HERIOT-WATT UNIVERSITY
MSC IN ACTUARIAL SCIENCE

Life Insurance Mathematics I
Tutorial 1 Solutions

1. (a)

$$\begin{aligned} P\ddot{a}_{[60]:\overline{5}|} &= SA_{[60]:\overline{5}|} + 100 + 0.4P + 0.05P(\ddot{a}_{[60]:\overline{5}|} - 1) + 250 \\ \Rightarrow P(0.95 \times 4.559 - 0.35) &= 25000 \times 0.82465 + 100 + 250 \\ \Rightarrow P &= 5266.51 \end{aligned}$$

(b) Note that $q_{[60]} = 0.005774$, $q_{[60]+1} = 0.008680$, $q_{62} = 0.010112$, $q_{63} = 0.011344$, $q_{64} = 0.012716$.
The recursive formula indicates that $(V_t + P_t - E_t)(1+i) = V_{t+1} + q_{[60]+t}(S_{t+1} - V_{t+1})$ where $P_t = P$ for $t = 0, 1, 2, 3, 4$, $E_0 = 100 + 0.4P$, $E_t = 0.05P$ for $t = 1, 2, 3, 4$, and $S_{t+1} = S$ for $t = 0, 1, 2, 3, 4$.

$$\begin{aligned} V_5 &= 25000 \\ (V_4 + P_4 - E_4)(1+i) &= V_5 + q_{64}(S_5 - V_5) \\ \Rightarrow (V_4 + 0.95 \times 5266.51) \times 1.04 &= 25000 + q_{64} \times 0 \\ \Rightarrow V_4 &= 19035.28 \\ (V_3 + 0.95P) \times 1.04 &= V_4 + q_{63}(S - V_4) \\ \Rightarrow V_3 &= 13365.03 \\ (V_2 + 0.95P) \times 1.04 &= V_3 + q_{62}(S - V_3) \\ \Rightarrow V_2 &= 7960.93 \\ (V_1 + 0.95P) \times 1.04 &= V_2 + q_{[60]+1}(S - V_2) \\ \Rightarrow V_1 &= 2793.77 \\ (V_0 + 0.6P - 100) \times 1.04 &= V_1 + q_{[60]}(S - V_1) \\ \Rightarrow V_0 &= -250.30 \end{aligned}$$

(c) V_0 is (apart from rounding errors) equal to -1 times the expected present value of the profit on the contract. That is the reserve indicates that this profit could be taken immediately at time 0 in the knowledge that the future gross premiums will be sufficient to pay for the benefits (if experience is according to the reserve basis) even if there is an initial reserve of -250.

2. Let the annual amount of premium be P so that each monthly premium is $P/12$.

(a) Equation of value:

$$P\ddot{a}^{(12)}_{55:\overline{5}|} = 50,000\overline{A}^{1}_{55:\overline{5}|}$$

$$
\begin{aligned}
P\left(\ddot{a}_{55:\overline{5}|} - \frac{11}{24}(1 - D_{60}/D_{55})\right) &= 50,000\frac{i}{\delta}A^{1}_{55:\overline{5}|} \\
P\left(4.423 - \frac{11}{24}(1 - 0.72610)\right) &\approx 50,000 \times 1.029709 \times \\
&\qquad (0.26092 - 0.72610 \times 0.32692) \\
\rightarrow P &\approx £282.06 \\
\rightarrow P/12 &\approx £23.50(5)
\end{aligned}
$$

(b)

$$
\begin{aligned}
V_2 &= 50,000\overline{A}^{1}_{57:\overline{3}|} - 282.06\ddot{a}^{(12)}_{57:\overline{3}|} \\
&\approx 50,000\frac{i}{\delta}(0.28614 - 0.82365 \times 0.32692) - \\
&\qquad 282.06 \times (2.817 - \frac{11}{24}(1 - 0.82365)) \\
&\approx £96.91(5)
\end{aligned}
$$

(c) Consider the reserve prospectively. Immediately after the payment of the monthly premium, the expected present value (EPV) of future premiums decreases by the amount of the monthly premium while the EPV of the future benefits is unchanged. Hence, the reserve must increase by the amount of the monthly premium, and so:

$$V_{2+} = 96.915 + 23.505 = £120.42$$

(d) The reserve at time 2 represents the amount of money the office needs at that time so that with future premiums it can pay the future benefits. At time 2+ it has one less monthly premium to come in the future (compared to time 2-) and so needs an extra $P/12$ in its reserves.

3. (a) The death benefit in the (t + 1)th year is 100,000 + (V_{t+1})

so we have by the recursive formula:
$(V_t + P)(1 + i) = (V_{t+1} + q100,000)$
and putting t = 0, 1, 2, 3, 4, 5 with $V_0 = 0$ and $V_5 = 100,000$ gives

$$
\begin{aligned}
P(1+i) &= V_1 + q_{40}100,000 \\
(V_1 + P)(1+i) &= V_2 + q_{41}100,000 \\
(V_2 + P)(1+i) &= V_3 + q_{42}100,000 \\
(V_3 + P)(1+i) &= V_4 + q_{43}100,000 \\
(V_4 + P)(1+i) &= 100,000 + q_{44}100,000
\end{aligned}
$$

Multiplying the first equation by v, the second by v^2 etc., gives

$$\begin{aligned} P &= vV_1 + vq_{40}100,000 \\ vV_1 + vP &= v^2V_2 + v^2q_{41}100,000 \\ v^2V_2 + v^2P &= v^3V_3 + v^3q_{42}100,000 \\ v^3V_3 + v^3P &= v^4V_4 + v^4q_{43}100,000 \\ v^4V_4 + v^4P &= v^5100,000 + v^5q_{44}100,000 \end{aligned}$$

Adding all the equations will remove the intermediate values of V_t giving $P\ddot{a}_{\overline{5}|} = v^5100,000 + 100,000\sum_{t=1}^{5} v^t q_{39+t}$.
As there are only 5 years it is easy to calculate this last term exactly:

$$\begin{aligned} vq_{40} &= (0.96154)(0.00144) = 0.00138 \\ v^2q_{41} &= (0.92456)(0.00162) = 0.00150 \\ v^3q_{42} &= (0.88900)(0.00183) = 0.00163 \\ v^4q_{43} &= (0.85480)(0.00207) = 0.00177 \\ v^5q_{44} &= (0.82193)(0.00234) = 0.00192 \\ Total &= 0.00820 \end{aligned}$$

Then

$$\begin{aligned} P(4.6299) &= 82193 + 820 \\ P &= 17,930 \end{aligned}$$

(b) The premium without the special benefit can be calculated from

$$P = 100,000\frac{M_{40} - M_{45} + D_{45}}{N_{40} - N_{45}} = 100,000\frac{1909.50 - 1852.39 + 5689.18}{132001 - 99756} = 17,820.$$

4. (a)

$$\begin{aligned} P\ddot{a}_{[20]} &= 100,000A_{[20]} + 3,000(IA)_{[20]} + 200 + 0.05Pa_{[20]} \\ P(16.877) &= 100,000(0.04472) + 3,000(2.00874) + 200 - .05(16.877 - 1) \\ P &= £665.18 \end{aligned}$$

(b) Required provision is

$$\begin{aligned} &110,000A_{23} + 4,000(IA)_{23} - 0.95(665.18)\ddot{a}_{23} \\ &= 110,000(0.12469) + 4,000(6.09644) - 0.95(665.18)(22.758) \\ &= £23,720 \end{aligned}$$

5. (a)

$$\begin{aligned} P\ddot{a}_{35:\overline{5}|} - SA^{1}_{35:\overline{5}|} &= 200 \\ \Rightarrow 4.62151P - 0.0046380 \times 150000 &= 200 \\ \Rightarrow P &= 193.81 \end{aligned}$$

(b) We need to calculate the retrospective and prospective reserves separately, starting from $V_0^R = 0$ and $V_5^P = 0$. We also need:

$$\begin{aligned} q_{35} &= 0.00085577 \\ q_{36} &= 0.00093663 \\ q_{37} &= 0.00103409 \\ q_{38} &= 0.00114973 \\ q_{39} &= 0.00128528 \\ (V_t^R + P)(1+i) &= p_{35+t}V_{t+1}^R + q_{35+t}S \\ \Rightarrow V_{t+1}^R &= \frac{1}{1-q_{35+t}}\Big[(V_t^R + P)(1+i) - q_{35+t}S\Big] \\ \Rightarrow V_1^R &= 73.26 \\ V_2^R &= 137.39 \\ V_3^R &= 189.53 \\ V_4^R &= 226.47 \\ V_5^R &= 244.61 = \frac{D_{35}}{D_{40}} \times 200 \\ (V_t^P + P)(1+i) &= p_{35+t}V_{t+1}^P + q_{35+t}S \\ \Rightarrow V_t^P &= v(q_{35+t}(S - V_{t+1}^P) + V_{t+1}^P) - P \\ \Rightarrow V_4^P &= -8.43 \\ V_3^P &= -36.08 \\ V_2^P &= -79.32 \\ V_1^P &= -134.92 \\ V_0^P &= -200.00 = -E[PV_\pi] \end{aligned}$$