(\mu_{x+t}+\delta^{'}\right)\right] \\
&\quad +G(t)\left[\left(\delta^{'}+\mu_{x+t}\right)\left(V^{'}(t)-V(t)\right)+c_t\right] \\
&= G(t)\,c_t.
\end{aligned}$$

Therefore

$$\frac{d}{dt}\left[G(t)\left(V^{'}(t)-V(t)\right)\right] = G(t)\,c_t \tag{7}$$

as required.

(b) Point 1: From $c_t = (\pi^{'} - \pi) + V(t)(\delta^{'} - \delta)$, since $\delta^{'} > \delta$, then we know that if $V(t)$ increases as t increases, then c_t increases as t increases.
Point 2: By its definition $G(t)$ is non-negative.
Point 3: Integrating the L.H.S. of equation 7 we have

$$\int_0^n \left\{\frac{d}{dt}\left[G(t)\left(V^{'}(t)-V(t)\right)\right]\right\}dt = G(n)\left(V^{'}(n)-V(n)\right) = 0$$

since at the boundary $V^{'}(n) = V(n) = E$. Equating this to the integral of the R.H.S. of equation 7 gives

$$\int_0^n G(t)\,c_t\,dt = 0.$$

This cannot be true according to Points 1 and 2, unless c_t changes sign from negative to positive at some time t_0 such that $0 < t_0 < n$.
We can now investigate the features of the function $G(t)\left(V^{'}(t)-V(t)\right)$.
Feature 1: Since $V^{'}(0) = V(0) = 0$, then at $t = 0 \quad G(t)\left(V^{'}(t)-V(t)\right) = 0$.

Feature 2: Since $V^{'}(n) = V(n) = E$, then at $t = n \quad G(t)\left(V^{'}(t)-V(t)\right) = 0$.

Feature 3: Since c_t is negative for $t < t_0$ then the gradient of $G(t)\left(V^{'}(t)-V(t)\right)$ which is $\frac{d}{dt}\left[G(t)\left(V^{'}(t)-V(t)\right)\right]$ is negative.

Feature 3: Since c_t is positive for $t > t_0$ then the gradient of $G(t)\left(V^{'}(t)-V(t)\right)$ which is $\frac{d}{dt}\left[G(t)\left(V^{'}(t)-V(t)\right)\right]$ is positive.

We can then plot the function $G(t)\left(V^{'}(t)-V(t)\right)$ as a function of time t which shows that it is always negative in $0 < t < n$. Therefore if $\delta^{'} > \delta$, then $V^{'}(t) < V(t)$ (Lidstone's theorem).