Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 7 Solutions

1. **Off-period** is the period which must be spent without sickness claims for later sickness to be considered separate from earlier spells for purpose of calculating sickness benefit. This is necessary if the sickness benefit falls after a certain period of sickness.

   Waiting period is the period after joining a sickness benefit scheme during which no sickness benefit is allowed.

   Deferred period is the period of sickness which must elapse before sickness benefit is payable.

2. (a) Consider the probability of staying in state 1 for time $t + dt$:

$$\begin{aligned} {}_{t+dt}p_x^{\overline{11}} &= {}_tp_x^{\overline{11}} \; {}_{dt}p_{x+t}^{11} \\ &= {}_tp_x^{\overline{11}} \left\{1 - (\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14})\, dt\right\} + \mathrm{o}(dt) \end{aligned}$$

Therefore:

$$\frac{{}_{t+dt}p_x^{\overline{11}} - {}_tp_x^{\overline{11}}}{dt} = -(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14})\, {}_tp_x^{\overline{11}} + \frac{\mathrm{o}(dt)}{dt}$$

and on letting $dt \to 0$ we get:

$$\frac{d}{dt}{}_tp_x^{\overline{11}} = -(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14})\, {}_tp_x^{\overline{11}}$$

(b)

$$\begin{aligned} {}_0p_x^{\overline{11}} &= \exp\left\{-\int_0^0 \mu_{x+r}^{12} + \mu_{x+r}^{13} + \mu_{x+r}^{14} dr\right\} \\ &= \exp(0) = 1 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}{}_tp_x^{\overline{11}} &= \frac{d}{dt}\exp\left\{-\int_0^t \mu_{x+r}^{12} + \mu_{x+r}^{13} + \mu_{x+r}^{14} dr\right\} \\ &= -(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14}) \times \exp\left\{-\int_0^t \mu_{x+r}^{12} + \mu_{x+r}^{13} + \mu_{x+r}^{14} dr\right\} \\ &= -(\mu_{x+t}^{12} + \mu_{x+t}^{13} + \mu_{x+t}^{14})\, {}_tp_x^{\overline{11}} \end{aligned}$$

(c) Since there are no recoveries back to state 1, we have that:

$$ {}_tp_x^{11} = {}_tp_x^{\overline{11}} = \exp\left\{-\int_0^t \mu_{x+r}^{12} + \mu_{x+r}^{13} + \mu_{x+r}^{14} dr\right\} $$

3. (a) The equation of value is:

$$ \begin{aligned} 0.92\,\bar{P} \int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{11}\, dt &= \frac{\bar{B}}{2} \int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{12}\, dt \\ &\quad + \bar{B} \int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{13}\, dt \end{aligned} $$

Putting $\delta = 0.05$ gives the desired result.

(b) Now:

$$ \begin{aligned} \int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{11}\, dt &= \int_0^{40} \left[\frac{100-60-t}{100-60}\right]^3 dt &&= \left[-\frac{1}{4}\frac{(40-t)^4}{40^3}\right]_0^{40} \\ &= (0) - (-10) &&= 10 \end{aligned} $$

and:

$$ \begin{aligned} \int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{12}\, dt &= \int_0^{40} \frac{t\,(100-60-t)}{4000}\, dt \\ &= \left[\frac{1}{4000}\left(20\,t^2 - \frac{t^3}{3}\right)\right]_0^{40} \\ &= \frac{1}{4000}\left[\left(20(40^2) - \frac{40^3}{3}\right) - (0)\right] \\ &= \frac{8}{3}\ (= 2.66667) \end{aligned} $$

and:

$$ \int_0^{40} e^{-\delta t}\, e^{0.05\,t}\, {}_tp_{60}^{13}\, dt = \frac{16}{3} \quad \text{since} \quad {}_tp_{60}^{13} = 2 \times {}_tp_{60}^{12} $$

hence:

$$ 0.92\,\bar{P}\,(10) = 5{,}000\left(\frac{8}{3}\right) + 10{,}000\left(\frac{16}{3}\right) $$

Which gives: $\bar{P} = \frac{66{,}666.667}{0.92\times 10} = 7{,}246.38$ to 2 d.p.

4. (a) Consider the two routes of getting to state 2 in time $t + dt$:

$$ \begin{aligned} {}_{t+dt}p_x^{12} &= {}_tp_x^{11}\, {}_{dt}p_{x+t}^{12} + {}_tp_x^{12}\, {}_{dt}p_{x+t}^{22} \\ &= {}_tp_x^{11}\, \mu_{x+t}^{12}\, dt + {}_tp_x^{12}\,(1 - (\mu_{x+t}^{23} + \mu_{x+t}^{24})\, dt) + o(dt) \end{aligned} $$

Rearranging:

$$\frac{{}_{t+dt}p_x^{12} - {}_tp_x^{12}}{dt} = {}_tp_x^{11}\,\mu_{x+t}^{12} - {}_tp_x^{12}\,(\mu_{x+t}^{23} + \mu_{x+t}^{24}) + \frac{\mathrm{o}(dt)}{dt}$$

and on letting $dt \to 0$ we get:

$$\frac{d}{dt}{}_tp_x^{12} = {}_tp_x^{11}\,\mu_{x+t}^{12} - {}_tp_x^{11}\,(\mu_{x+t}^{23} + \mu_{x+t}^{24})$$

(b) Using Euler's method we get:

$$\begin{aligned} {}_sp_x^{12} &\approx {}_0p_x^{12} + s\left.\frac{d}{dt}{}_tp_x^{12}\right|_{t=0} \\ &= 0 + s\,({}_0p_x^{11}\,\mu_x^{12} - {}_0p_x^{12}\,(\mu_x^{23} + \mu_x^{24}) \\ &= s\,(\mu_x^{12} - 0) = s\,\mu_x^{12} \end{aligned}$$

Hence, ${}_1p_x^{12} \approx 1 \times \mu_x^{12} = 0.025$

(c) We can take another Euler step by using the boundary condition from the previous step: ${}_sp_x^{12} = s\,\mu_x^{12}$. Hence:

$$\begin{aligned} {}_{2s}p_x^{12} &\approx {}_sp_x^{12} + s\left.\frac{d}{dt}{}_tp_x^{12}\right|_{t=s} \\ &= s\,\mu_{x+s}^{12} + s\,\left[{}_sp_x^{11}\,\mu_{x+s}^{12} - {}_sp_x^{12}\,(\mu_{x+s}^{23} + \mu_{x+s}^{24})\right] \\ &= s\,\mu_{x+s}^{12}\,\left[1 + \mathrm{e}^{-(\mu_{x+s}^{12}+\mu_{x+s}^{14})\,t} - s\,(\mu_{x+s}^{23} + \mu_{x+s}^{24})\right] \end{aligned}$$

Hence, using a stepsize of $s = 0.5$, we get:

$$\begin{aligned} {}_1p_x^{12} &\approx 0.5(0.025)\,(1 + \mathrm{e}^{-0.5\,(0.025+0.01)} - 0.5(0.05 + 0.02)) \\ &= 0.024346 \quad \text{to 6 d.p.} \end{aligned}$$

(d) Using the formula given we get:

$$\begin{aligned} {}_1p_x^{12} &= \frac{0.025}{0.025 + 0.01 - 0.05 - 0.02}\,(\mathrm{e}^{-1\,(0.05+0.02)} - \mathrm{e}^{-1\,(0.025+0.01)}) \\ &= 0.023723 \quad \text{to 6 d.p.} \end{aligned}$$

Both answers from (b) and (c) are quite close to the actual value, but both overstate it. Answer (c) is closer — as expected since it uses a smaller stepsize. (or other sensible comment)

To get a more accurate answer using Euler's method, use a smaller stepsize.