Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 9 Solutions

1. The formula was derived in the notes. To obtain numerical values we choose a small stepsize and use Euler's method with the boundary condition ${}_{0,s}p_x^{12} = 0$ and we get

$$ {}_{s,s}p_x^{12} = {}_{0,s}p_x^{12} + {}_0p_x^{11}\,(\sigma_x \cdot s) \cdot {}_0p_{x+s}^{\overline{22}} = \sigma_x \cdot s $$

since ${}_{0,s}p_x^{12} = 0$, ${}_0p_x^{11} = 1$ and ${}_0p_{x+s}^{\overline{22}} = 1$. Subsequently we can derive

$$ {}_{s,2s}p_x^{12} = {}_{0,2s}p_x^{12} + {}_sp_x^{11}\,(\sigma_{x+s} \cdot s) \cdot {}_0p_{x+2s}^{\overline{22}} = {}_sp_x^{11}\,(\sigma_{x+s} \cdot s)\,, $$

$$ {}_{2s,2s}p_x^{12} = {}_{s,2s}p_x^{12} + {}_0p_x^{11}\,(\sigma_x \cdot s) \cdot {}_sp_{x+s}^{\overline{22}} = {}_{s,2s}p_x^{12} + (\sigma_x \cdot s) \cdot {}_sp_{x+s}^{\overline{22}}, $$

repeating thie procedure until the required probabilities are obtained.

2. In the first integral, $x + u$ represents the age at which the life falls sick for the last time before age $x + t$. The probability that this happens at age $x + u$ and that the life remains sick until age $x + t$ is:

$$ {}_up_x^{11}.\sigma_{x+u}.du.{}_{t-u}p_{x+u}^{\overline{22}} $$

Summing (integrating) over all possible values of u gives the required integral.

In the second integral, $x+u$ represents the age at which the life fell sick for the first time before age $x + t$.

3. (a) Suppose first that $d < t + z$. Then the life cannot stay sick from age x to age $x+t$, since the duration of sickness at age $x+t$ would be $t+z$, which is greater than d. Hence, the life must recover and subsequently fall sick, at some age $x + u$, for the last time before age $x + t$. The probability of this is:

$$ {}_up_{x,z}^{21}.\sigma_{x+u}.du.{}_{t-u}p_x^{\overline{22}} $$

The possible values of u are $0 \to t$ if $d \geq t$, and $t - d \to t$ if $d < t$. Hence, the possible values are $\max(t - d, 0) \to t$. Integrating the above probability over the range of possible values of u gives the required answer.
If $d > t + z$, we need to allow for the extra probability $({}_tp_{x,z}^{\overline{22}})$ that the life remains sick from age x to age $x + t$.

(b) The formula for P is:

$$P\int_{t=0}^{65-x} v^t({}_tp_{x,z}^{21} + {}_tp_{x,z}^{\overline{22}} + {}_{d,t}p_{x,z}^{22})dt = B\int_{t=0}^{65-x} v^t\left({}_tp_{x,z}^{22} - {}_{d,t}p_{x,z}^{22} - {}_tp_{x,z}^{\overline{22}}\right)dt$$

Note that the premium is payable at time t if the life is healthy, if the life is still sick with the current sickness or if the life is sick with duration of sickness $\leq d$, which must be a new sickness since $z > d$.
The benefit is payable at time t if the life is sick with duration of sickness $> d$ as long as that sickness is not part of the current episode.

4. The expected present value of the profit is:

EPV of premiums–EPV of death benefit (from Healthy)–EPV of death benefit (from Sick)–EPV of sickness benefit

$$500\int_0^{30} e^{-\delta t}\,{}_tp_{35}^{11}\,dt - 20,000\int_0^{30} e^{-\delta t}\,{}_tp_{35}^{11}\,\mu_{35+t}\,dt - 30,000\int_0^{30} e^{-\delta t}\,{}_tp_{35}^{12}\,\nu_{35+t}\,dt - 3,000\int_0^{30} e^{-\delta t}\,{}_tp_{35}^{12}\,dt.$$

5. (a) Temporary initial selection refers to the effect on mortality rates (or other characteristic of interest), at any specified age, of a process of selection of lives from the general population which excludes those in ill-health or subject to other risks. The most important example is the selection of lives acceptable at normal assurance premium rates by the underwriting process of a life office. The difference between the mortality rates of the selected selected group and the general population may be expected to decrease as the time since selection increases, but even the 'ultimate' rates are lower than for the general population.

(b) Anti selection occurs when a life office's business contains a disproportionate share of lives subject to some risk. For example, a life office which offers the same premium rates to smokers and non-smokers (when other offices charged higher life assurance premiums for smokers) would tend to find that most of its policyholders are smokers.

(c) Class selection is the effect on mortality rates (or other characteristics of interest) of treating as one population, populations which have different mortality rates due to differences in some permanent features of the populations. An example is the difference in mortality between males and females such that two populations which have differing proportions of males and females may exhibit different mortality rates due to this difference in proportions.

6. (a) The **Crude Death Rate** is heavily influenced by mortality at older ages.

(i) OK if population structures by age and sex are reasonably stable. Beware of large scale emigration/immigration. Easy and practical.

(ii) Not likely to be suitable. Age and sex distributions in occupational groups are likely to vary significantly.

The **Standardised Death Rate** is also influenced by mortality at older ages.

(i) OK but need age specific mortality rates at each time point. Changing population structure has no effect.

(ii) Copes well with age/sex variations provided age specific rates are available for occupational groups. Use of a fixed age structure may be unrepresentative of given occupation.

The **Standardised Mortality Ratio** is heavily influenced by relative mortality at older ages.

(i) Fine but ensure standard rates are same each time.

(ii) Good except for possible problems gathering data on age distributions. Use of age structure maintains relevance.

(b) Crude Death Rate

$$= \frac{235}{37,000} = 0.00635.$$

Standardised Death Rate:

$$= \frac{1}{3,100,000}\left(960,000 \times \frac{52}{15000} + 1,400,000 \times \frac{74}{12000} + 740,000 \times \frac{109}{10000}\right) = 0.00646$$

Standardised Mortality Ratio:

$$235/\left(15,000 \times \frac{3,100}{960,000} + 12,000 \times \frac{7,500}{1,400,000} + 10,000 \times \frac{7,100}{740,000}\right) = \frac{235}{208.68} = 1.126$$

7. (a)

$$\mathrm{CDR}_{\mathrm{A}} = \frac{86,520}{3,800,000} = 0.0228, \quad \mathrm{CDR}_{\mathrm{B}} = \frac{78,360}{4,000,000} = 0.0196, \quad \mathrm{CDR}_{\mathrm{W}} = \frac{814,000}{40,000,000} = 0.0204.$$

(b) First calculate the table of mortality rates.

Mortality Rates

Age	Area A	Area B	Country
30-39	0.0035	0.0045	0.0040
40-49	0.0070	0.0090	0.0080
50-59	0.0125	0.0150	0.0140
60-69	0.0320	0.0370	0.0354
70-79	0.0700	0.0900	0.0800

$$\begin{aligned}\mathrm{SDR}_{\mathrm{A}} &= (0.0035 \times 10,000 + 0.0070 \times 10,000 + 0.0125 \times 9,000 + 0.0320 \times 7,000 + \\ &\quad 0.0700 \times 4,000)/40,000 \\ &= \frac{721.5}{40,000} = 0.0180\end{aligned}$$

$$\mathrm{SDR}_{\mathrm{B}} = \frac{889}{40,000} = 0.0222$$

(c)

$$\text{SMR}_{\text{A}} = \frac{86,520}{97,808} = 0.8846 \quad \text{and} \quad \text{SMR}_{\text{B}} = \frac{78,360}{71,720} = 1.0926.$$

8. (a) The crude death rate for Region 1 is 9,800/570,000 = 0.0172 and for Region 2 is 19,620/1,020,000 = 0.0192.
   The standardised mortality ratio for Region 1 is

   $$9,800/(0.00106 \ \times 220,000 + \cdots + 0.051 \ \times 100,000) = 1.4482$$

   and the standardised mortality ratio for Region 2 is

   $$19,620/(0.00106 \ \times 180,000 + \cdots + 0.051 \ \times 360,000) = 0.9015.$$

   (b) The indices give conflicting indications. The CDR suggests mortality is heavier in Region 2, whereas the SMR suggests it is heavier in Region 1.
   (c) The SMR gives us a comparison against a common standard. It is therefore a more reliable measure for comparisons than the CDR. Hence we would say that Region 1 experiences the heavier mortality. Note that the exposure in Region 2 is more heavily weighted towards the older ages, while the opposite is true for younger ages.

9. I would disagree with this statement. Social class impacts on income and education, and consequently on living standards. Each of these impacts on mortality.

   Income: Those on high incomes can afford to eat properly and live in decent conditions. Those on low incomes are constrained by their budget. Diet affects mortality, as does quality of living - clean, warm, dry, etc. Population density may increase exposure to infectious diseases.

   Education: Those who are better educated tend to have professional or white collar employment, compared with blue collar and manual employment. Mortality rates are lower for the former. Education also affects personal behaviour. In general, those with a higher level of education heed health warnings on issues like smoking, drinking and safe sex.

10. The order is not surprising. There are links between mortality, occupation, education and income.

    Those in the first three groups will be tertiary educated, well paid (teachers - maybe not as much compared to the others), working in a safe environment, likely to live in good housing with the means to afford a balanced diet.

    Ministers of religion are also educated people, but not well paid, and subject to stress. They have, however, a reasonably good working environment.

    The last three groups will all be subject to some hazard caused by their occupation, e.g. dangerous chemicals, working at heights, operating machinery. They will be on low incomes, affecting housing and diet. On the positive side for these groups, manual work may lead to an improved level of fitness.

Foremen are in supervisory role, so would not be exposed to the same risks as those in the last three groups and hence we would expect them to have lighter mortality. Also, their incomes will generally be higher than for those in the last three groups.

11.

$$\begin{aligned} P_t &= \frac{e^{0.05t}}{c_1 + c_2 e^{0.05t}} \\ P_0 &= 100,000 \text{ so } c_1 + c_2 = 10^{-5} \\ P_\infty &= 250,000 \text{ so } c_2 = 4 \times 10^{-6} \end{aligned}$$

Hence, $c_1 = 6 \times 10^{-6}$ and

$$P_{10} = \frac{e^{0.5}}{6 \times 10^{-6} + 4 \times 10^{-6} \times e^{0.5}} = 130,904$$

12. Suppose we know the population of Edinburgh and the Lothians, by age and sex, as at the beginning of 2001.

    We need a baseline population. Part of this will consist of the female population of the area who are between 11 and 46. This is based on the assumption of zero fertility before age 15 and after age 50. (Age 46 at the start of projection period implies could give birth after 4 years, and the child could survive to be of school age at the beginning of 2010, and a similar argument for age 11.)

    The second part of the baseline population consists of everyone in the population aged up to 3 last birthday (who will then be greater than 5 (actually, 9) and less than 12 at the beginning of 2010).

    We need to make assumptions about female mortality and fertility, and male mortality. We also need to make assumptions about net migration into the area (and possibly the mortality and fertility of these migrants, although it would probably be reasonable to assume the net number of immigrants is sufficiently small that separate assumptions are not warranted; negative net migration is possible!). We would also make assumptions on trends in mortality, fertility and migration. OPCS (Office of Population, Censuses and Surveys) data would be the best guide for all three, including mortality improvement factors.

    An additional assumption to be made is the proportion of primary school age children in the area who will be in state-controlled education as opposed to private education. This could vary by age.