5 Markov Multiple-State Models

5.1 Two-state alive-dead model

Firstly, we review for single lives two equivalent model formulations. We also keep in the background that the ultimate aim is to derive the present value of insurance payments.

Formulation 1: Random Variable

T_x represents the complete future lifetime of (x). Then:

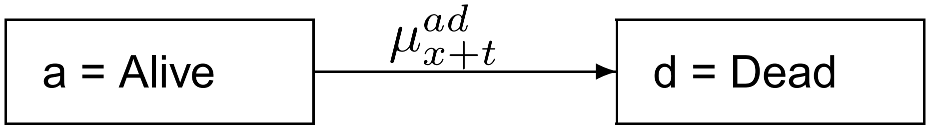
(a) the c.d.f. is $F_x(t)$

(b) the p.d.f. is $f_x(t)$

(c) the force of mortality is $\dfrac{f_x(t)}{1 - F_x(t)}$.

Formulation 2: Multiple-State Model

There are two states, ‘alive’ and ‘dead’, and a single transition between them.



Define ${}_tp_s^{ij}$ to be the **probability of being in state j at age $s+t$, conditional on being in state i at age s.**

The time of death is governed by the **transition intensity** μ_{x+t}^{ad} by making the following assumptions:

(i) The intensity μ_{x+t}^{ad} depends only on the current age $x+t$ and **not on any other aspect of the life's past history.**

(ii) The probability of dying before age $x+t+dt$, conditional on being alive at age $x+t$, is

$$ {}_{dt}p_{x+t}^{ad} = \mu_{x+t}^{ad} dt + o(dt). $$

Notes: (1) A function $f(t)$ is said to be '$o(dt)$' if:

$$\lim_{dt \to 0} \frac{f(dt)}{dt} = 0.$$

(2) The superscript 'ad' on the transition intensity indicates that it refers to the transition from the state labelled 'a' to the state labelled 'd'.

(3) Assumption (i) is the **Markov property.**

(4) Assumption (ii) defines the behaviour of the model over infinitesimal time intervals dt. The key question then is: how does this determine the model's behaviour over extended time intervals, e.g. years?

The answer to this key question is that Assumptions (i) and (ii) allow us to derive the **the Kolmogorov differential equation**and **Thiele's differential equation**and we have already seen, in Section 3, that **these can be solved numerically for all probabilities and EPVs that we may need.**

Derivation of the Kolmogorov Equation from Assumptions (i) and (ii):

Consider ${}_{t+dt}p_x^{ad}$ **(note that this is the same as the life table ${}_{t+dt}q_x$).**Condition on the state occupied at age $x + t$:

$$
\begin{aligned}
{}_{t+dt}p_x^{ad} &= {}_tp_x^{ad} \; {}_{dt}p_{x+t}^{dd} + {}_tp_x^{aa} \; {}_{dt}p_{x+t}^{ad} \\
&= {}_tp_x^{ad} \times 1 + {}_tp_x^{aa} \left(\mu_{x+t} dt + o(dt) \right)
\end{aligned}
$$

Therefore:

$$
\frac{{}_{t+dt}p_x^{ad} - {}_tp_x^{ad}}{dt} = {}_tp_x^{aa} \, \mu_{x+t} + \frac{o(dt)}{dt}.
$$

Take limits as $dt \longrightarrow 0$ and:

$$
\frac{d}{dt} {}_tp_x^{ad} = {}_tp_x^{aa} \, \mu_{x+t}.
$$

Since ${}_tp_x^{aa} + {}_tp_x^{ad} = 1$, this is equivalent to:

$$\frac{d}{dt}{}_tp_x^{aa} = -{}_tp_x^{aa}\,\mu_{x+t}$$

which is the Kolmogorov equation as in Section 3.

5.2 The general Markov multiple-state model

The real importance of reformulating the life table model as a Markov model is that this generalises to more complicated problems that form the basis of other insurance contracts and problems.

Our aim in any given case is to model the **life history**of a person initially age (x), of which ‘alive or dead’ is merely the simplest possible example. In general, we have a finite set of M states $\mathcal{S}$. The states in $\mathcal{S}$ may be labelled by numbers:

$$\mathcal{S} = \{1, 2, \ldots, M\}$$

or by letters:

$$\mathcal{S} = \{a, b, c, \ldots\}$$

or in any other convenient way. For each pair of distinct states i and j in $\mathcal{S}$, the probability of making a transition from state i to state j at age $x+t$ (conditional on then being in state i) is governed by a transition intensity μ_{x+t}^{ij}. This statement is given precise meaning by making the following assumptions:

(i) All intensities μ_{x+t}^{ij} depend only on the current age $x+t$ and **not on any other aspect of the life's past history.**

(ii) The probability of making a transition $i \longrightarrow j$ before age $x+t+dt$, conditional on being in state i at age $x+t$, is

$$_{dt}p_{x+t}^{ij} = \mu_{x+t}^{ij} dt + o(dt).$$

(iii) The probability of making any two or more transitions **in time dt is $o(dt)$.**

5.3 More examples of Markov models

Figure 4 shows a model of a life insurance contract including the possibility that the policyholder chooses to terminate the contract early **(known as 'withdrawal').** We have chosen to label the states and transition intensities by numbers.

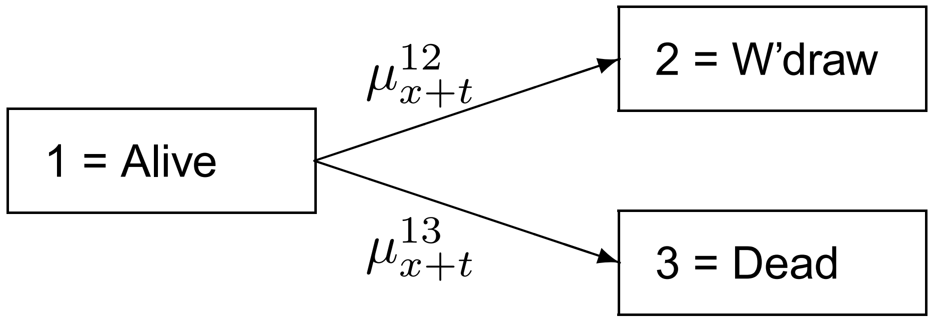


Figure 4: Multiple decrement model: a single life subject to more than one decrement.

Figure 5 shows a model suitable for underwriting disability insurance, which **replaces part of the policyholder's earnings while too ill to work.** We have chosen to denote the transition intensities individually by Greek letters.

Note that in Figure 5 **the number of movements between the 'Able' and 'Ill' states is not bounded but the model is fully specified in terms of just four intensities.** This hints at the problems we would enountered were we to attempt to specify this model **in terms of random times between transitions, i.e. the analogues of the random lifetime T_x.**

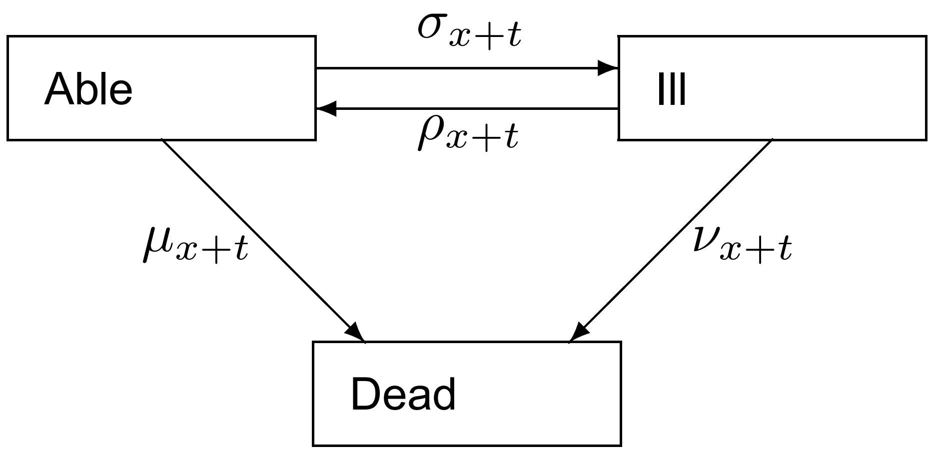


Figure 5: Disability insurance: premiums are paid while 'able' and an annuity-type benefit is payable while 'ill'.

Figure 6 shows a restricted version of the disability insurance model, that covers only permanent, irrecoverable illnesses.

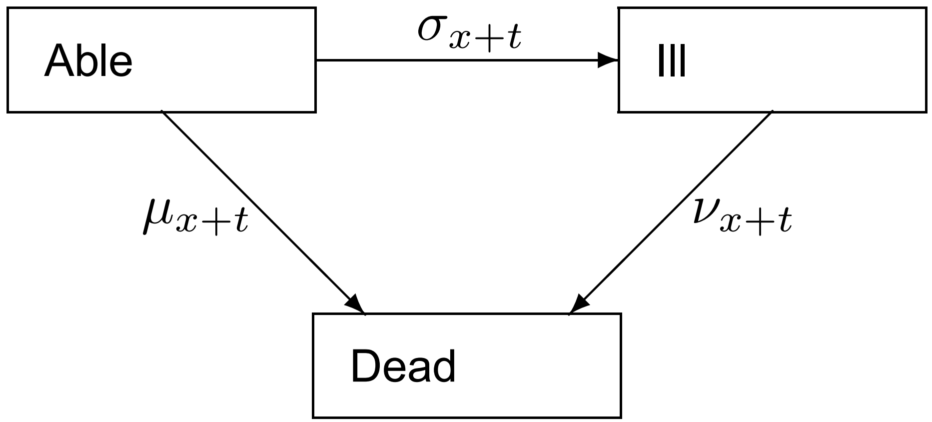


Figure 6: Permanent disability/Terminal Illness: This model is similar to that shown for disability insurance but only covers irrecoverable illnesses.

Figures 7 and 8 show two possible models for **long-term care (LTC) insurance**, which provides for the cost of care at home or in a nursing institution in old age (usually). Claims may be made upon the loss of a certain number of **activities of daily living (ADLs)**which are essential to be able to care for oneself properly.

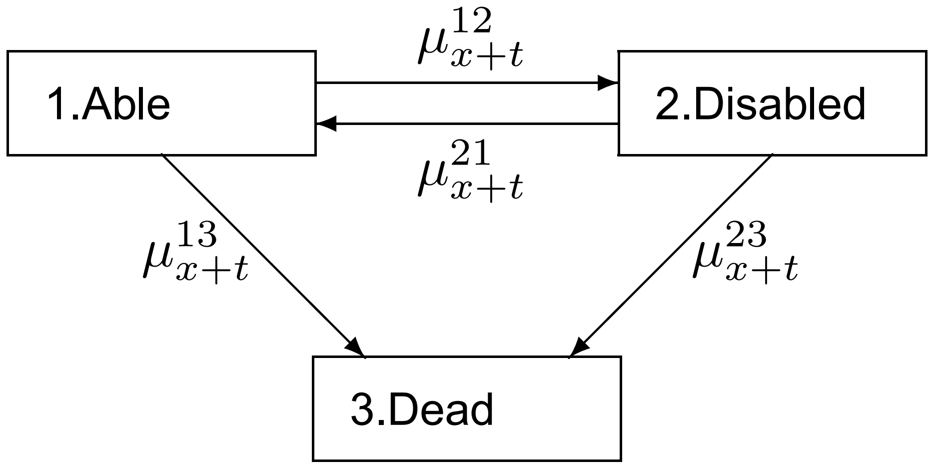


Figure 7: Long-term care: This model uses the loss of activities of daily living (ADLs) as a definition of disability and for the purposes of validating claims. ABI benchmark ADLs are: washing, dressing, mobility, toiletting, feeding and transferring.

Note that, from a mathematical point of view, the LTC model in Figure 7 is identical to the disability model in Figure 5. The only difference lies in the values of the intensities, **which must be parameterised using suitable data.**

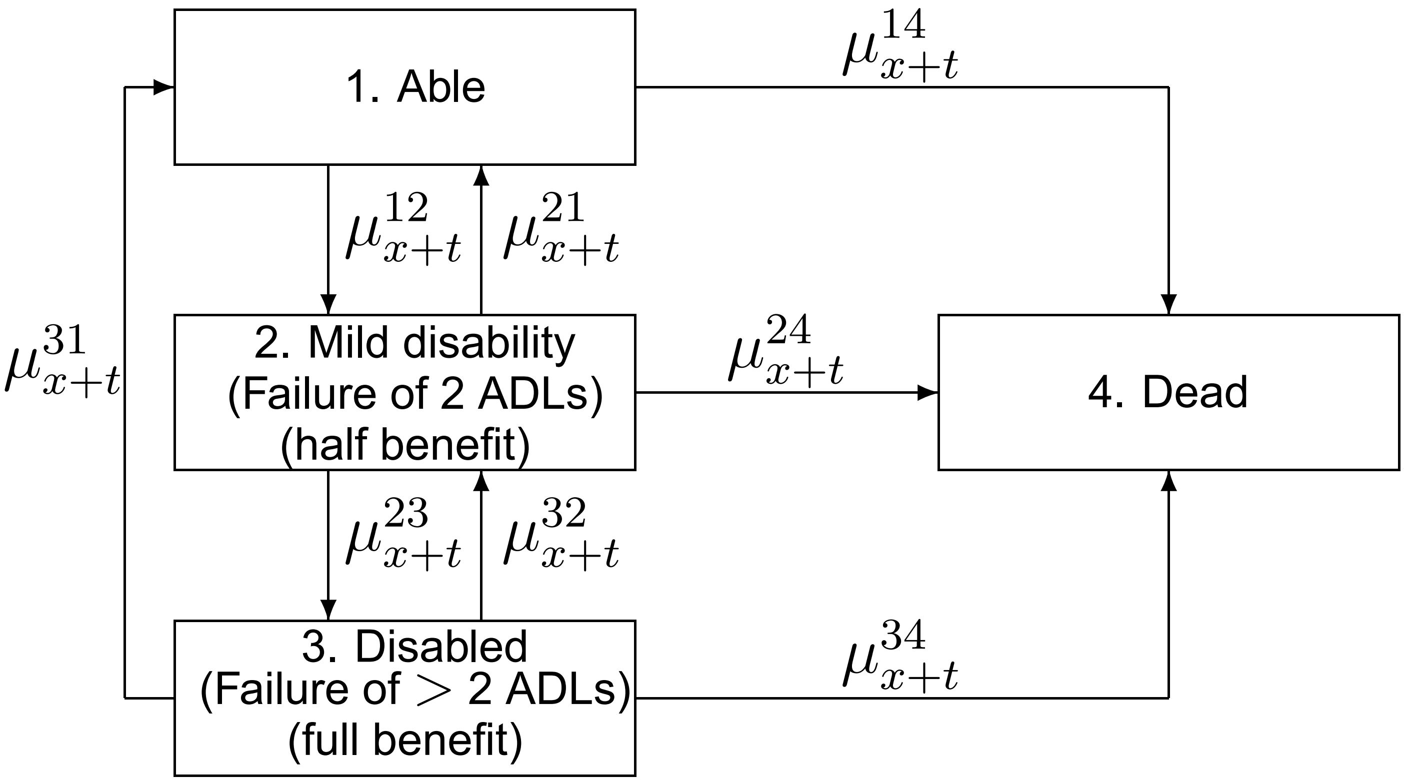


Figure 8: Long-term care, expanded: This model is similar to the simple model of long-term care, except that it allows for some benefits to be paid on partial disability (defined as the

In general, multiple state models specified in terms of intensities are important because:

(a) They give a general method can be applied to different problems;

(b) Statistical inference **(i.e. estimating the intensities)**is relatively easy (this is covered in the *Survival Models* modules;

(c) All such models are **specified in terms of a finite number of intensities,**whereas the number of events (i.e. transfers between states) **need not be bounded;**and

(d) Very general methods (solving ODEs) are available for calculating probabilities and EPVs.

5.4 The Kolmogorov equations: Derivation

Define the following **occupancy probabilities**(the first of which we have already seen).

$$
\begin{aligned}
{}_tp_x^{ij} &= \mathbf{P}\big[\text{in state } j \text{ at age } x+t \mid \\
&\qquad\qquad \text{in state } i \text{ at age } x\big] \\
{}_tp_x^{\overline{ii}} &= \mathbf{P}\big[\text{in state } i \text{ for ages } x \rightarrow x+t \mid \\
&\qquad\qquad \text{in state } i \text{ at age } x\big]
\end{aligned}
$$

Note that ${}_tp_x^{\overline{ii}} \neq {}_tp_x^{ii}$ in general. The first means that the life **never leaves state** $i$while the second allows the life to **leave and then return to state** i**.**They are equal if and only if **return to state** i **is impossible.**

We have the following results for ${}_tp_x^{\overline{ii}}$:

$$\text{(5.10)} \qquad {}_{dt}p_x^{\overline{ii}} = {}_{dt}p_x^{ii} + o(dt)$$

$$\text{(5.11)} \qquad {}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j\neq i} \mu_{x+s}^{ij} ds\right).$$

Proof of (5.10):

If the life is in state i at age x and at age $x + dt$, there are just two possibilities:

(i) The life never left state i. This has probability ${}_{dt}p_x^{\overline{ii}}$ by definition.

(ii) The life left state i and returned to it, which implies two or more transitions in time dt. This has probability $o(dt)$ by assumption (iii).

Therefore, by the law of total probability:

$$ {}_{dt}p_x^{ii} = {}_{dt}p_x^{\overline{ii}} + o(dt)$$

which is equivalent to:

$$ {}_{dt}p_x^{\overline{ii}} = {}_{dt}p_x^{ii} + o(dt). $$

Proof of (5.11):

Condition on being in state i at time t.

$$
\begin{aligned}
{}_{t+dt}p_x^{\overline{ii}} &= {}_tp_x^{\overline{ii}} \times {}_{dt}p_{x+t}^{\overline{ii}} \\
&= {}_tp_x^{\overline{ii}} \times \left({}_{dt}p_{x+t}^{ii} + o(dt) \right) \\
&= {}_tp_x^{\overline{ii}} \times \left(1 - \sum_{j \neq i} {}_{dt}p_{x+t}^{ij} + o(dt) \right) \\
&= {}_tp_x^{\overline{ii}} \times \left(1 - \sum_{j \neq i} \mu_{x+t}^{ij}\, dt + o(dt) \right).
\end{aligned}
$$

Hence:

$$\frac{{}_{t+dt}p_x^{\overline{ii}} - {}_{dt}p_x^{\overline{ii}}}{dt} = -{}_tp_x^{\overline{ii}} \left(\sum_{j \neq i} \mu_{x+t}^{ij} \right) + \frac{o(dt)}{dt}$$

and on taking limits as $dt \rightarrow 0$:

$$\frac{d}{dt} {}_tp_x^{\overline{ii}} = -{}_tp_x^{\overline{ii}} \left(\sum_{j \neq i} \mu_{x+t}^{ij} \right).$$

This is a familiar ODE (think of the Kolmogorov equation of the ordinary life table) which has boundary condition ${}_0p_x^{\overline{ii}} = 1$ and solution:

$$
{}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \sum_{j\neq i} \mu_{x+s}^{ij}\, ds\right).
$$

The **Kolmogorov (forward) differential equations** are a **system of simultaneous equations** for *all* the probabilities ${}_tp_x^{ij}$, including the case $i = j$. They are as follows:

$$\frac{d}{dt}\,{}_tp_x^{ij} = \sum_{k \neq j} {}_tp_x^{ik}\, \mu_{x+t}^{kj} - {}_tp_x^{ij} \sum_{k \neq j} \mu_{x+t}^{jk}$$

for all i and j in $\mathcal{S}$.

Proof:

Consider ${}_{t+dt}p_x^{ij}$ and **Condition on the state occupied at age $x + t$:**

$$\begin{aligned}
{}_{t+dt}p_x^{ij} &= \sum_{k \in \mathcal{S}} {}_tp_x^{ik}\, {}_{dt}p_{x+t}^{kj} \\
&= \sum_{k \neq j} {}_tp_x^{ik}\, {}_{dt}p_{x+t}^{kj} + {}_tp_x^{ij}\, {}_{dt}p_{x+t}^{jj} \\
&= \sum_{k \neq j} {}_tp_x^{ik}\, (\mu_{x+t}^{kj}\, dt + o(dt))
\end{aligned}$$

$$
\begin{aligned}
& + \;\; {}_t p_x^{ij} \left(1 - \sum_{k \neq j} {}_{dt} p_{x+t}^{jk} \right) \\
= \;\; & {}_t p_x^{ij} + \sum_{k \neq j} {}_t p_x^{ik} \, (\mu_{x+t}^{kj} \, dt + o(dt)) \\
& - \;\; {}_t p_x^{ij} \sum_{k \neq j} (\mu_{x+t}^{jk} \, dt + o(dt)).
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\frac{{}_{t+dt} p_x^{ij} - {}_t p_x^{ij}}{dt} \;\; = \;\; & \sum_{k \neq j} {}_t p_x^{ik} \, \mu_{x+t}^{kj} - {}_t p_x^{ij} \sum_{k \neq j} \mu_{x+t}^{jk} \\
& + \;\; \frac{o(dt)}{dt}.
\end{aligned}
$$

Take limits as $dt \rightarrow 0$ and:

$$\frac{d}{dt}\,{}_tp_x^{ij} = \sum_{k \neq j} {}_tp_x^{ik}\,\mu_{x+t}^{kj} - {}_tp_x^{ij} \sum_{k \neq j} \mu_{x+t}^{jk} \tag{5.12}$$

as required.

5.5 The Kolmogorov equations: Numerical solution

The Kolmogorov equations form a **system of simultaneous ODEs**because the right-hand side of Equation (5.12) contains probabilities other than ${}_tp_x^{ij}$ whose derivative appears on the left.

Most numerical methods of solving a single ODE can be extended to solve a system of ODEs very simply. All that is necessary is that at each step, the solution of the **entire system of equations is advanced before going to the next step.**

For example, suppose there are three states, $\mathcal{S} = \{1, 2, 3\}$. There are therefore nine equations of the form (5.12). Some of these may be trivial. Now suppose, for example, we wish to solve these over a period of 10 years, with step size $h = 0.01$ years, therefore **1,000 steps in total.**

The **WRONG**approach is to take the first ODE, for $\frac{d}{dt}\,{}_tp_x^{11}$, and try to advance its solution for the entire 1,000 steps, to obtain:

$${}_hp_x^{11},\, {}_{2h}p_x^{11},\, \ldots,\, {}_{1{,}000h}p_x^{11}$$

before considering the other equations in the system. This will **FAIL**because the other probabilities are needed at each step.

The **RIGHT**approach is to advance **ALL**the equations one step, to obtain:

$${}_hp_x^{11},\, {}_hp_x^{12},\, \ldots,\, {}_hp_x^{33}.$$

Then, using these values, advance the whole system one more step to obtain:

$$ {}_{2h}p_x^{11}, {}_{2h}p_x^{12}, \ldots, {}_{2h}p_x^{33} $$

and so on.

For a concrete example, we use the disability model of Figure 5. This has transition intensities labelled μ_{x+t}, ν_{x+t}, σ_{x+t} and ρ_{x+t}. It is easily shown that the Kolmogorov equations for ${}_tp_x^{11}$ and ${}_tp_x^{12}$ depend on each other but not on any other occupancy probabilities (see tutorial). They are:

$$
\begin{aligned}
\frac{d}{dt}{}_tp_x^{11} &= {}_tp_x^{12}\,\rho_{x+t} - {}_tp_x^{11}\,(\sigma_{x+t} + \mu_{x+t}) \\
\frac{d}{dt}{}_tp_x^{12} &= {}_tp_x^{11}\,\sigma_{x+t} - {}_tp_x^{12}\,(\rho_{x+t} + \nu_{x+t}).
\end{aligned}
$$

We assume that the life is healthy at age x when the disability insurance policy is sold, so the **boundary conditions are ${}_0p_x^{11} = 1$ and ${}_0p_x^{12} = 0$.**We use Euler's method with **step size h.**The first step is:

$$
\begin{aligned}
{}_hp_x^{12} &\approx {}_0p_x^{12} + h\,\frac{d}{dt}\,{}_tp_x^{12}\Big|_{t=0} \\
&= {}_0p_x^{12} + h\left[{}_0p_x^{11}\cdot\sigma_x - {}_0p_x^{12}\cdot(\rho_x+\nu_x)\right] \\
&= 0 + h\left[1\cdot\sigma_x - 0\right] \\
&= h\cdot\sigma_x \qquad (5.13)
\end{aligned}
$$

and:

$$
\begin{aligned}
{}_hp_x^{11} &\approx {}_0p_x^{11} + h\,\frac{d}{dt}\,{}_tp_x^{11}\Big|_{t=0} \\
&= {}_0p_x^{11} + h\left[{}_0p_x^{12}\cdot\rho_x - {}_0p_x^{11}\cdot(\sigma_x+\mu_x)\right] \\
&= 1 + h\left[0 - 1\cdot(\sigma_x+\mu_x)\right] \\
&= 1 - h(\sigma_x+\mu_x). \qquad (5.14)
\end{aligned}
$$

Using ${}_hp_x^{12}$ and ${}_hp_x^{11}$ as our new boundary conditions we can perform another Euler step to get approximate values of ${}_{2h}p_x^{12}$ **and** ${}_{2h}p_x^{11}$**:**

$$
\begin{aligned}
{}_{2h}p_x^{12} &\approx {}_{h}p_x^{12} + h \, \frac{d}{dt} {}_{t}p_x^{12} \Big|_{t=h} \\
&= {}_{h}p_x^{12} + h \left[{}_{h}p_x^{11} \, \sigma_{x+h} \right. \\
&\qquad \left. - {}_{h}p_x^{12} \left(\rho_{x+h} + \nu_{x+h} \right) \right] \\
&= {}_{h}p_x^{12} \left(1 - h \left(\rho_{x+h} + \nu_{x+h} \right) \right) + {}_{h}p_x^{11} \, h \, \sigma_{x+h}
\end{aligned}
$$

and

$$
\begin{aligned}
{}_{2h}p_x^{11} &\approx {}_{h}p_x^{11} + h \, \frac{d}{dt} {}_{t}p_x^{11} \Big|_{t=h} \\
&= {}_{h}p_x^{11} + h \left[{}_{h}p_x^{12} \, \rho_{x+h} \right. \\
&\qquad \left. - {}_{h}p_x^{11} \left(\sigma_{x+h} + \mu_{x+h} \right) \right] \\
&= {}_{h}p_x^{11} \left(1 - h \left(\mu_{x+h} + \sigma_{x+h} \right) \right) + {}_{h}p_x^{12} \, h \, \rho_{x+h}
\end{aligned}
$$

where ${}_hp_x^{12}$ and ${}_hp_x^{11}$ are given by equations (5.13) and (5.14) respectively.

We repeat this process for any required policy term.

5.6 Thiele's equations: Informal derivation

When we consider the reserves that need to be held for an insurance policy more general than life insurance, e.g. disability insurance, it is **clear that a different reserve needs to be held, depending on the state currently occupied.**For example, under disability insurance:

- If the life is currently healthy, it is **certain**that they are currently paying premiums and **possible**that they might, in future, receive benefits.
- If the life is currently sick, it is **certain**that they are currently receiving benefits and **possible**that they might, in future, resume paying premiums.

The life office's liability is different in each case. Define $V^i(t)$to be the policy value, on a given valuation basis, in respect of a life in state $i \in \mathcal{S}$ at time t.

Given a Markov model with states $\mathcal{S}$, a **general insurance contract**is defined by specifying the following cashflows, by analogy with a life insurance policy whose benefits are payable immediately on death and whose premiums are payable continuously:

- For all i in $\mathcal{S}$, **an annuity-type benefit payable continuously at rate $b_i(t)$ per annum if the life is in state i at time t.**Premiums payable by the policyholder are just treated as a negative benefit.
- For all distinct i, j in $\mathcal{S}$, **a sum assured of $b_{ij}(t)$ payable immediately on a transition from state i to state j at time t.**

If, as is often the case, cashflows do not depend on t, we just write b_i **and** b_{ij}.

Given a Markov model with states $\mathcal{S}$, a **valuation basis**is defined by:

- A force of interest $\delta(t)$, which we often assume to be a constant δ.
- A complete set of transition intensities μ_{x+t}^{ij} for all distinct i, j in $\mathcal{S}$.

- Possibly, expenses payable continuously at rate $e_i(t)$ per annum if in state i at time t, or as a lump sum $e_{ij}(t)$ on transition from state i to state j at time t. Clearly these are analagous to the benefits $b_i(t)$ and $b_{ij}(t)$.

In what follows we ignore expenses.

These policy values are obtained as the **solution of Thiele's differential equations.**We apply exactly the same logic as for the whole life policy, by supposing the life to be in state i at time t (i.e. age $x + t$) and asking, **what happens in the next time dt?**

(1) The reserve currently held is **equal to the policy value $V^i(t)$, by definition.**

(2) In time dt, interest of $V^i(t)\,\delta(t)\,dt$will be earned by these assets.

(3) In time dt, a cashflow of $b_i(t)\,dt$will be paid by the office.

(4) For *each* state $j \neq i$ in $\mathcal{S}$, **a transition to state j may occur, with probability $\mu_{x+t}^{ij}\,dt$.**If it does, the following happens:

   - **the sum assured $b_{ij}(t)$ is paid;**
   - **the reserve necessary while in state j, equal to the policy value $V^j(t)$, must**

be set up;

– **the reserve being held, $V^i(t)$, is available to offset these costs.**

(Some of these may be zero, depending on the policy design.) The expected cost of a transition into state j is therefore:

$$\mu_{x+t}^{ij}\, dt\, (b_{ij}(t) + V^j(t) - V^i(t)).$$

Putting these together, the **expected change in the reserve held**is:

$$\begin{aligned} & V^i(t+dt) - V^i(t) \\ = \quad & V^i(t)\,\delta(t)\,dt - b_i(t)\,dt \\ & - \sum_{j \neq i} \mu_{x+t}^{ij}\, dt\, (b_{ij}(t) + V^j(t) - V^i(t)). \end{aligned}$$

Divide by dt and take limits as $dt \to 0$, and we obtain the general form of Thiele's equations:

$$\begin{aligned}\frac{d}{dt}V^i(t) &= V^i(t)\,\delta(t) - b_i(t) \\ &\quad - \sum_{j \neq i} \mu_{x+t}^{ij}(b_{ij}(t) + V^j(t) - V^i(t)).\end{aligned}$$

Note that this is a **system of simultaneous ODEs, one for each state i.** If, as is usually the case, benefits and force of interest do not depend on t, we get the simpler system:

$$\begin{aligned}\frac{d}{dt}V^i(t) &= V^i(t)\,\delta - b_i \\ &\quad - \sum_{j \neq i} \mu_{x+t}^{ij}(b_{ij} + V^j(t) - V^i(t)).\end{aligned}$$

Note: This is not a rigorous mathematical derivation of Thiele's equations. To give one would require a deeper background in a certain class of stochastic processes called *counting processes.*

For a concrete example, return to the disability insurance contract of Figure 5. Suppose the premiums and benefits are defined as follows:

- Premiums at rate $\bar{P}$ per annum are payable while able, i.e. $b_1(t) = -\bar{P}$.
- Sickness benefits at rate $\bar{B}$ per annum are payable while sick, i.e. $b_2(t) = +\bar{B}$.
- A death benefit of S is payable immediately on death, i.e. $b_{13}(t) = b_{23}(t) = S$.
- The policy expires after n years.

Suppose the valuation basis is as follows:

- Constant force of interest δ.
- Transition intensities μ_{x+t}, ν_{x+t}, σ_{x+t} and ρ_{x+t} as in Figure 5.
- No expenses.

Then Thiele's differential equations are:

$$
\begin{aligned}
\frac{d}{dt}V^1(t) &= V^1(t)\,\delta + \bar{P} - \mu_{x+t}\left(S - V^1(t)\right) \\
&\quad -\sigma_{x+t}\left(V^2(t) - V^1(t)\right) \\
\frac{d}{dt}V^2(t) &= V^2(t)\,\delta - \bar{B} - \nu_{x+t}\left(S - V^2(t)\right) \\
&\quad -\rho_{x+t}\left(V^1(t) - V^2(t)\right) \\
\frac{d}{dt}V^3(t) &= 0
\end{aligned}
$$

5.7 Thiele's equations: Numerical solution

We will always solve Thiele's differential equation **backwards from terminal boundary values of the $V^i(t)$.** This is because these are easy to state. Suppose a policy expires at duration n years. Then:

- If there is a maturity benefit (pure endowment type) of $\pounds M_i$ if the policy expires with the life in state i, then $V^i(n) = M_i$.
- If there is no maturity benefit if the policy expires with the life in state i, then $V^i(n) = 0$.

It would be very difficult to specify initial values $V^i(0)$ in advance.

We use an Euler scheme by analogy with Section 3.5, advancing the solution of the entire system forward one step at a time, as we did for the Kolmogorov equations (in the other direction).

The following are the first few steps of an Euler scheme with step size $-h$ for the disability insurance policy and valuation basis discussed in the last section. We note that the boundary conditions are $V^i(n) = 0$ **for all** i,and we ignore $V^3(t)$ since it is clearly always zero. The first step is:

$$V^1(n-h) \quad \approx \quad V^1(n) - h \, \frac{d}{dt} V^1(t)\bigg|_{t=n}$$

$$
\begin{aligned}
&= V^1(n) - h\Big[V^1(n)\,\delta + \bar{P} \\
&\qquad -\mu_{x+n}\left(S - V^1(n)\right) \\
&\qquad -\sigma_{x+n}\left(V^2(n) - V^1(n)\right)\Big]
\end{aligned}
$$

and:

$$
\begin{aligned}
V^2(n-h) &\approx V^2(n) - h\,\frac{d}{dt}V^2(t)\Big|_{t=n} \\
&= V^2(n) - h\Big[V^2(n)\,\delta - \bar{B} \\
&\qquad -\nu_{x+n}\left(S - V^2(n)\right) \\
&\qquad -\rho_{x+n}\left(V^1(n) - V^2(n)\right)\Big],
\end{aligned}
$$

the second step is:

$$
\begin{aligned}
& V^1(n-2h) \\
\approx\ & V^1(n-h) - h\,\frac{d}{dt}V^1(t)\Big|_{t=n-h} \\
=\ & V^1(n-h) - h\Big[V^1(n-h)\,\delta + \bar{P} \\
& \quad -\mu_{x+n-h}\left(S - V^1(n-h)\right) \\
& \quad -\sigma_{x+n-h}\left(V^2(n-h) - V^1(n-h)\right)\Big]
\end{aligned}
$$

and:

$$
\begin{aligned}
& V^2(n-2h) \\
\approx\ & V^2(n-h) - h\,\frac{d}{dt}V^2(t)\Big|_{t=n-h} \\
=\ & V^2(n-h) - h\Big[V^2(n-h)\,\delta - \bar{B}
\end{aligned}
$$

$$-\nu_{x+n-h}\left(S - V^2(n-h)\right)$$
$$-\rho_{x+n-h}\left(V^1(n-h) - V^2(n-h)\right)\Big],$$

and so on.

5.8 Comments

We finish with two brief but important observations.

(1) The multiple state models illustrated here are all models of the **life history**of a given person, where states and transitions define the events that may be of interest. Models of various insurance contracts are built upon these by defining the insurance cashflows, here denoted $b_i(t)$ and $b_{ij}(t)$, and interest and expenses, but these are not themselves part of the underlying life history models.

(2) The Markov assumption was essential in the above development. In particular we used it when we assumed we could **define policy values $V^i(t)$ that depended on the**

state occupied at time t and nothing else.