Heriot-Watt University

<.Sc. in Actuarial Science

Life Insurance Mathematics I

Tutorial 2 Solutions

1. (a) ${}_{10}p_{30:40}$ is the probability that (30) and (40) both survive 10 years: ${}_{10}p_{30} \cdot {}_{10}p_{40} = 0.97853$.

   (b) $q_{30:40}$ is the probability that one or both of (30) and (40) die within one year: $1 - p_{30:40} = 0.001526$

   (c) $\mu_{40:50}$ multiplied by a small time element dt is interpreted as the probability that (40) or (50) or both die within time dt: $\mu_{40} + \mu_{50} = 0.003274$

   (d) ${}_{10}p_{[30]:[40]}$ is as for (a) but on a select basis: ${}_{10}p_{[30]} \cdot {}_{10}p_{[40]} = 0.97887$

   (e) $q_{[30]:[40]}$ is as for (b) but on a select basis: $1 - p_{[30]:[40]} = 0.001264$

   (f) $\mu_{[40]:[50]}$ is as for (c) but on a select basis: $\mu_{[40]} + \mu_{[50]} = 0.002293$

   (g) $\mu_{[40]+1:[60]+1}$ as for (f) but on select basis for (41) and (61) both with select duration 1: $\mu_{[40]+1:[60]+1} = 0.008129$.

   (h) ${}_{3|}q_{[30]+1:[40]+1}$ is the probability that one or both of (31) and (41), each with select duration of 1, will die within one year deferred for three years: ${}_{3|}q_{[30]+1:[40]+1} = 0.001976$

2. (a) The CDF of T_{max}, $P(T_{max} \le t)$, denoted ${}_tq_{\overline{xy}}$, can be given as ${}_tq_x{}_tq_y$. The density is therefore

$$\begin{aligned} f_{\overline{xy}}(t) &= \frac{d}{dt}{}_tq_{\overline{xy}} = \frac{d}{dt}{}_tq_x{}_tq_y = \frac{d}{dt}\left(1 - {}_tp_x - {}_tp_y + {}_tp_x{}_tp_y\right) \\ &= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_x{}_tp_y\left(\mu_{x+t} + \mu_{y+t}\right) \\ &= {}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t} \end{aligned}$$

   (b) The density of T_{min} is ${}_tp_{xy}\mu_{x+t:y+t}$. Therefore its the expected value is given by $\mathrm{E}[T_{min}] = \int_{t=0}^{\infty} t \cdot {}_tp_{xy}\mu_{x+t:y+t}dt$. Applying integration by parts, we let $u = t$ such that $\frac{du}{dt} = 1$ and we let $\frac{dv}{dt} = {}_tp_{xy}\mu_{x+t:y+t}$ such that $v = -{}_tp_{xy}$.

$$\mathrm{E}[T_{min}] = -t \cdot {}_tp_{xy}\Big|_{t=0}^{t=\infty} - \int_{t=0}^{\infty} -{}_tp_{xy}dt = \int_{t=0}^{\infty} {}_tp_{xy}dt.$$

(c)

$$\begin{aligned}
\mathrm{COV}(T_{min}, T_{max}) &= \mathrm{E}[T_{min}T_{max}] - \mathrm{E}[T_{min}] \cdot \mathrm{E}[T_{max}] \\
&= \mathrm{E}[T_x] \cdot \mathrm{E}[T_y] - \mathring{e}_{xy}\left(\mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}\right) \\
&= \mathring{e}_x\mathring{e}_y - \mathring{e}_{xy}\left(\mathring{e}_x + \mathring{e}_y - \mathring{e}_{xy}\right) = \left(\mathring{e}_x - \mathring{e}_{xy}\right)\left(\mathring{e}_y - \mathring{e}_{xy}\right).
\end{aligned}$$

(d) i. The density of K_{min} is ${}_{t|}q_{xy}$ such that

$$\begin{aligned}
\mathrm{E}[K_{min}] &= \sum_{k=0}^{k=\infty} k \cdot {}_{k|}q_{xy} = \sum_{k=0}^{k=\infty} k\left({}_kp_{xy} - {}_{k+1}p_{xy}\right) \\
&= 0\left({}_0p_{xy} - {}_1p_{xy}\right) + 1\left({}_1p_{xy} - {}_2p_{xy}\right) + 2\left({}_2p_{xy} - {}_3p_{xy}\right) + \cdots \\
&= {}_1p_{xy} + {}_2p_{xy} + {}_3p_{xy} + \cdots = \sum_{k=1}^{k=\infty} {}_kp_{xy}.
\end{aligned}$$

ii.

$$\begin{aligned}
\mathring{e}_{xy} &= \int_{t=0}^{\infty} {}_tp_{xy}dt = \int_{t=0}^{1} {}_tp_{xy}dt + \int_{t=1}^{2} {}_tp_{xy}dt + \int_{t=2}^{3} {}_tp_{xy}dt + \cdots \\
&\approx 0.5({}_0p_{xy} + {}_1p_{xy}) + 0.5({}_1p_{xy} + {}_2p_{xy}) + 0.5({}_2p_{xy} + {}_3p_{xy}) + \cdots \\
&= 0.5 + {}_1p_{xy} + {}_2p_{xy} + {}_2p_{xy} + \cdots = 0.5 + \sum_{k=1}^{k=\infty} {}_kp_{xy} = 0.5 + e_{xy}.
\end{aligned}$$

(e) The 'force of mortality' associated with T_{max} can be defined as

$$\mu_{\overline{x+t:y+t}} = \frac{f_{\overline{xy}}}{1 - F_{\overline{xy}}} = \frac{{}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}}{{}_tp_{\overline{xy}}}.$$

This way of defining a force is valid for any continuous random variable defining the time to a future event. For $t = 0$ we have $\mu_{\overline{x:y}} = 0$. This is surprising at first sight. However it is correct; the force $\mu_{\overline{x+t:y+t}}$ is defined in respect of the random variable $T_{max} = \max[T_x, T_y]$. If we choose another pair of ages $x' = x+s$ and $y' = y+s$ say, the force $\mu'_{\overline{x'+t:y'+t}}$ defined in respect of the random variable $T'_{max} = \max[T_{x'}, T_{y'}]$ is *not* the same as $\mu_{\overline{x+s+t:y+s+t}} = \mu_{\overline{x'+t:y'+t}}$. For the single life case, the random variables T_x and $T_{x'} = T_{x+s}$ were related by $\mathrm{P}[T_{x'} \leq t] = \mathrm{P}[T_x \leq s + t | T_x > s]$, and this was required in the proof that the forces defined in respect of T_x and $T_{x'}$ were equal at equal ages, namely $\mu'_{x'+t} = \mu_{x+s+t}$. There is no such relationship between $T_{max} = \max[T_x, T_y]$ and $T'_{max} = \max[T_{x'}, T_{y'}]$.

3. We have ${}_{10}p_x = \frac{l_{x+10:y}}{l_{x:y}} = 0.96$ and ${}_{10}p_y = \frac{l_{x:y+10}}{l_{x:y}} = 0.92$. Therefore the required probability is

$${}_{10}p_x\left(1 - {}_{10}p_y\right) + {}_{10}p_y\left(1 - {}_{10}p_x\right) = 0.1136.$$

4. (a) We note that this is the expected value of the random variable $\ddot{a}_{\overline{K_{min}+1}|}$. Therefore

$$\begin{aligned}
\ddot{a}_{xy} &= \sum_{k=0}^{\infty} \ddot{a}_{\overline{k+1}|} {}_{k|}q_{xy} = \sum_{k=0}^{\infty} \frac{1-v^{k+1}}{d} \left({}_kp_{xy} - {}_{k+1}p_{xy}\right) \\
&= \frac{1}{d} \sum_{k=0}^{\infty} \left({}_kp_{xy} - {}_{k+1}p_{xy} - v^{k+1}{}_kp_{xy} + v^{k+1}{}_{k+1}p_{xy}\right) \\
&= \frac{1}{d} \left(\sum_{k=0}^{\infty} {}_kp_{xy} - \sum_{k=0}^{\infty} {}_{k+1}p_{xy} - v \sum_{k=0}^{\infty} v^k {}_kp_{xy} + \sum_{k=0}^{\infty} v^{k+1}{}_{k+1}p_{xy} \right)
\end{aligned}$$

But $\sum_{k=0}^{\infty} {}_{k+1}p_{xy} = \sum_{k=0}^{\infty} {}_kp_{xy} - 1$ and $\sum_{k=0}^{\infty} v^{k+1}{}_{k+1}p_{xy} = \sum_{k=0}^{\infty} v^k{}_kp_{xy} - 1$. Substituting gives

$$\ddot{a}_{xy} = \frac{1}{d}\left((1-v)\sum_{k=0}^{\infty} v^k{}_kp_{xy}\right) = \sum_{k=0}^{\infty} v^k{}_kp_{xy} \quad \text{since} \quad d = 1-v.$$

(b) $\ddot{a}_{xy:\overline{n}|}$ is the expected value of the random variable $\ddot{a}_{\overline{\min(K_{min}+1,n)}|}$.

$$\begin{aligned}
\ddot{a}_{xy:\overline{n}|} &= \sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_{xy} + {}_{n-1}p_{xy} \cdot \ddot{a}_{\overline{n}|} = \sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|} \left[{}_kp_{xy} - {}_{k+1}p_{xy}\right] + {}_{n-1}p_{xy} \cdot \ddot{a}_{\overline{n}|} \\
&= \sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|} \left[\left({}_kp_x + {}_kp_y - {}_kp_{\overline{xy}}\right) - \left({}_{k+1}p_x + {}_{k+1}p_y - {}_{k+1}p_{\overline{xy}}\right)\right] \\
&\qquad + \left({}_{n-1}p_x + {}_{n-1}p_y - {}_{n-1}p_{\overline{xy}}\right) \cdot \ddot{a}_{\overline{n}|} \\
&= \sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|} \left[\left({}_kp_x - {}_{k+1}p_x\right) + \left({}_kp_y - {}_{k+1}p_y\right) - \left({}_kp_{\overline{xy}} - {}_{k+1}p_{\overline{xy}}\right)\right] \\
&\qquad + \left({}_{n-1}p_x + {}_{n-1}p_y - {}_{n-1}p_{\overline{xy}}\right) \cdot \ddot{a}_{\overline{n}|} \\
&= \left(\sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_x + {}_{n-1}p_x \cdot \ddot{a}_{\overline{n}|}\right) + \left(\sum_{k=0}^{n-2} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_y + {}_{n-1}p_y \cdot \ddot{a}_{\overline{n}|}\right) \\
&\qquad - \left(\sum_{k=0}^{n} \ddot{a}_{\overline{k+1}|} \cdot {}_{k|}q_{\overline{xy}} + {}_{n-1}p_{\overline{xy}} \cdot \ddot{a}_{\overline{n}|}\right) = \ddot{a}_{x:\overline{n}|} + \ddot{a}_{y:\overline{n}|} - \ddot{a}_{\overline{xy}:\overline{n}|}
\end{aligned}$$

(c) $A_{\overline{xy}}$ is the expected value of the random variable $v^{K_{max}+1}$.

$$\begin{aligned}
A_{\overline{xy}} &= 1 - d\ddot{a}_{\overline{xy}} = 1 - d\left(\ddot{a}_x + \ddot{a}_y - \ddot{a}_{xy}\right) \\
&= (1 - d\ddot{a}_x) + (1 - d\ddot{a}_y) - (1 - d\ddot{a}_{xy}) = A_x + A_y - A_{xy}.
\end{aligned}$$

5. We note that the expected value of the random variable $v^{K_{max}+1}$ is

$$A_{\overline{xy}} = \sum_{k=0}^{\infty} v^{k+1} \cdot {}_{k|}q_{\overline{xy}} \quad \text{where} \quad v = \frac{1}{1+i}.$$

The variance of $v^{K_{max}+1}$ is given by

$$\begin{aligned} \mathrm{Var}[v^{K_{max}+1}] &= \mathrm{E}\left[\left(v^{K_{max}+1}\right)^2\right] - \left(\mathrm{E}\left[v^{K_{max}+1}\right]\right)^2 \\ &= \sum_{k=0}^{\infty}\left(v^{k+1}\right)^2 \cdot {}_{k|}q_{\overline{xy}} - (A_{\overline{xy}})^2 = \sum_{k=0}^{\infty}\left(v^2\right)^{k+1} \cdot {}_{k|}q_{\overline{xy}} - (A_{\overline{xy}})^2 . \end{aligned}$$

For a rate of interest j we define $V = \frac{1}{1+j}$ and let $V = v^2$. This means that $j = i^2 + 2i$. Substituting in the above we get

$$\mathrm{Var}[v^{K_{max}+1}] = \sum_{k=0}^{\infty} V^{k+1} \cdot {}_{k|}q_{\overline{xy}} - (A_{\overline{xy}})^2_{(@\ i)} = A_{\overline{xy}_{@\ j=i^2+2i}} - (A_{\overline{xy}})^2_{(@\ i)} .$$

6. (a) $\ddot{a}_{70:67} = 10.233$ (from tables).
   (b) $\ddot{a}^{(12)}_{70:67} \approx \ddot{a}_{70:67} - 0.458 = 9.775.$
   (c) $\ddot{a}_{70:67:\overline{10}|} = \ddot{a}_{70:67} - v^{10}\,{}_{10}p^m_{70} \cdot {}_{10}p^f_{67} . \ddot{a}_{80:77} = 7.458.$
   (d) $\ddot{a}^{(12)}_{70:67:\overline{10}|} = (\ddot{a}_{70:67} - 0.458) - v^{10}\,{}_{10}p^m_{70} \cdot {}_{10}p^f_{67} . (\ddot{a}_{80:77} - 0.458) = 7.204.$
   (e) $\ddot{a}_{\overline{70:67}} = \ddot{a}^m_{70} + \ddot{a}^f_{67} - \ddot{a}_{70:67} = 15.44.$
   (f) $\ddot{a}^{(12)}_{\overline{70:67}} = \ddot{a}_{\overline{70:67}} - 0.458 = 14.982.$

7. (a) $A_{\overline{xy}:\overline{n}|}$ is the EPV of an assurance of 1 payable at the end of the year in which the second of (x) and (y) dies, if that death occurs within n years, or 1 payable in n years exact if one or both (x) and (y) survive n years.

$$A_{\overline{xy}:\overline{n}|} = \mathrm{E}[v^{\min[K_{\max}+1,n]}] = \mathrm{E}[1 - d\ddot{a}_{\overline{\min[K_{\max}+1,n]}|}] = 1 - d\mathrm{E}[\ddot{a}_{\overline{\min[K_{\max}+1,n]}|}] = 1 - d\ddot{a}_{\overline{xy}:\overline{n}|}.$$

   (b) $\bar{A}_{xy:\overline{n}|}$ is the EPV of an assurance of 1 payable immediately upon the first death of (x) or (y), if that death occurs within n years.

$$\bar{A}_{xy:\overline{n}|} = \mathrm{E}[v^{\min[T_{\min},n]}] = \mathrm{E}[1 - \delta\bar{a}_{\overline{\min[T_{\min},n]}|}] = 1 - \delta\bar{a}_{xy:\overline{n}|}.$$