Heriot-Watt University

M.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 6 Solutions

1. See notes.

2. (a) The variance of the present value of the future loss is:

$$
\begin{aligned}
& \operatorname{Var}\left[v^{\min(K_{35}+1,n)} - P_{35:\overline{25|}}.\ddot{a}_{\overline{\min(K_{35}+1,n)|}}\right] \\
= & \operatorname{Var}\left[v^{\min(K_{35}+1,n)} - P_{35:\overline{25|}} - \frac{1-v^{\min(K_{35}+1,n)}}{d}\right] \\
= & \left(1+\frac{P_{35:\overline{25|}}}{d}\right)^2 \operatorname{Var}\left[v^{\min(K_{35}+1,n)}\right] \\
= & \left(1+\frac{P_{35:\overline{25|}}}{d}\right)^2 \left({}^2A_{35:\overline{25|}} - A^2_{35:\overline{25|}}\right)
\end{aligned}
$$

where ${}^2A_{35:\overline{25|}}$ is calculated at rate $i^2 + 2i = 10.25\%$. The answer is 0.008737.

(b) The net premium, by definition, sets the EPV of the future loss at outset to zero. Let L be the total loss on all the policies sold. If N policies are sold, then by the Central Limit Theorem:

$$\frac{L}{\sqrt{0.008737N}} \longrightarrow X$$

where X is a random variable with a Normal(0,1) distribution. The 95^{th} percentile of the Normal(0,1) distribution is 1.645 so approximately:

$$\mathrm{P}\left[\frac{L}{\sqrt{0.008737N}} > 1.645\right] = \mathrm{P}\left[L > \sqrt{0.008737N} \times 1.645\right] = 0.05.$$

So the answer is $\sqrt{0.008737N} \times 1.645$. With $N = 10, 100$ or $1,000$ this is 0.48624, 1.53761 or 4.86236 respectively. Expressed as a percentage of one year's premium income, $N \times P_{35:\overline{25|}} = 0.02143N$, they are 227%, 72% and 23% respectively.

(c) The 75^{th} percentile of the Normal(0,1) distribution is 0.674 so approximately:

$$\mathrm{P}\left[\frac{L}{\sqrt{0.008737N}} > 0.674\right] = \mathrm{P}\left[L > \sqrt{0.008737N} \times 0.674\right] = 0.25.$$

So the answer is $\sqrt{0.008737N} \times 0.674$. With $N = 10, 100$ or $1,000$ this is 0.19922, 0.63000 or 1.99224 respectively. Expressed as a percentage of one year's premium income, $N \times P_{35:\overline{25|}} = 0.02143N$, they are 93%, 29% or 9%.

3. (a) The office should ascertain the state of health of the twin brother and whether he is likely to engage in risky pursuits in the fairly near future since his death (before that of the other twin's death) would render the policy worthless.
   (b) What is required is a net premium reserve. The reserve is

$$
\begin{aligned}
{}_{10}V &= 300,000\bar{A}^{\;1}_{60:60} - P\ddot{a}_{60:60} \\
&= 300,000\left[\bar{A}^{\;1}_{60:60} - \frac{\bar{A}^{\;1}_{50:50}}{\ddot{a}_{50:50}}\ddot{a}_{60:60}\right]
\end{aligned}
$$

Using

$$\bar{A}^{\;1}_{60:60} = 0.5\bar{A}_{60:60} \approx 0.5\left(1+i\right)^{0.5}(1 - d\ddot{a}_{60:60}\right) = 0.31490$$

and

$$\bar{A}^{\;1}_{50:50} = 0.5\bar{A}_{50:50} \approx 0.5\left(1+i\right)^{0.5}(1 - d\ddot{a}_{50:50}\right) = 0.24187,$$

we have ${}_{10}V = 41,682$. Therefore the surrender value is 39,598.

4. (a) From Thiele's differential equations we get

$$\frac{d}{dt}{}_tV = {}_tV\cdot\delta + \pi - (S - {}_tV)\mu_{x+t} = {}_tV\cdot\delta + {}_tV\cdot\delta^{'} - {}_tV\cdot\delta^{'} + \pi - (S - {}_tV)\mu_{x+t} \quad (1)$$

and

$$\frac{d}{dt}{}_tV^{'} = {}_tV^{'}\cdot\delta^{'} + \pi^{'} - (S - {}_tV^{'})\mu_{x+t}. \quad (2)$$

Subtracting equation 1 from equation 2 gives us

$$
\begin{aligned}
\frac{d}{dt}\left({}_tV^{'} - {}_tV\right) &= {}_tV^{'}(\delta^{'} + \mu_{x+t}) + \pi^{'} - S\mu_{x+t} - \left[{}_tV(\delta^{'} + \mu_{x+t}) + \pi - {}_tV(\delta^{'} - \delta) - S\mu_{x+t}\right] \\
&= \left(\delta^{'} + \mu_{x+t}\right)\left({}_tV^{'} - {}_tV\right) + (\pi^{'} - \pi) + {}_tV(\delta^{'} - \delta) \\
&= \left(\delta^{'} + \mu_{x+t}\right)\left({}_tV^{'} - {}_tV\right) + c_t \quad (3)
\end{aligned}
$$

where $c_t = (\pi^{'} - \pi) + {}_tV(\delta^{'} - \delta)$ as required.
Now we note that, by the chain rule of differentiation

$$\frac{d}{dt}\left[G(t)\left({}_tV^{'} - {}_tV\right)\right] = \left({}_tV^{'} - {}_tV\right)\frac{d}{dt}G(t) + G(t)\frac{d}{dt}\left({}_tV^{'} - {}_tV\right) \quad (4)$$

But

$$\begin{aligned}\frac{d}{dt}G(t) &= \frac{d}{dt}\exp\left[-\int_0^t \left(\mu_{x+r}+\delta'\right)dr\right] \\ &= \exp\left[-\int_0^t \left(\mu_{x+r}+\delta'\right)dr\right]\cdot\frac{d}{dt}\left[-\int_0^t \left(\mu_{x+r}+\delta'\right)dr\right] \\ &= -G(t)\left(\mu_{x+t}+\delta'\right) \qquad (5)\end{aligned}$$

Substituting equations 5 and 3 into equation 4, then

$$\begin{aligned}\frac{d}{dt}\left[G(t)\left({}_tV'-{}_tV\right)\right] &= \left({}_tV'-{}_tV\right)\frac{d}{dt}G(t)+G(t)\frac{d}{dt}\left({}_tV'-{}_tV\right) \\ &= \left({}_tV'-{}_tV\right)\left[-G(t)\left(\mu_{x+t}+\delta'\right)\right]+G(t)\left[\left(\delta'+\mu_{x+t}\right)\left({}_tV'-{}_tV\right)+c_t\right] \\ &= G(t)c_t\end{aligned}$$

Therefore

$$\frac{d}{dt}\left[G(t)\left({}_tV'-{}_tV\right)\right]=G(t)c_t \qquad (6)$$

as required.

(b) Point 1: From $c_t=(\pi'-\pi)+{}_tV(\delta'-\delta)$, since $\delta'>\delta$, then we know that if ${}_tV$ increases as t increases, the c_t increases as t increases.
Point 2: By its definition $G(t)$ is non-negative.
Point 3: Integrating the L.H.S. of equation 6 we have

$$\int_0^n\left\{\frac{d}{dt}\left[G(t)\left({}_tV'-{}_tV\right)\right]\right\}dt=G(n)\left({}_nV'-{}_nV\right)=0$$

since at the boundary ${}_nV'={}_nV=E$. Equating this to the integral of the R.H.S. of equation 6 gives

$$\int_0^n G(t)c_t=0.$$

This can not be true according to Points 1 and 2, unless c_t changes sign from negative to positive at some time t_0 such that $0<t_0<n$.

We can now investigate the features of the function $G(t)\left({}_tV'-{}_tV\right)$.
Feature 1: Since ${}_0V'={}_0V=0$, then at $t=0$ $\quad G(t)\left({}_tV'-{}_tV\right)=0$.

Feature 2: Since ${}_nV'={}_nV=E$, then at $t=n$ $\quad G(t)\left({}_tV'-{}_tV\right)=0$

Feature 3: Since c_t is negative for $t < t_0$ then the gradient of $G(t)\left({}_tV^{'} - {}_tV\right)$ which is $\frac{d}{dt}\left[G(t)\left({}_tV^{'} - {}_tV\right)\right]$ is negative.

Feature 3: Since c_t is positive for $t > t_0$ then the gradient of $G(t)\left({}_tV^{'} - {}_tV\right)$ which is $\frac{d}{dt}\left[G(t)\left({}_tV^{'} - {}_tV\right)\right]$ is positive.

We can then plot the function $G(t)\left({}_tV^{'} - {}_tV\right)$ as a function of time t which shows that it is always negative in $0 < t < n$. Therefore if $\delta^{'} > \delta$, then ${}_tV^{'} < {}_tV$. (Lidstone's theorem)!!!